THE SHIFT OF THE PRINCIPAL EIGENVALUE FOR THE NEUMANN LAPLACIAN IN A DOMAIN WITH MANY SMALL HOLES IN R^d, $d \geq 2$

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Abstract. In this paper we introduce a new probabilistic method to evaluate the asymptotic behavior of the principal eigenvalue for the Neumann Laplacian in a region from which many small holes have been deleted and on which the Dirichlet boundary condition has been imposed. We assume that the centers of the holes form a regular lattice with spacing $\delta \to 0$. If there are $n = n(\delta)$ holes, let $\{x_j^{(n)}\}_{j=1}^n$ denote their centers and let $\epsilon_n \in (0, \frac{\delta}{2})$ denote the common length of their radii. Define $A_n = \cup_{j=1}^n \bar{B}_{\epsilon_n}(x_j^{(n)})$, where $B_r(x)$ denotes the open ball of radius r centered at $x \in R^d$. We evaluate the asymptotic behavior of the principal eigenvalue $\lambda_N^{(n)}$ for $-\Delta$ in $\Omega - A_n$ with the Neumann boundary condition on $\partial\Omega$ and the Dirichlet boundary condition on ∂A_n. Assuming that

$$\begin{cases} \lim_{n\to\infty} n\epsilon_n^{d-2} = \alpha \in [0,\infty], \text{ if } d \geq 3; \\ \lim_{n\to\infty} \frac{n}{-\log \epsilon_n} = \alpha \in [0,\infty], \text{ if } d = 2, \end{cases}$$

we prove that

$$\lim_{n\to\infty} \lambda_N^{(n)} = \begin{cases} \frac{d(d-2)\omega_d\alpha}{|\Omega|}, \text{ if } d \geq 3; \\ \frac{2\pi\alpha}{|\Omega|}, \text{ if } d = 2. \end{cases}$$

Let $\Omega \subset R^d$, $d \geq 2$, be a bounded domain with smooth boundary. For each positive integer n, let $\{x_j^{(n)}\}_{j=1}^n$ be points in Ω. Let $\{\epsilon_n\}_{n=1}^\infty$ be a sequence of positive numbers decreasing to 0, and define $A_n = \cup_{j=1}^n \bar{B}_{\epsilon_n}(x_j^{(n)})$, where $B_r(x)$ denotes the open ball of radius r centered at $x \in R^d$. Denote by $\lambda_N^{(n)}$ the principal eigenvalue for $-\Delta$ in $\Omega - A_n$ with the Neumann boundary condition on $\partial\Omega$ and the

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Dirichlet boundary condition on ∂A_n, and denote by $\lambda_D^{(n)}$ the principal eigenvalue for $-\Delta$ in $\Omega - A_n$ with the Dirichlet boundary condition on both $\partial\Omega$ and ∂A_n. In the sequel we will refer to these two situations as the Neumann case and the Dirichlet case respectively.

For the Dirichlet case with $d = 3$, Kac showed that if the points $\{x_j^{(n)}\}_{j=1}^n$ are independently distributed according to the uniform distribution on Ω, and $\lim_{n\to\infty} n\epsilon_n = \alpha \in [0, \infty]$, then the principal eigenvalue $\lambda_D^{(n)}$ satisfies

$$\lim_{n\to\infty} \lambda_D^{(n)} = \lambda_D + \frac{4\pi\alpha}{|\Omega|} \quad \text{in probability,}$$

where λ_D is the principal eigenvalue for $-\Delta$ in Ω with the Dirichlet boundary condition on $\partial\Omega$, and $|\Omega|$ is the volume of Ω [5]. Kac used probabilistic methods involving the asymptotics of the Wiener sausage. (See Simon [12, chapter 22] for a good exposition of this method, some of whose details are omitted in Kac's paper.)

Still assuming that $d = 3$ and that $\lim_{n\to\infty} n\epsilon_n = \alpha \in [0, \infty]$, but choosing the points $\{x_j^{(n)}\}_{j=1}^n$ independently according to a density function V (with $\int_\Omega V(x)dx = 1$), Rauch and Taylor [11] and Ozawa [7] showed for the Dirichlet case that

$$\lim_{n\to\infty} \lambda_D^{(n)} = \lambda_D^V \quad \text{in probability,}$$

where λ_D^V is the principal eigenvalue for $-\Delta + 4\pi\alpha V$ in Ω with the Dirichet boundary condition on $\partial\Omega$. Rauch and Taylor used functional analytic methods and the Feynman-Kac formula, whereas Ozawa used Green's function perturbation methods. (In the above-cited results, the actual constant appearing in the papers differs by a factor of 2 because the operator $\frac{1}{2}\Delta$ is used instead of Δ.)

The first treatment of the Neumann case was in a paper by Rauch [10] where the following bounds were obtained for $\lambda_N^{(n)}$ when $d = 3$: $\lambda_N^{(n)} \le cn\epsilon_n$, for small $n\epsilon_n$ and some $c > 0$, and, if the holes satisfy a kind of even spacing assumption, $\lambda_N^{(n)} \ge c_1 n\epsilon_n - c_2$, for all n and some positive constants c_1, c_2. Later Chavel and Feldman [1] proved that if the points $\{x_j^{(n)}\}_{j=1}^n$ are independently distributed according to the uniform distribution on Ω, and if

$$\begin{cases} \lim_{n\to\infty} n\epsilon_n^{d-2} = \alpha \in [0,\infty], \text{ if } d \ge 3; \\ \lim_{n\to\infty} \frac{n}{-\log \epsilon_n} = \alpha \in [0,\infty], \text{ if } d = 2, \end{cases}$$

then

$$\lim_{n\to\infty} \lambda_N^{(n)} = \begin{cases} \frac{d(d-2)\omega_d \alpha}{|\Omega|} \text{ in probability, if } d \ge 3; \\ \frac{2\pi\alpha}{|\Omega|} \text{ in probability, if } d = 2, \end{cases}$$

where ω_d is the volume of the unit ball in R^d. Chavel and Feldman used Wiener sausage techniques along with delicate analytic estimates that go back to Minakshisundaram and Weyl [6] for the fundamental solution of the heat equation in a domain with the Neumann boundary condition.

In this paper, we use completely different probabilistic methods to obtain the deterministic version of the above result of Chavel and Feldman under the assumption of deterministic lattice spacing, which we now describe. In the future we hope to apply the methods developed here to treat eigenvalue asymptotics for other problems in domains with many small holes.

For $\delta > 0$, consider the δ-lattice in R^d consisting of all points of the form δx where $x \in R^d$ has integral coefficients. A lattice point contained in Ω will be called a boundary point if at least one of its nearest neighbors in the lattice is not contained in Ω, and will be called an interior point otherwise. Let $n = n(\delta)$ denote the number of interior lattice points. Clearly,

$$\lim_{\delta\to 0} n\delta^d = |\Omega|. \tag{1.1}$$

In the sequel it will be more natural to treat n rather than δ as the independent parameter, even though it is possible that certain positive integers may not be realizable through the above scheme and even though a particular n corresponds to more than one value of δ. Denote the interior lattice points by $\{x_j^{(n)}\}_{j=1}^n$ (the order of the labeling is arbitrary), and for $\epsilon = \epsilon_n \in (0, \frac{\delta}{2})$, let

$$A_n \equiv \cup_{j=1}^n \bar{B}_{\epsilon_n}(x_j^{(n)}).$$

By construction, the balls $\{\bar{B}_{\epsilon_n}(x_j^{(n)})\}_{j=1}^n$ are disjoint and do not intersect the boundary $\partial\Omega$. Note that

$$w - \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n \delta_{x_j^{(n)}} = \text{ the uniform distribution on } \Omega.$$

Recall that ω_d is the volume of the unit ball in R^d.

Theorem 1. *Assume that*

$$\begin{cases} \lim_{n\to\infty} n\epsilon_n^{d-2} = \alpha \in [0,\infty], \ \text{if } d \geq 3; \\ \lim_{n\to\infty} \frac{n}{-\log \epsilon_n} = \alpha \in [0,\infty], \ \text{if } d = 2. \end{cases}$$

Then

$$\lim_{n\to\infty} \lambda_N^{(n)} = \begin{cases} \frac{d(d-2)\omega_d\alpha}{|\Omega|}, \ \text{if } \ d \geq 3; \\ \frac{2\pi\alpha}{|\Omega|}, \ \text{if } d = 2. \end{cases}$$

In light of the above-cited work of Rauch and Taylor and of Ozawa, we expect that in the Neumann case, if instead of imposing lattice-spacing, one assumed that $\frac{1}{n}\sum_{j=1}^n \delta_{x_j^{(n)}}$ converged weakly to a probability measure with density V, then $\lambda_N^{(n)}$ would converge to the principal eigenvalue for $-\Delta + c_d\alpha V$ in Ω with the Neumann boundary condition on $\partial\Omega$, where $c_d = d(d-2)\omega_d$, if $d \geq 3$, and $c_2 = 2\pi$. In fact, we can show that for any choice of $\{x_j^{(n)}\}_{j=1}^n$ which maintain an appropriate minimal spacing, $\limsup_{n\to\infty} \lambda_N^{(n)}$ is less than or equal to the answer obtained in Theorem 1 (see the end of section 2). This is consistent with the above-stated expectation since it is easy to see from the Rayliegh-Ritz formula that the principal eigenvalue $\lambda_N(c_d\alpha V)$ for $-\Delta + c_d\alpha V$ in Ω with the Neumann boundary condition on $\partial\Omega$, where $V \geq 0$ satisfies $\int_\Omega V(x)dx = 1$, is maximized when $V \equiv \frac{1}{|\Omega|}$ is the uniform density.

With the Neumann boundary condition and $d = 3$, the problem can be viewed as an idealized model for the cooling effect of crushed ice: consider an insulated container filled with a liquid occupying the region Ω and containing n spherical coolers of radius ϵ_n which are centered at the points $\{x_j^{(n)}\}_{j=1}^n$ and maintain temperature 0. Then heat flow in Ω starting from a temperature distribution $f \geq 0$ (and with diffusion coefficient normalized to 1) is modeled by

$$\begin{aligned} &u_t = \Delta u, \ x \in \Omega, \ t > 0; \\ &u(x,0) = f(x), \ x \in \Omega; \\ &\frac{\partial u}{\partial N} = 0 \text{ on } \partial\Omega; \\ &u = 0 \text{ on } \partial A_n, \end{aligned}$$

where N denotes the unit outward normal to Ω at $\partial\Omega$. The exponential decay rate of the solution is then equal to the principal eigenvalue $\lambda_N^{(n)}$. Since the volume of the holes is on the order $n\epsilon_n^3$ and the surface area is on the order $n\epsilon_n^2$, while the shift of the principal eigenvalue is on the order $n\epsilon_n$, one concludes that crushed ice is a more effective coolant than ice cubes. This model was suggested by Rauch [10].

Our approach is almost completely probabilistic, but as already noted, it is completely different from the methods of Kac and of Chavel and Feldman, which utilized the Wiener sausage. We use the probabilistic representation for the principal eigenvalue; namely, $\lambda_N^{(n)}$ is the exponential decay rate in t of the probability that a normally reflected Brownian motion in Ω has not hit A_n by time t. The proof involves large deviations, but its most novel part exploits Hasminskii's representation for the invariant measure of a recurrent Markov process along with Theorem 2 below.

We now recall the probabilistic representation of the principal eigenvalue. Let $X(t)$ be a Brownian motion in Ω with normal reflection at $\partial\Omega$. Denote by P_x the probability measure corresponding to the reflected Brownian motion starting from $x \in \bar{\Omega}$. Let $\tau_{A_n} = \inf\{t \geq 0 : X(t) \in \partial A_n\}$. Then

(1.2)
$$\lim_{t\to\infty} \frac{1}{t} \log P_x(\tau_{A_n} > t) = \lim_{t\to\infty} \frac{1}{t} \log \sup_{z\in\bar{\Omega}-A_n} P_z(\tau_{A_n} > t) = -\frac{1}{2}\lambda_N^{(n)}, \text{ for } x \in \bar{\Omega} - A_n.$$

To verify (1.2), recall that the semigroup $T_t^{(n)}$ for the operator $-\frac{1}{2}\Delta$ on $\Omega - A_n$ with the Neumann boundary condition on $\partial\Omega$ and the Dirichlet boundary condition on ∂A_n can be represented by

$$(T_t^{(n)} f)(x) = E_x(f(X(t)); \tau_{A_n} > t) = \int_{\Omega - A_n} p^{(n)}(t,x,y) f(y) dy,$$

where $p^{(n)}(t,x,y) = P_x(X(t) \in dy, \tau_{A_n} > t)$. Letting $\phi_0^{(n)} > 0$ denote the eigenfunction corresponding to the principal eigenvalue $\frac{1}{2}\lambda_N^{(n)}$ for $-\frac{1}{2}\Delta$, we have

$$\begin{aligned} P_x(\tau_{A_n} > t) &= (T_t^{(n)} 1)(x) \geq \frac{1}{\sup_{y\in\Omega-A_n} \phi_0^{(n)}(y)} (T_t \phi_0^{(n)})(x) \\ &= \frac{1}{\sup_{y\in\Omega-A_n} \phi_0^{(n)}(y)} \phi_0^{(n)}(x) \exp(-\frac{1}{2}\lambda_N^{(n)} t). \end{aligned}$$

Thus,

$$\liminf_{t\to\infty}\frac{1}{t}\log P_x^{(n)}(\tau_{A_n}>t)\ge -\frac{1}{2}\lambda_N^{(n)},\ \text{for } x\in\bar{\Omega}-A_n. \tag{1.3}$$

On the other hand, using the semigroup property and the fact that $p^{(n)}(t,z,x)$ is symmetric in z and x by self-adjointness, we have for $t>1$,

$$\begin{aligned}
&P_z(\tau_{A_n}>t)=\int_{\Omega-A_n}p^{(n)}(t,z,y)dy\\
&=\int_{\Omega-A_n}dy\int_{\Omega-A_n}dx\ p^{(n)}(1,z,x)p^{(n)}(t-1,x,y)\\
&\le \sup_{x\in\Omega-A_n}\frac{p^{(n)}(1,z,x)}{\phi_0^{(n)}(x)}\int_{\Omega-A_n}dy\int_{\Omega-A_n}dx\ \phi_0^{(n)}(x)p^{(n)}(t-1,x,y)\\
&=\sup_{x\in\Omega-A_n}\frac{p^{(n)}(1,z,x)}{\phi_0^{(n)}(x)}(\int_{\Omega-A_n}\phi_0^{(n)}(y)dy)\exp(-\frac{1}{2}\lambda_N^{(n)}(t-1)).
\end{aligned} \tag{1.4}$$

By the Hopf maximum principal, there exists a $\delta>0$ such that

$$\phi_0^{(n)}(x)\ge \delta dist(x,\partial A_n). \tag{1.5}$$

As a function of $t>0$ and of z or x, $p^{(n)}(t,z,x)$ is a smooth solution to the heat equation $u_t=\frac{1}{2}\Delta u$ in $\Omega-A_n$ with the homogeneous Neumann boundary condition on $\partial\Omega$ and the homogeneous Dirichlet boundary condition on ∂A_n; thus,

$$\begin{aligned}
&p^{(n)}(1,z,x)=0,\ \text{for } z\in\Omega-A_n \text{ and } x\in\partial A_n;\\
&\sup_{z,x\in\Omega-A_n}|\nabla_x p^{(n)}(1,z,x)|<\infty.
\end{aligned} \tag{1.6}$$

From (1.5) and (1.6) we have

$$\sup_{z,x\in\Omega-A_n}\frac{p^{(n)}(1,z,x)}{\phi_0^{(n)}(x)}<\infty. \tag{1.7}$$

From (1.4) and (1.7) we conclude that

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{z\in\bar{\Omega}-A_n}P_z(\tau_{A_n}>t)\le -\frac{1}{2}\lambda_N^{(n)}. \tag{1.8}$$

Now (1.2) follows from (1.3) and (1.8).

We now present a result which is used in the proof of Theorem 1 and which may be of independent interest. Fix a positive integer $N_0\ge 1$ and consider N_0 pairs of concentric nonoverlapping balls compactly imbedded in Ω. To be concrete, let $\{y_j\}_{j=1}^{N_0} \subset \Omega$ and let $\{R_j\}_{j=1}^{N_0}$ and $\{L_j\}_{j=1}^{N_0}$ be positive numbers satisfying $\bar{B}_{R_j}(y_j) \subset B_{L_j}(y_j) \subset \bar{B}_{L_j}(y_j) \subset \Omega$, for all j, and $\bar{B}_{L_i}(y_i) \cap \bar{B}_{L_j}(y_j) = \emptyset$, for $i \neq j$. Let $A^R = \cup_{j=1}^{N_0} \bar{B}_{R_j}(y_j)$ and $A^L = \cup_{j=1}^{N_0} \bar{B}_{L_j}(y_j)$. Let $X(t)$ and P_x be as above. As is well-known, $X(t)$ is positive recurrent and its invariant probability measure is normalized Lebesgue measure in Ω. (The invariant density satisfies the homogeneous adjoint equation. The operator $\frac{1}{2}\Delta$ in Ω with the Neumann boundary condition is self-adjoint, and clearly positive constants satisfy the homogeneous adjoint equation.) Define inductively stopping times $\sigma_1 = \inf\{t \geq 0 : X(t) \in \partial A^L\}$, $\eta_n = \inf\{t > \sigma_n : X(t) \in \partial A^R\}$, and $\sigma_{n+1} = \inf\{t > \eta_n : X(t) \in \partial A^L\}$, for $n \geq 1$. Note that under P_x with $x \in \partial A_n^L$, we have $\eta_1 = \inf\{t \geq 0 : X(t) \in \partial A_n^R\}$. Then $(X(\sigma_1), X(\eta_1), X(\sigma_2), X(\eta_2), ...)$ is a Markov process on $\partial A^L \cup \partial A^R$. Since the process is irreducible and defined on a compact state space, it possesses a unique invariant probability measure. From this it follows that there exist probability measures m_R on ∂A^R and m_L on ∂A^L such that $P_{m_R}(X(\sigma_1) \in \cdot) = m_L$ and $P_{m_L}(X(\eta_1) \in \cdot) = m_R$. The following result gives an explicit calculation of these measures as well as an evaluation of the expected length of one cycle; that is, of $E_{m_R}\eta_1 = E_{m_R}\sigma_1 + E_{m_L}\eta_1$.

Theorem 2.

i. The conditional distribution of m_R on $\partial B_{R_j}(y_j)$ and of m_L on $\partial B_{L_j}(y_j)$ is uniform. Furthermore

$$m_R(\partial B_{R_j}(y_j)) = m_L(\partial B_{L_j}(y_j)) = \begin{cases} \frac{(R_j^{2-d} - L_j^{2-d})^{-1}}{\sum_{k=1}^{N_0} (R_k^{2-d} - L_k^{2-d})^{-1}}, & \text{if } d \geq 3; \\ \frac{(\log \frac{L_j}{R_j})^{-1}}{\sum_{k=1}^{N_0} (\log \frac{L_k}{R_k})^{-1}}, & \text{if } d = 2. \end{cases} \tag{1.9}$$

ii. The expected time for one cycle is given by

$$E_{m_R}\eta_1 = \begin{cases} \frac{2|\Omega|}{d(d-2)\omega_d \sum_{j=1}^{N_0} (R_j^{2-d} - L_j^{2-d})^{-1}}, & \text{if } d \geq 3; \\ \frac{|\Omega|}{\pi \sum_{j=1}^{N_0} (\log \frac{L_j}{R_j})^{-1}}, & \text{if } d = 2. \end{cases} \tag{1.10}$$

We prove Theorem 2 in section 2. In section 3 we construct the probabilistic setup for the proof of Theorem 1, state three propositions and proof Theorem 1 based on those propositions. The three propositions are proved in sections 4-7.

We conclude this section with a couple of additional historical notes. In fact, the papers of Rauch and Taylor and of Ozawa showed that for each m, the m-th eigenvalue for $-\Delta$ in $\Omega - A_n$ with the Dirichlet boundary condition on both $\partial\Omega$ and ∂A_n (arranged in nondecreasing order and counting multiplicities) converges to the m-th eigenvalue for $-\Delta + 4\pi\alpha V$ in Ω with the Dirichlet boundary condition on $\partial\Omega$. Kac also showed this in the particular case he treated ($V \equiv \frac{1}{|\Omega|}$.) This is the so-called *Lenz* shift.

In a related work, Papanicolaou and Varadhan [8] showed that when $\Omega = R^3$ and $\frac{1}{n}\sum_{j=1}^{n} \delta_{x_j^{(n)}}$ converges weakly to a probability measure with compactly supported density V, then the solution to the Cauchy problem for $u_t = \Delta u$ in $R^3 - A_n$ with the Dirichlet boundary condition on ∂A_n and compactly supported initial data converges in an appropriate sense to the solution of $u_t = \Delta u - 4\pi\alpha V u$ in R^3 with the same initial data.

2. Proof of Theorem 2.

i. We use the notation in the paragraph preceding Theorem 2. It is enough to prove that with m_L and m_R as prescribed in the theorem, $P_{m_L}(X(\eta_1) \in \cdot) = m_R$. The other equality, $P_{m_R}(X(\sigma_1) \in \cdot) = m_L$, follows automatically by symmetry.

For $f \in C(\partial A^R)$, let u_f denote the solution to

$$\begin{aligned} &\frac{1}{2}\Delta u_f = 0 \text{ in } \Omega - A^R; \\ &u_f = f \text{ on } \partial A^R; \\ &\frac{\partial u_f}{\partial N} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.1}$$

Letting $\tau_{A^R} = \inf\{t \geq 0 : X(t) \in \partial A^R\}$, we have the probabilistic representation $u_f(x) = E_x f(X(\tau_{A^R}))$, for $x \in \Omega - A^R$. From its definition, $\eta_1 = \tau_{A^R}$ under P_x with $x \in \partial A^L$; thus,

$$u_f(x) = E_x f(X(\eta_1)), \text{ for } x \in \partial A^L. \tag{2.2}$$

Let $v=\{v_j\}_{j=1}^{N_0}$ be positive constants satisfying $\sum_{j=1}^{N_0}v_j=1$, and for each $j\in\{1,...,N_0\}$, let $r_j(t),\ 0\le t\le 1$, be an increasing, smooth function satisfying $r_j(0)=R_j$ and $r_j(1)=L_j$. Letting (r,θ) denote polar coordinates in R^d and letting σ denote Lebesgue surface measure on the unit sphere S^{d-1}, normalized to be a probability measure, define

$$H_v(t)=\sum_{j=1}^{N_0}v_j\int_{S^{d-1}}u_f(y_j+(r_j(t),\theta))d\sigma(\theta).$$

Let $\Gamma_t\equiv\cup_{j=1}^{N_0}B_{r_j(t)}(y_j)$ and for $t\in[0,1]$, let $\sigma_{v,t}$ denote the probability measure on $\partial\Gamma_t\equiv\cup_{j=1}^{N_0}\partial B_{r_j(t)}(y_j)$ whose conditional distribution on each $\partial B_{r_j(t)}(y_j)$ is uniform and which satisfies $\sigma_{v,t}(\partial B_{r_j(t)}(y_j))=v_j$. Note that $\partial\Gamma_0=\partial A^R$ and $\partial\Gamma_1=\partial A^L$. From the boundary condition on u_f it follows that

$$H_v(0)=\int_{\partial A^R}fd\sigma_{v,0}, \tag{2.3}$$

and from (2.2) it follows that

$$H_v(1)=E_{\sigma_{v,1}}f(X(\eta_1)). \tag{2.4}$$

Note that if

$$v_j=\begin{cases}\frac{(R_j^{2-d}-L_j^{2-d})^{-1}}{\sum_{k=1}^{N_0}(R_k^{2-d}-L_k^{2-d})^{-1}}, & \text{if } d\ge 3;\\ \frac{(\log\frac{L_j}{R_j})^{-1}}{\sum_{k=1}^{N_0}(\log\frac{L_k}{R_k})^{-1}}, & \text{if } d=2,\end{cases} \tag{2.5}$$

then $\sigma_{v,0}$ and $\sigma_{v,1}$ will be equal to m_R and m_L respectively as they are defined in (1.9). We will show that if v is chosen as in (2.5), then $H_v(t)$ is constant in t, and thus $H_v(1)=H_v(0)$. It will then follow from (2.3) and (2.4) that $E_{m_L}f(X(\eta_1))=\int_{\partial A^R}fdm_R$. Since f is an arbitrary continuous function on ∂A_n^R, it will follow that $P_{m_L}(X(\eta_1)\in\cdot)=m_R$, completing the proof.

It remains to show that $H_v(t)$ is constant if v is chosen as in (2.5). For the time being, we continue to consider v as arbitrary. Treating y_j as a constant, let $v_f(r,\theta)=u_f(y_j+(r,\theta))$. In the sequel, when N appears in an integral over the boundary of a domain, it will denote the outward unit normal to the domain. Let

$\hat{\omega}_d$ denote the Lebesgue surface measure of S^{d-1}. Using the generic notation S for the standard Lebesgue surface measure on a $(d-1)-$ dimensional hyperspace and recalling that σ is normalized Lebesgue surface measure on S^{d-1}, we have

$$\begin{aligned} H_v'(t) &= \sum_{j=1}^{N_0} v_j \int_{S^{d-1}} \frac{\partial v_f}{\partial r}(r_j(t),\theta) r_j'(t) d\sigma(\theta) \\ &= \frac{1}{\hat{\omega}_d} \sum_{j=1}^{N_0} v_j \frac{r_j'(t)}{r_j^{d-1}(t)} \int_{\partial B_{r_j(t)}(y_j)} \frac{\partial u_f}{\partial N}(x) dS(x). \end{aligned} \tag{2.6}$$

Suppose $v = \{v_j\}_{j=1}^{N_0}$ is chosen so that for some $c > 0$

$$v_j \frac{r_j'(t)}{r_j^{d-1}(t)} = c, \text{ for all } t \in [0,1] \text{ and } j = 1,2,...,N_0. \tag{2.7}$$

If (2.7) holds, then we have from (2.6), (2.1) and the divergence theorem that

$$\begin{aligned} H_v'(t) &= \frac{c}{\hat{\omega}_d} \int_{\partial \Gamma_t} \frac{\partial u_f}{\partial N}(x) dS(x) \\ &= -\frac{c}{\hat{\omega}_d} \int_{\Omega - \Gamma_t} \Delta u(x) dx + \frac{c}{\hat{\omega}_d} \int_{\partial \Omega} \frac{\partial u_f}{\partial N}(x) dS(x) = 0. \end{aligned}$$

Choosing $r_j(t) = (R_j^{2-d}(1-t) + L_j^{2-d} t)^{\frac{1}{2-d}}$, if $d \geq 3$, and $r_j(t) = \exp((1-t)\log R_j + t \log L_j)$, if $d = 2$, one can now check that (2.7) will hold if v_j is chosen as in (2.5).

ii. We exploit Hasminskii's representation for the invariant measure of a recurrent Markov process. (See [4] where the construction is carried out in the case of a diffusion on all of R^d; the same construction works for any recurrent, irreducible Markov process.) The invariant probability measure for the normally reflected Brownian motion is normalized Lebesgue measure. By Hasminskii's representation the invariant probability measure (normalized Lebesgue measure) has the following representation:

$$\frac{|C|}{|\Omega|} = \frac{E_{m_R} \int_0^{\eta_1} 1_C(X(t)) dt}{E_{m_R} \eta_1}, \text{ for } C \subset \Omega. \tag{2.8}$$

In particular, if we choose $C = A^R$, then $\int_0^{\eta_1} 1_C(X(t))dt = \int_0^{\sigma_1} 1_C(X(t))dt$, so we obtain from (2.8)

$$E_{m_R} \eta_1 = \frac{|\Omega|}{|A^R|} E_{m_R} \int_0^{\sigma_1} 1_{A^R}(X(t)) dt. \tag{2.9}$$

From part (i) we have

(2.10)

$$E_{m_R}\int_0^{\sigma_1} 1_{A^R}(X(t))dt$$

$$=\begin{cases} \frac{1}{\sum_{k=1}^{N_0}(R_k^{2-d}-L_k^{2-d})^{-1}}\sum_{j=1}^{N_0}(R_j^{2-d}-L_j^{2-d})^{-1}E_{x_j}\int_0^{\sigma_1}1_{A^R}(X(t))dt, \text{ if } d\ge 3;\\ \frac{1}{\sum_{k=1}^{N_0}(\log\frac{L_k}{R_k})^{-1}}\sum_{j=1}^{N_0}(\log\frac{L_j}{R_j})^{-1}E_{x_j}\int_0^{\sigma_1}1_{A^R}(X(t))dt, \text{ if } d=2,\end{cases}$$

for any choice of $x_j\in\partial B_{R_j}(y_j),\ j=1,2,...,N_0$.

From the standard probabilistic representation of solutions of elliptic pde's, it follows that

$$u_j(R_j)=E_{x_j}\int_0^{\sigma_1}1_{A^R}(X(t))dt, \text{ for } x_j\in\partial B_{R_j}(y_j), \tag{2.11}$$

where $u_j(r)$ is the solution to

$$\begin{aligned}&\frac{1}{2}u_j''+\frac{d-1}{2r}u_j'=-1_{[0,R_j]}(r),\ r\in[0,L_j];\\ &u_j'(0)=0,\ u_j(L_j)=0.\end{aligned} \tag{2.12}$$

Writing the left hand side of (2.12) in the form $\frac{1}{2r^{d-1}}(r^{d-1}u_j')'$, solving separately on $[0,R_j]$ and on $[R_j,L_j]$, and matching the solutions and their first derivatives at R_j, we obtain

$$u_j(R_j)=\begin{cases}\frac{2}{d(d-2)}R_j^d(R_j^{2-d}-L_j^{2-d}), \text{ if } d\ge 3;\\ R_j^2\log\frac{L_j}{R_j}, \text{ if } d=2.\end{cases} \tag{2.13}$$

Now (1.10) follows from (2.9)-(2.11), (2.13) and the fact that $|A^R|=\omega_d\sum_{k=1}^{N_0}R_k^d$. □

3. Probabilistic Set Up, Statement of Three Propositions and Proof of Theorem 1. In the sequel, the notation $f\sim g$ will be used to indicate that there exist constants $c_1,c_2>0$ such that $c_1g\le f\le c_2g$. We will always assume that $\alpha\in(0,\infty)$ since the cases $\alpha=0$ and $\alpha=\infty$ follow from the other cases by comparison. By (1.1) and the fact that we have deleted boundary lattice points, the distance between any $x_i^{(n)}$ and $x_j^{(n)}$ or between $x_j^{(n)}$ and $\partial\Omega$ is at least $cn^{-\frac{1}{d}}$ for some $c>0$. Since $\alpha\in(0,\infty)$, $\epsilon_n\sim n^{-\frac{1}{d-2}}$ when $d\ge3$. Thus, when $d\ge3$, we may choose $L_n=Cn^{\frac{2}{d(d-2)}}$, with an appropriate $C>0$, so that the balls

$\{\bar{B}_{L_n\epsilon_n}(x_j^{(n)})\}_{j=1}^n$ are disjoint and do not intersect the boundary $\partial\Omega$. When $d=2$, we have $\lim_{n\to\infty}\frac{n}{-\log\epsilon_n}=\alpha$, and we can choose $L_n=\frac{C}{\epsilon_n}n^{-\frac{1}{2}}$, for an appropriate $C>0$, so that the balls $\{\bar{B}_{L_n\epsilon_n}(x_j^{(n)})\}_{j=1}^n$ are disjoint and do not intersect the boundary $\partial\Omega$. Note that $L_n\epsilon_n\sim n^{-\frac{1}{d}}$, for all $d\geq 2$. Fix $\kappa\in(0,\frac{1}{2}]$ and let

$$R_n=\kappa L_n.$$

The parameter κ will remain fixed until section 6 when we will need to let $\kappa\to 0$. Thus, in our notation until section 6, we will suppress the dependence on κ. We record some of the above facts for later reference.

$$\begin{aligned}
&L_n\epsilon_n\sim n^{-\frac{1}{d}},\ d\geq 2;\\
&L_n=\begin{cases} Cn^{\frac{2}{d(d-2)}}, & \text{if } d\geq 3;\\ \frac{C}{\epsilon_n}n^{-\frac{1}{2}} & \text{if } d=2;\end{cases}\\
&R_n=\kappa L_n,\ \kappa\in(0,\frac{1}{2}].
\end{aligned}\tag{3.1}$$

From now on we work with n sufficiently large so that $R_n>1$. Let A_n^R and A_n^L denote the collection of holes thickened by a factor of R_n and L_n respectively; that is, $A_n^R=\cup_{j=1}^n\bar{B}_{R_n\epsilon_n}(x_j^{(n)})$ and $A_n^L=\cup_{j=1}^n\bar{B}_{L_n\epsilon_n}(x_j^{(n)})$. Define inductively stopping times $\sigma_1=\inf\{t\geq 0: X(t)\in\partial A_n^L\}$, $\eta_m=\inf\{t\geq\sigma_m: X(t)\in\partial A_n^R\}$, and $\sigma_{m+1}=\inf\{t\geq\eta_m: X(t)\in\partial A_n^L\}$, $m=1,2,\ldots$. Note that under P_z with $z\in\partial A_n^L$, the stopping time η_1 is the first hitting time of ∂A_n^R. Consider the sequence $X(\sigma_1),X(\eta_1),X(\sigma_2),X(\eta_2),\ldots$ under the following Markov transition measure: let the transition from $X(\sigma_m)$ to $X(\eta_m)$ be according to the distribution $P_z(X(\eta_1)\in\cdot)$, when $z=X(\sigma_m)\in\partial A_n^L$, and let the transition from $X(\eta_m)$ to $X(\sigma_{m+1})$ be according to the conditional distribution $P_z(X(\sigma_1)\in\cdot|\sigma_1<\tau_{A_n})$ when $z=X(\eta_m)\in\partial A_n^R$. Since the above process is defined on the compact state space $\partial A_n^R\cup\partial A_n^L$, it possesses an invariant probability measure. Thus, there exist probability measures $m_{1,n}$ and $m_{2,n}$ on ∂A_n^R and ∂A_n^L respectively such that $P_{m_{1,n}}(X(\sigma_1)\in\cdot|\sigma_1<\tau_{A_n})=m_{2,n}(\cdot)$ and $P_{m_{2,n}}(X(\eta_1)\in\cdot)=m_{1,n}(\cdot)$.

Define

$$\rho_n=P_{m_{1,n}}(\sigma_1<\tau_{A_n})\tag{3.2}$$

and

(3.3)

$$\mu_n = E_{m_{1,n}}(\sigma_1|\sigma_1 < \tau_{A_n}) + E_{m_{2,n}}\eta_1 = E_{m_{1,n}}(\eta_1|\sigma_1 < \tau_{A_n}) = E_{m_{1,n}}(\eta_1|\eta_1 < \tau_{A_n}).$$

In sections 4-7 we will prove the following three propositions.

Proposition 1.

$$\lambda_N^{(n)} \le 2\frac{|\log \rho_n|}{\mu_n}.$$

Proposition 2.

$$\lim_{n\to\infty} \frac{|\log \rho_n|}{\mu_n} = \begin{cases} \frac{d(d-2)\omega_d\alpha}{2|\Omega|}, & \text{if } d \ge 3; \\ \frac{\pi\alpha}{|\Omega|}, & \text{if } d = 2. \end{cases}$$

Remark. Note that whereas μ_n and ρ_n depend on the suppressed parameter κ, $\lim_{n\to\infty} \frac{|\log \rho_n|}{\mu_n}$ does not depend on κ.

Proposition 3. *Assume that* $\lim_{n\to\infty} \frac{|\log \rho_n|}{\mu_n} \in (0,\infty)$. *Then*

$$\liminf_{n\to\infty} \lambda_N^{(n)} \ge 2 \lim_{n\to\infty} \frac{|\log \rho_n|}{\mu_n}.$$

Proof of Theorem 1. Since we are assuming that $\alpha \in (0,\infty)$, it follows from Proposition 2 that the assumption in Proposition 3 holds. Thus,

$$\lim_{n\to\infty} \lambda_N^{(n)} \le \begin{cases} \frac{d(d-2)\omega_d\alpha}{|\Omega|}, & \text{if } d \ge 3 \\ \frac{2\pi\alpha}{|\Omega|}, & \text{if } d = 2 \end{cases}$$

follows directly from Propositions 1 and 2 while

$$\lim_{n\to\infty} \lambda_N^{(n)} \ge \begin{cases} \frac{d(d-2)\omega_d\alpha}{|\Omega|}, & \text{if } d \ge 3 \\ \frac{2\pi\alpha}{|\Omega|}, & \text{if } d = 2 \end{cases}$$

follows directly from Propositions 3 and 2. □

The proof of Proposition 1 only requires the ergodic theorem and is given in section 4. The proofs of Propositions 2 and 3 use Theorem 2. The proof of Proposition 2, which in our opinion is the most novel and interesting part of the proof of Theorem 1, is given in section 5. The rather intricate and delicate proof of Proposition 3 accounts for over half the length of the paper. The proof, which uses three auxiliary lemmas, is given in section 6, while the auxiliary lemmas are proved in section 7.

An inspection of the proofs of Propositions 1 and 2 reveals that they do not require the lattice-spacing assumption, but only a minimal spacing assumption to insure that the construction of the collection of nonoverlapping balls A_n^L may be carried out. One can verify that the following minimal spacing assumption (along with the fact that $\alpha \in (0,\infty)$) is enough:

$$\begin{cases} \lim_{n\to\infty} \min_{1\le i,j\le n} n^{\frac{1}{d-2}} dist(x_i^{(n)}, x_j^{(n)}) = \infty \\ \text{and } \lim_{n\to\infty} \min_{1\le j\le n} n^{\frac{1}{d-2}} dist(x_j^{(n)}, \partial\Omega) = \infty, \text{ if } d \ge 3; \\ \lim_{n\to\infty} \min_{1\le i,j\le n} \frac{1}{n} \log dist(x_i^{(n)}, x_j^{(n)}) = 0 \\ \text{and } \lim_{n\to\infty} \min_{1\le j\le n} \frac{1}{n} \log dist(x_j^{(n)}, \partial\Omega) = 0, \text{ if } d = 2. \end{cases}$$

Thus, with this minimal spacing assumption, we have

$$\limsup_{n\to\infty} \lambda_N^{(n)} \le \begin{cases} \frac{d(d-2)\omega_d\alpha}{|\Omega|}, \text{ if } d \ge 3; \\ \frac{2\pi\alpha}{|\Omega|}, \text{ if } d = 2. \end{cases}$$

4. Proof of Proposition 1. Letting $\eta_0 = 0$, we note that for each k, the random variables $\{\eta_j - \eta_{j-1}\}_{j=1}^k$ and the random variables $\{X(\eta_j)\}_{j=0}^k$ have the same distribution under $P_{m_{1,n}}(\cdot|\eta_l < \tau_{A_n} < \eta_{l+1})$ as they have under $P_{m_{1,n}}(\cdot|\eta_l < \tau_{A_n})$, for any $l \ge k$, and this distribution does not depend on l. Furthermore, $\{\eta_j - \eta_{j-1}\}_{j=1}^k$ and $\{X(\eta_j)\}_{j=0}^k$ are stationary sequences under the above measures. The proof of this follows from the invariance of $m_{1,n}$ and from the crucial fact that $P_z(\sigma_1 < \tau_{A_n})$ is the same for all $z \in \partial A_n^R$. Therefore, there exists a probability measure, which we will denote by $P^{stat}(\cdot)$, under which $\{\eta_j - \eta_{j-1}\}_{j=1}^\infty$ and $\{X(\eta_j)\}_{j=0}^\infty$ are stationary and such that

$$P^{stat}(\{\eta_j - \eta_{j-1}\}_{j=1}^k \in \cdot, \{X(\eta_j)\}_{j=0}^k \in \cdot\ )$$
$$= P_{m_{1,n}}(\{\eta_j - \eta_{j-1}\}_{j=1}^k \in \cdot, \{X(\eta_j)\}_{j=0}^k \in \cdot\ |\eta_k < \tau_{A_n}),$$

for all $k \ge 1$. In fact, $\{\eta_j - \eta_{j-1}\}_{j=1}^\infty$ is ergodic under $P^{stat}(\cdot)$. Indeed, the dependence of the random variables $\{\eta_j - \eta_{j-1}\}_{j=1}^\infty$ enters only through the hitting locations $\{X(\eta_j)\}_{j=0}^\infty$ and this latter sequence is ergodic since it is easily seen to satisfy a uniform Doeblin condition. From the definition of μ_n it follows that

$$E^{stat}(\eta_j - \eta_{j-1}) = \mu_n. \tag{4.1}$$

Using the strong Markov property, the fact that $P_z(\sigma_1 < \tau_{A_n})$ is the same for all $z \in \partial A_n^R$, and the invariance of $m_{1,n}$, it follows that for $t > 0$ and any positive integer k, we have

(4.2)
$$P_{m_{1,n}}(\tau_{A_n} > t) \geq P_{m_{1,n}}(\tau_{A_n} > \eta_k > t) = P_{m_{1,n}}(\tau_{A_n} > \eta_k)P_{m_{1,n}}(\eta_k > t|\tau_{A_n} > \eta_k)$$
$$= P_{m_{1,n}}(\tau_{A_n} > \eta_k)P^{stat}(\eta_k > t),$$

while from (3.2) we have

$$P_{m_{1,n}}(\tau_{A_n} > \eta_k) = \rho_n^k. \tag{4.3}$$

Fix $\epsilon > 0$. For each $t > 0$, let k_t be an integer such that

$$\lim_{t\to\infty} \frac{t}{k_t} = \mu_n - \epsilon. \tag{4.4}$$

We write

$$P^{stat}(\eta_{k_t} > t) = P^{stat}\big(\frac{1}{k_t}\sum_{j=1}^{k_t}(\eta_j - \eta_{j-1}) > \frac{t}{k_t} \big). \tag{4.5}$$

Letting $t \to \infty$ in (4.5), using (4.4) and (4.1), and applying the ergodic theorem gives

$$\lim_{t\to\infty} P^{stat}(\eta_{k_t} > t) = 1. \tag{4.6}$$

From (4.2)-(4.4) and (4.6) we obtain

$$\liminf_{t\to\infty} \frac{1}{t}\log P_{m_{1,n}}(\tau_{A_n} > t) \geq -\frac{|\log \rho_n|}{\mu_n - \epsilon}. \tag{4.7}$$

The proposition now follows from (4.7), (1.2) and the fact that $\epsilon > 0$ is arbitrary. □

5. Proof of Proposition 2. We begin with an estimate for $|\log \rho_n|$. From (3.2) it follows that ρ_n is equal to the probability that a d-dimensional Brownian motion starting from a point on $\partial B_{R_n\epsilon_n}(0)$ will hit $\partial B_{L_n\epsilon_n}(0)$ before hitting $\partial B_{\epsilon_n}(0)$. This probability is $\frac{(\epsilon_n)^{2-d}-(R_n\epsilon_n)^{2-d}}{(\epsilon_n)^{2-d}-(L_n\epsilon_n)^{2-d}} = \frac{1-R_n^{2-d}}{1-L_n^{2-d}}$, if $d \geq 3$, and $\frac{\log\epsilon - \log R_n\epsilon_n}{\log\epsilon_n - \log L_n\epsilon_n} = \frac{\log R_n}{\log L_n}$, if

$d=2$. From (3.1) it follows that $\lim_{n\to\infty}\rho_n=1$; therefore $\lim_{n\to\infty}\frac{|\log\rho_n|}{1-\rho_n}=1$, and we conclude that

$$\text{(5.1)}\qquad \begin{cases} \lim_{n\to\infty}\frac{1-L_n^{2-d}}{R_n^{2-d}-L_n^{2-d}}|\log\rho_n|=1, \text{ if } d\geq 3;\\ \lim_{n\to\infty}\frac{\log L_n}{\log\frac{L_n}{R_n}}|\log\rho_n|=1, \text{ if } d=2. \end{cases}$$

Now we must estimate μ_n defined in (3.3). Note that whereas $|\log\rho_n|$ only involves Brownian motion in a ball, μ_n includes the term $E_{m_{2,n}}\eta_1$, and this term depends on the reflecting Brownian motion in the entire domain with holes $\Omega-A_n$. We circumvent this problem by exploiting Hasminskii's representation for the invariant measure of a recurrent Markov process, as was done in the proof of Theorem 2-ii. We apply Hasminskii's construction to the original unconditioned normally reflected Brownian motion. The same argument that led to the existence of $m_{1,n}$ and $m_{2,n}$ leads to the existence of probability measures $M_{1,n}$ and $M_{2,n}$ on ∂A_n^R and ∂A_n^L respectively satisfying $P_{M_{1,n}}(X(\sigma_1)\in\cdot)=M_{2,n}$ and $P_{M_{2,n}}(X(\eta_1)\in\cdot)=M_{1,n}$. The invariant probability measure for the normally reflected Brownian motion in Ω is normalized Lebesgue measure. By Hasminskii's construction, the invariant probability measure (normalized Lebesgue measure) has the following representation:

$$\frac{|C|}{|\Omega|}=\frac{E_{M_{1,n}}\int_0^{\eta_1}1_C(X(t))dt}{E_{M_{1,n}}\eta_1}, \text{ for } C\subset\Omega.$$

In particular, if we choose $C=A_n^R$, then $\int_0^{\eta_1}1_C(X(t))dt=\int_0^{\sigma_1}1_C(X(t))dt$, so we have

$$\text{(5.2)}\qquad E_{M_{1,n}}\eta_1=\frac{|\Omega|}{|A_n^R|}E_{M_{1,n}}\int_0^{\sigma_1}1_{A_n^R}(X(t))dt.$$

A crucial point now is that

$$\text{(5.3)}\qquad m_{i,n}=M_{i,n}.$$

In fact, after making the appropriate changes in notation, both $m_{1,n}$ and $M_{1,n}$ are distributed according to the distribution of m_R in (1.9) and both $m_{2,n}$ and $M_{2,n}$ are distributed according to the distribution of m_L in (1.9). (The changes in notation are $x_j^{(n)}$ in place of y_j, $R_n\epsilon_n$ in place of R_j, $L_n\epsilon_n$ in place of L_j, and n in place of N_0.) For $M_{i,n}$, this claim follows directly from Theorem 2. To prove the claim for $m_{i,n}$, recall that $m_{1,n}$ and $m_{2,n}$ are the unique pair of probability measures satisfying $P_{m_{1,n}}(X(\sigma_1) \in \cdot | \sigma_1 < \tau_{A_n}) = m_{2,n}(\cdot)$ and $P_{m_{2,n}}(X(\eta_1) \in \cdot) = m_{1,n}(\cdot)$. When $m_{1,n}$ and $m_{2,n}$ are as prescribed in (1.9), the second equation above holds by Theorem 2 and the first one holds by symmetry.

It follows from the definitions that

$$E_{M_{1,n}}\eta_1 = E_{M_{1,n}}\sigma_1 + E_{M_{2,n}}\eta_1. \tag{5.4}$$

From (5.3), (5.4) and (3.3), we obtain

$$\mu_n = E_{M_{1,n}}\eta_1 + E_{m_{1,n}}(\sigma_1 | \sigma_1 < \tau_{A_n}) - E_{M_{1,n}}\sigma_1. \tag{5.5}$$

From (5.2) and (5.5) we conclude that

$$\mu_n = \frac{|\Omega|}{|A_n^R|} E_{M_{1,n}} \int_0^{\sigma_1} 1_{A_n^R}(X(t))dt + E_{m_{1,n}}(\sigma_1 | \sigma_1 < \tau_{A_n}) - E_{M_{1,n}}\sigma_1. \tag{5.6}$$

We have now obtained an expression for μ_n which can be calculated explicitly since it only depends on Brownian motion in balls.

The first expectation on the right hand side of (5.6) is the expected amount of time a Brownian motion starting from a point on $\partial B_{R_n\epsilon_n}(0)$ spends in $B_{R_n\epsilon_n}(0)$ before exiting $B_{L_n\epsilon_n}(0)$. The second expectation is the expected exit time from $B_{L_n\epsilon_n}(0)$ for a Brownian motion starting from a point on $\partial B_{R_n\epsilon_n}(0)$ and conditioned to exit $B_{L_n\epsilon_n}(0)$ before hitting $B_{\epsilon_n}(0)$, and the third expectation is the expected exit time from $B_{L_n\epsilon_n}(0)$ for a Brownian motion starting from a point on $\partial B_{R_n\epsilon_n}(0)$.

We first calculate $E_{M_{1,n}} \int_0^{\sigma_1} 1_{A_n^R}(X(t))dt$. Let $A = \frac{1}{2}\frac{d^2}{dr^2} + \frac{d-1}{2r}\frac{d}{dr}$ and let $u_{1,n} = u_{1,n}(r)$, be the solution to the following elliptic problem:

$$Au_{1,n} = -1_{[0,R_n\epsilon_n]}(r), \ r \in [0, L_n\epsilon_n];$$

$$u'_{1,n}(0) = 0, \ u_{1,n}(L_n\epsilon_n) = 0;$$

Then it follows from the standard probabilistic representation of solutions of elliptic pde's that

$$u_{1,n}(R_n\epsilon_n) = E_{M_{1,n}} \int_0^{\sigma_1} 1_{A_n^R}(X(t))dt. \tag{5.7}$$

Writing $Au_{1,n}=\frac{1}{2r^{d-1}}(r^{d-1}u'_{1,n})'$, solving separately on $[0,R_n\epsilon_n]$ and on $[R_n\epsilon_n,L_n\epsilon_n]$, and matching the solutions and their first derivatives at $R_n\epsilon_n$, we obtain

$$u_{1,n}(R_n\epsilon_n)=\begin{cases}\frac{2}{d(d-2)}(R_n\epsilon_n)^d((R_n\epsilon_n)^{2-d}-(L_n\epsilon_n)^{2-d}), \text{ if } d\ge 3;\\ (R_n\epsilon_n)^2\log\frac{L_n}{R_n}, \text{ if } d=2.\end{cases}\tag{5.8}$$

Using (5.7) and (5.8) along with the fact that $|A_n^R|=n\omega_d(R_n\epsilon_n)^d$, we obtain after some cancellations

$$\frac{|\Omega|}{|A_n^R|}E_{M_{1,n}}\int_0^{\sigma_1}1_{A_n^R}(X(t))dt=\begin{cases}\frac{2|\Omega|(R_n^{2-d}-L_n^{2-d})}{d(d-2)\omega_d n\epsilon_n^{d-2}}, \text{ if } d\ge 3;\\ \frac{|\Omega|}{n\pi}\log\frac{L_n}{R_n}, \text{ if } d=2.\end{cases}\tag{5.9}$$

We will show below that

$$E_{m_{1,n}}(\sigma_1|\sigma_1<\tau_{A_n})-E_{M_{1,n}}\sigma_1=o(\frac{|\Omega|}{|A_n^R|}E_{M_{1,n}}\int_0^{\sigma_1}1_{A_n^R}(X(t))dt), \text{ as } n\to\infty.\tag{5.10}$$

From (5.10) and (5.6) it follows that the first term on the right hand side of (5.6) gives the leading order of μ_n. We now complete the proof of the proposition and then return to show (5.10).

Consider first the case $d\ge 3$. Since $\lim_{n\to\infty}n\epsilon_n^{d-2}=\alpha\in(0,\infty)$, it follows from (5.1), (5.6), (5.9) and (5.10) that

$$\lim_{n\to\infty}\frac{|\log\rho_n|}{\mu_n}=\lim_{n\to\infty}\frac{R_n^{2-d}-L_n^{2-d}}{1-L_n^{2-d}}\frac{d(d-2)\omega_d n\epsilon_n^{d-2}}{2|\Omega|(R_n^{2-d}-L_n^{2-d})}=\frac{d(d-2)\omega_d\alpha}{2|\Omega|},$$

which concludes the proof of the proposition when $d\ge 3$.

Now consider the case $d=2$. By (3.1) and the assumption on ϵ_n, we have $\lim_{n\to\infty}\frac{1}{n}\log L_n=\frac{1}{\alpha}$. Thus, we conclude from (5.1), (5.6), (5.9) and (5.10) that

$$\lim_{n\to\infty}\frac{|\log\rho_n|}{\mu_n}=\lim_{n\to\infty}\frac{\log\frac{L_n}{R_n}}{\log L_n}\frac{n\pi}{|\Omega|\log\frac{L_n}{R_n}}=\frac{\pi\alpha}{|\Omega|},$$

which completes the proof of the proposition in the case $d=2$.

In remains to prove (5.10). Recall that because of symmetry, the particular measures $m_{1,n}$ and $M_{1,n}$ on the left hand side of (5.10) play no role, and recall that $\rho_n=P_{m_{1,n}}(\sigma_1<\tau_{A_n})$. We have

$$\begin{aligned}E_{m_{1,n}}(\sigma_1|\sigma_1<\tau_{A_n})-E_{M_{1,n}}\sigma_1&=(\frac{1}{\rho_n}-1)E_{m_{1,n}}(\sigma_1,\sigma_1<\tau_{A_n})\\&-E_{M_{1,n}}(\sigma_1;\sigma_1>\tau_{A_n}),\end{aligned}\tag{5.11}$$

and by the strong Markov property,

$$E_{M_{1,n}}(\sigma_1;\sigma_1>\tau_{A_n})=(1-\rho_n)(E_{m_{1,n}}(\tau_{A_n}|\tau_{A_n}<\sigma_1)+E_{\epsilon_n e_1}\sigma_1), \tag{5.12}$$

where e_1 is a unit vector in R^d.

Now let $u_{2,n}$ be the solution to

$$Au_{2,n}=-1,\ r\in[0,L_n\epsilon_n];$$
$$u'_{2,n}(0)=0,\ u_{2,n}(L_n\epsilon_n)=0.$$

It follows from the standard probabilistic representation of solutions of elliptic pde's that

$$u_{2,n}(r)=E_{re_1}\sigma_1,\ \text{for } 0\le r\le L_n\epsilon_n.$$

A straightforward calculation reveals that

$$u_{2,n}(r)=\frac{1}{d}((L_n\epsilon_n)^2-r^2).$$

Thus,

$$E_{re_1}\sigma_1\le\frac{1}{d}(L_n\epsilon_n)^2,\ \text{for } 0\le r\le L_n\epsilon_n. \tag{5.13}$$

Since $\lim_{n\to\infty}\rho_n=1$, it follows from (5.13) that

$$(\frac{1}{\rho_n}-1)E_{m_{1,n}}(\sigma_1,\sigma_1<\tau_{A_n})+(1-\rho_n)E_{\epsilon_n e_1}\sigma_1=o((L_n\epsilon_n)^2),\ \text{as } n\to\infty. \tag{5.14}$$

Since $L_n=\kappa R_n$ for some $\kappa\in(0,\frac{1}{2}]$, the right hand side of (5.9) is on the order L_n^{2-d}, when $d\ge 3$. Recalling from (3.1) that $L_n\epsilon_n\sim n^{-\frac{1}{d}}$ and that $L_n\sim n^{\frac{2}{d(d-2)}}$, it follows that $L_n^{2-d}\sim(L_n\epsilon_n)^2$. In the case $d=2$, the right hand side of (5.9) is on the order $\frac{1}{n}$. From (3.1), $(L_n\epsilon_n)^2=O(\frac{1}{n})$. Thus, we conclude that the right hand side of (5.9) is on the order $(L_n\epsilon_n)^2$ for all $d\ge 2$. Using this with (5.14), it follows that

$$\begin{aligned}&(\frac{1}{\rho_n}-1)E_{m_{1,n}}(\sigma_1,\sigma_1<\tau_{A_n})+(1-\rho_n)E_{\epsilon_n e_1}\sigma_1\\&=o(\frac{|\Omega|}{|A_n^R|}E_{M_{1,n}}\int_0^{\sigma_1}1_{A_n^R}(X(t))dt),\ \text{as } n\to\infty.\end{aligned} \tag{5.15}$$

In light of (5.11), (5.12) and (5.15) and the fact that the right hand side of (5.9) is on the order $(L_n\epsilon_n)^2$, to complete the proof of (5.10) it is enough to show that

$$(1-\rho_n)E_{m_{1,n}}(\tau_{A_n}|\tau_{A_n}<\sigma_1)=o((L_n\epsilon_n)^2),\ \text{as } n\to\infty. \tag{5.16}$$

We calculate $E_{m_{1,n}}(\tau_{A_n}|\tau_{A_n}<\sigma_1)$. Let

$$h_n(r)=\begin{cases}\frac{r^{2-d}-(L_n\epsilon_n)^{2-d}}{(\epsilon_n)^{2-d}-(L_n\epsilon_n)^{2-d}}, & \text{if } d\ge 3;\\ \frac{\log\frac{L_n\epsilon_n}{r}}{\log L_n}, & \text{if } d=2,\end{cases} \tag{5.17}$$

and let $u_{3,n}$ denote the solution to

$$\begin{aligned}&Au_{3,n}=-h_n(r),\ r\in[\epsilon_n,L_n\epsilon_n];\\ &u_{3,n}(\epsilon_n)=u_{3,n}(L_n\epsilon_n)=0.\end{aligned} \tag{5.18}$$

Note that $h_n(r)=P_{re_1}(\tau_{A_n}<\sigma_1)$, for $r\in[\epsilon_n,L_n\epsilon_n]$, and that $h_n(R_n\epsilon_n)=1-\rho_n$. Then it follows from the standard probabilistic representation of solutions of elliptic pde's and from the connection between conditioned diffusions and h-transforms [9, section 7.2]) that

$$\frac{u_{3,n}(R_n\epsilon_n)}{h_n(R_n\epsilon_n)}=E_{m_{1,n}}(\tau_{A_n}|\tau_{A_n}<\sigma_1).$$

Thus,

$$(1-\rho_n)E_{m_{1,n}}(\tau_{A_n}|\tau_{A_n}<\sigma_1)=u_{3,n}(R_n\epsilon_n). \tag{5.19}$$

Writing $Au_{3,n}=\frac{1}{2r^{d-1}}(r^{d-1}u'_{1,n})'$ and using (5.17), we solve for $u_{3,n}$ in (5.18) to obtain

$$\begin{aligned}u_{3,n}(r)=&-\frac{r^{4-d}-\epsilon_n^{4-d}}{(4-d)(\epsilon_n^{2-d}-(L_n\epsilon_n)^{2-d})}+\frac{(r^2-\epsilon_n^2)(L_n\epsilon_n)^{2-d}}{d(\epsilon_n^{2-d}-(L_n\epsilon_n)^{2-d})}\\ &+\frac{r^{2-d}-\epsilon_n^{2-d}}{(L_n^{2-d}-1)\epsilon_n^{2-d}}\left(\frac{\epsilon_n^2(L_n^{4-d}-1)}{(4-d)(1-L_n^{2-d})}-\frac{\epsilon_n^2(L_n^2-1)L_n^{2-d}}{d(1-L_n^{2-d})}\right)\end{aligned} \tag{5.20}$$

when $d=3$ or $d\ge 5$. One solves similarly for $d=2$ and for $d=4$. Substituting $r=R_n\epsilon_n$ in (5.20), one finds that $u_{3,n}(R_n\epsilon_n)=O(\epsilon_n^2L_n^{4-d})$ for $d=3$ and $d\ge 5$. For $d=4$, it turns out that $u_{3,n}(R_n\epsilon_n)=O(\epsilon^2\log L_n)$. Therefore, it follows from (5.19) that (5.16) holds when $d\ge 3$. When $d=2$, it turns out that $u_{3,n}(R_n\epsilon_n)=O(L_n^2\epsilon_n^2\frac{\log L_n\epsilon_n}{\log L_n})$. From (3.1) and the fact that $\lim_{n\to\infty}\frac{n}{-\log\epsilon_n}=\alpha$, it follows that

$\lim_{n\to\infty} \frac{\log L_n\epsilon_n}{\log L_n} = 0$. Therefore, it follows from (5.19) that (5.16) also holds when $d = 2$. □

6. Proof of Proposition 3. Recall from (3.1) that $R_n = \kappa L_n$, for some $\kappa \in (0, \frac{1}{2}]$. Until now we have worked with an arbitrary fixed κ and have suppressed the dependence on κ. In this section we will need to let $\kappa \to 0$, so we will use the notation $R_{n,\kappa}, \mu_{n,\kappa}$, $\rho_{n,\kappa}$ and $A^R_{n,\kappa}$. Of course, $m_{i,n}$ and the hitting times σ_j and η_j now also depend on κ, but we will continue to suppress this dependence.

Letting $\eta_0 = 0$, we write

$$P_{m_{1,n}}(\tau_{A_n} > t) = \sum_{k=0}^{\infty} P_{m_{1,n}}(\tau_{A_n} > t, \eta_k < \tau_{A_n} < \eta_{k+1}). \tag{6.1}$$

By the same reasoning as in (4.2) and (4.3) we have

(6.2)

$$P_{m_{1,n}}(\tau_{A_n} > t, \eta_k < \tau_{A_n} < \eta_{k+1}) = (1-\rho_{n,\kappa})\rho^k_{n,\kappa} P_{m_{1,n}}(\tau_{A_n} > t | \eta_k < \tau_{A_n} < \eta_{k+1}).$$

Let $\beta \in (0, 1)$. We divide the sum on the right hand side of (6.1) into three parts—one with $k > \frac{t}{\mu_{n,\kappa}}$, one with $k \le \beta\frac{t}{\mu_{n,\kappa}}$ and one with $\beta\frac{t}{\mu_{n,\kappa}} < k \le \frac{t}{\mu_{n,\kappa}}$. Using (6.2) we have

$$\begin{aligned} &\sum_{k>\frac{t}{\mu_{n,\kappa}}} P_{m_{1,n}}(\tau_{A_n} > t, \eta_k < \tau_{A_n} < \eta_{k+1}) \le \sum_{k>\frac{t}{\mu_{n,\kappa}}} (1-\rho_{n,\kappa})\rho^k_{n,\kappa} \\ &\le \rho_{n,\kappa}^{\frac{t}{\mu_{n,\kappa}}} = \exp(-\frac{t}{\mu_{n,\kappa}}|\log \rho_{n,\kappa}|). \end{aligned} \tag{6.3}$$

In order to treat the other two parts of the sum, we need three lemmas. Let

$$\mu_{n,\kappa,x} = E_x(\eta_1|\eta_1 < \tau_{A_n}) = E_x(\eta_1|\eta_1 < \tau_{A_n} < \eta_2), \text{ for } x \in \partial A^R_{n,\kappa},$$

and note that $\mu_{n,\kappa} = \int_{\partial A^R_{n,\kappa}} \mu_{n,\kappa,x} m_{1,n}(dx)$. The main import of the first lemma is that under $P_x(\cdot\ |\eta_1 < \tau_{A_n})$, $\frac{\eta_1}{(L_n\epsilon_n)^2}$ is stochastically dominated by a geometric random variable with parameter p_κ independent of $x \in \bar{\Omega} - A_n$, and that $\mu_{n,\kappa,x}$ depends very little on $x \in \partial A^R_{n,\kappa}$ if x is not near $\partial\Omega$ and κ is small.

Lemma 1. *i. There exists a $C_0 > 0$ independent of κ such that*

$$\mu_{n,\kappa} \ge C_0(L_n\epsilon_n)^2;$$

ii. There exists a $p_\kappa \in (0,1)$ *independent of* n *such that*

$$P_x(\eta_1 > k(L_n\epsilon_n)^2)|\eta_1 < \tau_{A_n}) = P_x(\eta_1 > k(L_n\epsilon_n)^2)|\eta_1 < \tau_{A_n} < \eta_2)$$
$$< (1-p_\kappa)^k, \text{ for all } x \in \Omega - A_n \text{ and } k = 1, 2, ...;$$

thus, $\mu_{n,\kappa} = E_{m_{1,n}}(\eta_1|\eta_1 < \tau_{A_n}) \le \frac{1-p_\kappa}{p_\kappa}(L_n\epsilon_n)^2$.

iii. There exists a $\gamma_n(l,\kappa)$ *satisfying* $\lim_{\kappa\to 0}\lim_{l\to\infty}\lim_{n\to\infty}\gamma_n(l,\kappa) = 0$ *such that*

$$\mu_{n,\kappa,x} \le (1+\gamma_n(l,\kappa))\mu_{n,\kappa}, \text{ for all } x \in \partial A^R_{n,\kappa} \text{ which satisfy } dist(x,\partial\Omega) > lL_n\epsilon_n.$$

Lemma 2. *For each* n*, there exists a* c_n *independent of* κ *and a* t_n *such that*

$$P_x(\tau_{A_n} > t|\tau_{A_n} < \eta_1) \le \exp(-c_n t), \text{ for } x \in \partial A^R_{n,\kappa} \text{ and } t > t_n,$$
$$\text{where } \lim_{n\to\infty} c_n = \infty.$$

The third lemma concerns the locations $\{X(\eta_m)\}_{m=0}^\infty$ of the reflected Brownian motion at the hitting times $\{\eta_m\}_{m=0}^\infty$, and states that in an appropriate scale the probability that any fixed positive proportion of the first k locations will be close to the boundary $\partial\Omega$ decays asymptotically fast as $k \to \infty$.

Lemma 3. *Assume that* $\lim_{n\to\infty}\frac{|\log\rho_{n,\kappa}|}{\mu_{n,\kappa}} \in (0,\infty)$*. For* $l > 0$*, let* $D_{n,l} = \{x \in \bar{\Omega} : dist(x,\partial\Omega) \le lL_n\epsilon_n\}$*. For each* $q \in (0,1)$*, for each* $l > 0$ *and for each* $\kappa \in (0,\frac{1}{2}]$*, there exists a* $\zeta_n(q,l,\kappa)$ *satisfying* $\lim_{n\to\infty}\zeta_n(q,l,\kappa) = \infty$*, and a* k_n *such that*

$$P_{m_{1,n}}\left(\frac{1}{k}\sum_{j=0}^{k-1}1_{D_{n,l}}(X(\eta_j)) \ge q \;|\eta_k < \tau_{A_n} < \eta_{k+1}\right)$$
$$\le \exp(-\zeta_n(q,l,\kappa)\mu_{n,\kappa}k), \text{ for } k \ge k_n.$$

We now return to the proof of the proposition. The proof of the lemmas is postponed until the next section. We first treat the interval $0 \le k \le \beta\frac{t}{\mu_{n,\kappa}}$. Let $M_{n,x}(\lambda) = E_x(\exp(\lambda\eta_1)|\eta_1 < \tau_{A_n} < \eta_2)$, for $x \in \partial A^R_{n,\kappa}$. Lemma 1-ii indicates that $\frac{\eta_1}{(L_n\epsilon_n)^2}$ under $P_x(\cdot\;|\eta_1 < \tau_{A_n} < \eta_2)$ is stochastically dominated by a geometric random variable with parameter p_κ. Using this in conjunction with Lemma 1-i, it follows that for any $\lambda > 0$ and sufficiently large n, $M'_{n,x}(\lambda) = E_x(\eta_1\exp(\lambda\eta_1)|\eta_1 < \tau_{A_n} < \eta_2)$ satisfies $M'_{n,x}(\lambda) \le C_{\lambda,\kappa}\mu_{n,\kappa}$, for all $x \in \partial A^R_{n,\kappa}$ and some $C_{\lambda,\kappa} \in (1,\infty)$.

Let $N_n(\lambda) = E_x(\exp(\lambda\tau_{A_n})|\tau_{A_n} < \eta_1)$, for $x \in \partial A^R_{n,\kappa}$. (By symmetry, the expression is independent of $x \in \partial A^R_{n,\kappa}$.) By Lemma 2, it follows that for any $\lambda > 0$ and sufficiently large n, $N'_n(\lambda) = E_x(\tau_{A_n}\exp(\lambda\tau_{A_n})|\tau_{A_n} < \eta_1)$ satisfies $N'_n(\lambda) \le \hat{C}_{n,\lambda,\kappa}$, for some $\hat{C}_{n,\lambda,\kappa} \in (1,\infty)$. Since $M'_{n,x}(\lambda)$ and $N'_n(\lambda)$ are increasing, we have $\log M_{n,x}(\lambda) \le \lambda M'_{n,x}(\lambda)$ and $\log N_n(\lambda) \le \lambda N'_n(\lambda)$. Thus, there exists an n_λ such that

$$\begin{cases} E_x(\exp(\lambda\eta_1)|\eta_1 < \tau_{A_n} < \eta_2) \le \exp(\lambda C_{\lambda,\kappa}\mu_{n,\kappa}) \\ E_x(\exp(\lambda\tau_{A_n})|\tau_{A_n} < \eta_1) \le \exp(\lambda\hat{C}_{n,\lambda,\kappa}), \end{cases} \tag{6.4}$$
$$\text{for } x \in \partial A^R_{n,\kappa},\ \lambda > 0 \text{ and } n \ge n_\lambda.$$

By Chebyshev's inequality, we have for $\lambda > 0$,

(6.5)

$$P_{m_{1,n}}(\tau_{A_n} > t|\eta_k < \tau_{A_n} < \eta_{k+1}) \le \exp(-\lambda t)E_{m_{1,n}}(\exp(\lambda\tau_{A_n})|\eta_k < \tau_{A_n} < \eta_{k+1}).$$

Denoting by $\{\mathcal{F}_t\}_{t\ge 0}$ the filtration induced by $X(\cdot)$ and using (6.4) for the inequality below, we have

(6.6)

$$\begin{aligned}
&E_{m_{1,n}}(\exp(\lambda\tau_{A_n}); \eta_k < \tau_{A_n} < \eta_{k+1}) \\
&= E_{m_{1,n}}E_{m_{1,n}}(\exp(\lambda\tau_{A_n})1_{\{\eta_k<\tau_{A_n}<\eta_{k+1}\}}|\mathcal{F}_{\eta_k}) \\
&= E_{m_{1,n}}E_{m_{1,n}}(\exp(\lambda(\tau_{A_n}-\eta_k) + \lambda\sum_{j=1}^k(\eta_j-\eta_{j-1}))1_{\{\eta_k<\tau_{A_n}<\eta_{k+1}\}}|\mathcal{F}_{\eta_k}) \\
&= E_{m_{1,n}}\exp(\lambda\sum_{j=1}^k(\eta_j-\eta_{j-1}))1_{\{\eta_k<\tau_{A_n}\}}E_{m_{1,n}}(\exp(\lambda(\tau_{A_n}-\eta_k))1_{\{\tau_{A_n}<\eta_{k+1}\}}|\mathcal{F}_{\eta_k}) \\
&= E_{m_{1,n}}\exp(\lambda\sum_{j=1}^k(\eta_j-\eta_{j-1}))1_{\{\eta_k<\tau_{A_n}\}}E_{X(\eta_k)}(\exp(\lambda\tau_{A_n})|\tau_{A_n}<\eta_1)\times \\
&P_{X(\eta_k)}(\tau_{A_n}<\eta_1) \\
&\le \exp(\lambda\hat{C}_{n,\lambda,\kappa})E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^k(\eta_j-\eta_{j-1})); \eta_k < \tau_{A_n} < \eta_{k+1}).
\end{aligned}$$

From (6.6) we conclude that

$$\begin{aligned}
&E_{m_{1,n}}(\exp(\lambda\tau_{A_n})|\eta_k < \tau_{A_n} < \eta_{k+1}) \\
&\le \exp(\lambda\hat{C}_{n,\lambda,\kappa})E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^k(\eta_j-\eta_{j-1}))|\eta_k < \tau_{A_n} < \eta_{k+1}).
\end{aligned} \tag{6.7}$$

Using (6.4) for the inequality below, we estimate the expectation on the right hand side of (6.7) by writing

(6.8)

$$
\begin{aligned}
&E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k}(\eta_j-\eta_{j-1})); \eta_k<\tau_{A_n}<\eta_{k+1}) \\
&= E_{m_{1,n}}E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k}(\eta_j-\eta_{j-1}))1_{\{\eta_k<\tau_{A_n}<\eta_{k+1}\}}|\mathcal{F}_{\eta_{k-1}}) \\
&= E_{m_{1,n}}\exp(\lambda\sum_{j=1}^{k-1}(\eta_j-\eta_{j-1}))1_{\{\eta_{k-1}<\tau_{A_n}\}}E_{X(\eta_{k-1})}(\exp(\lambda\eta_1)1_{\eta_1<\tau_{A_n}<\eta_2}) \\
&= E_{m_{1,n}}\exp(\lambda\sum_{j=1}^{k-1}(\eta_j-\eta_{j-1}))1_{\{\eta_{k-1}<\tau_{A_n}\}}E_{X(\eta_{k-1})}(\exp(\lambda\eta_1)|\eta_1<\tau_{A_n}<\eta_2)\times \\
&P_{X(\eta_{k-1})}(\eta_1<\tau_{A_n}<\eta_2) \\
&\le \exp(\lambda C_{\lambda,\kappa}\mu_{n,\kappa})E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k-1}(\eta_j-\eta_{j-1})); \eta_k<\tau_{A_n}<\eta_{k+1}),
\end{aligned}
$$

for $\lambda>0$ and $n\ge n_\lambda$.

Since $E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k-1}(\eta_j-\eta_{j-1}))|\eta_l<\tau_{A_n}<\eta_{l+1})$ is independent of $l\ge k-1$, it follows from (6.8) that

$$
\begin{aligned}
&E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k}(\eta_j-\eta_{j-1}))|\eta_k<\tau_{A_n}<\eta_{k+1}) \\
&\le \exp(\lambda C_{\lambda,\kappa}\mu_{n,\kappa})E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k-1}(\eta_j-\eta_{j-1}))|\eta_{k-1}<\tau_{A_n}<\eta_k),
\end{aligned}
$$

and thus by induction we conclude that

$$
\begin{aligned}
&E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k}(\eta_j-\eta_{j-1}))|\eta_k<\tau_{A_n}<\eta_{k+1})\le\exp(k\lambda C_{\lambda,\kappa}\mu_{n,\kappa}), \\
&\text{for } \lambda>0 \text{ and } n\ge n_\lambda.
\end{aligned}
\tag{6.9}
$$

Using (6.2), (6.5),(6.7) and (6.9), we obtain

(6.10)

$$
\sum_{0\le k\le\beta\frac{t}{\mu_{n,\kappa}}}P_{m_{1,n}}(\tau_{A_n}>t,\eta_k<\tau_{A_n}<\eta_{k+1})\le\exp(\lambda\hat{C}_{n,\lambda,\kappa})\exp(-\lambda(1-\beta C_{\lambda,\kappa})t),
$$

for $\lambda>0$ and $n\ge n_\lambda$.

We now obtain an estimate for the part of the sum with $\beta\frac{t}{\mu_{n,\kappa}}<k\le\frac{t}{\mu_{n,\kappa}}$. Recalling that $\tau_{A_n}-\eta_k$ under $P_{m_{1,n}}(\cdot\ |\eta_k<\tau_{A_n}<\eta_{k+1})$ is distributed like τ_{A_n}

under $P_{m_{1,n}}(\cdot\ |\tau_{A_n} < \eta_1)$ and using Lemma 2, we have for $\delta \in (0, \frac{1}{3})$,

$$
\begin{aligned}
& P_{m_{1,n}}(\tau_{A_n} > t|\eta_k < \tau_{A_n} < \eta_{k+1}) \\
& = P_{m_{1,n}}(\eta_k + (\tau_{A_n} - \eta_k) > t|\eta_k < \tau_{A_n} < \eta_{k+1}) \\
& \leq P_{m_{1,n}}(\eta_k > (1-\delta)t|\eta_k < \tau_{A_n} < \eta_{k+1}) \\
& + P_{m_{1,n}}(\tau_{A_n} - \eta_k > \delta t|\eta_k < \tau_{A_n} < \eta_{k+1}) \\
& \leq P_{m_{1,n}}(\eta_k > (1-\delta)t|\eta_k < \tau_{A_n} < \eta_{k+1}) + \exp(-c_n\delta t),\ \text{for } t > t_n,
\end{aligned}
\tag{6.11}
$$

where $\lim\limits_{n\to\infty} c_n = \infty$.

To estimate the first term on the right hand side of (6.11), we will break it into four parts. Recalling the definition of $D_{n,l}$ in Lemma 3, let

$$
B_{n,l,q,k} = \{\frac{1}{k}\sum_{j=0}^{k-1} 1_{D_{n,l}}(X(\eta_j)) \geq q\}. \tag{6.12}
$$

Let $M > 0$. Since

$$
\begin{aligned}
\{\eta_k > (1-\delta)t\} \subset & \{\sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{D^c_{n,l}}(X(\eta_{j-1})) > (1-3\delta)t\} \\
& \cup \{\sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{\{\eta_j-\eta_{j-1}\leq M(L_n\epsilon_n)^2\}}1_{D_{n,l}}(X(\eta_{j-1})) > \delta t\} \\
& \cup \{\sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{\{\eta_j-\eta_{j-1}> M(L_n\epsilon_n)^2\}}1_{D_{n,l}}(X(\eta_{j-1})) > \delta t\},
\end{aligned}
$$

we have

$$
\begin{aligned}
& P_{m_{1,n}}(\eta_k > (1-\delta)t|\eta_k < \tau_{A_n} < \eta_{k+1}) \\
& \leq P_{m_{1,n}}(\sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{D^c_{n,l}}(X(\eta_{j-1})) > (1-3\delta)t|\eta_k < \tau_{A_n} < \eta_{k+1}) \\
& + P_{m_{1,n}}(\sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{\{\eta_j-\eta_{j-1}\leq M(L_n\epsilon_n)^2\}}1_{D_{n,l}}(X(\eta_{j-1})) > \delta t, B^c_{n,l,q,k}| \\
& \eta_k < \tau_{A_n} < \eta_{k+1}) + P_{m_{1,n}}(B_{n,l,q,k}|\eta_k < \tau_{A_n} < \eta_{k+1}) \\
& + P_{m_{1,n}}(\sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{\{\eta_j-\eta_{j-1}> M(L_n\epsilon_n)^2\}} > \delta t|\eta_k < \tau_{A_n} < \eta_{k+1}).
\end{aligned}
\tag{6.13}
$$

Using Lemma 1-i for the second inequality below, it follows from the definition of $D_{n,l}$ and $B_{n,l,q,k}$ that

$$
\sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{\{\eta_j-\eta_{j-1}\leq M(L_n\epsilon_n)^2\}}1_{D_{n,l}}(X(\eta_{j-1})) < q\frac{t}{\mu_{n,\kappa}}M(L_n\epsilon_n)^2 \leq \frac{tqM}{C_0},
$$

on $B^c_{n,l,q,k}$, if $k \leq \dfrac{t}{\mu_{n,\kappa}}$.

Thus, the second term on the right hand side of (6.13) satisfies

$$
\begin{aligned}
&P_{m_{1,n}}(\sum_{j=1}^{k}(\eta_j-\eta_{j-1})1_{\{\eta_j-\eta_{j-1}\le M(L_n\epsilon_n)^2\}}1_{D_{n,l}}(X(\eta_{j-1}))>\delta t, B^c_{n,l,q,k}| \\
&\eta_k<\tau_{A_n}<\eta_{k+1})=0,\ \text{if}\ \frac{qM}{C_0}\le\delta \text{ and } k\le\frac{t}{\mu_{n,\kappa}}.
\end{aligned}
\tag{6.14}
$$

By Lemma 3, the third term on the right hand side of (6.13) satisfies

(6.15)

$$
\begin{aligned}
&P_{m_{1,n}}(B_{n,l,q,k}|\eta_k<\tau_{A_n}<\eta_{k+1})\le\exp(-t\beta\zeta_n(q,l,\kappa)),\ \text{if}\ k\ge\beta\frac{t}{\mu_{n,\kappa}} \text{ and } t\ge t_n, \\
&\text{where}\ \lim_{n\to\infty}\zeta_n(q,l,\kappa)=\infty.
\end{aligned}
$$

Applying Chebyshev's inequality to the last term on the right hand side of (6.13), we have for $\lambda>0$

(6.16)

$$
\begin{aligned}
&P_{m_{1,n}}(\sum_{j=1}^{k}(\eta_j-\eta_{j-1})1_{\{\eta_j-\eta_{j-1}>M(L_n\epsilon_n)^2\}}>\delta t|\eta_k<\tau_{A_n}<\eta_{k+1}) \\
&\le\exp(-\lambda\delta t)E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k}(\eta_j-\eta_{j-1})1_{\{\eta_j-\eta_{j-1}>M(L_n\epsilon_n)^2\}})|\eta_k<\tau_{A_n}<\eta_{k+1}).
\end{aligned}
$$

Noting from Lemma 1-ii that $\frac{\eta_1}{(L_n\epsilon_n)^2}1_{\{\eta_1>M(L_n\epsilon_n)^2\}}$ under $P_x(\cdot\ |\eta_1<\tau_{A_n}<\eta_2)$ is stochastically dominated by a geometric random variable multiplied by the indicator of the set where it is greater than M, the same considerations that led to (6.4) give

$$
\begin{aligned}
&E_x(\exp(\lambda\eta_1 1_{\{\eta_1>M(L_n\epsilon_n)^2\}})|\eta_1<\tau_{A_n}<\eta_2)\le\exp(\lambda C_{\lambda,\kappa,M}\mu_{n,\kappa}), \\
&\text{for } x\in\partial A^R_{n,\kappa}, \lambda>0 \text{ and } n\ge n_\lambda,\ \text{where}\ \lim_{M\to\infty}C_{\lambda,\kappa,M}=0.
\end{aligned}
\tag{6.17}
$$

Using (6.17) for the inequality below, we estimate the expectation on the right hand side of (6.16) by writing

$$
\begin{aligned}
&E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k}(\eta_j-\eta_{j-1})1_{\{\eta_j-\eta_{j-1}>M(L_n\epsilon_n)^2\}});\eta_k<\tau_{A_n}<\eta_{k+1})\\
&=E_{m_{1,n}}\exp(\lambda\sum_{j=1}^{k-1}(\eta_j-\eta_{j-1})1_{\{\eta_j-\eta_{j-1}>M(L_n\epsilon_n)^2\}})1_{\{\eta_{k-1}<\tau_{A_n}\}}\times\\
&E_{X(\eta_{k-1})}\exp(\lambda\eta_1 1_{\{\eta_1>M(L_n\epsilon_n)^2\}})1_{\{\eta_1<\tau_{A_n}<\eta_2\}}\\
&=E_{m_{1,n}}\exp(\lambda\sum_{j=1}^{k-1}(\eta_j-\eta_{j-1})1_{\{\eta_j-\eta_{j-1}>M(L_n\epsilon_n)^2\}})1_{\{\eta_{k-1}<\tau_{A_n}\}}\times \qquad (6.18)\\
&E_{X(\eta_{k-1})}(\exp(\lambda\eta_1 1_{\{\eta_1>M(L_n\epsilon_n)^2\}})|\eta_1<\tau_{A_n}<\eta_2)P_{X(\eta_{k-1})}(\eta_1<\tau_{A_n}<\eta_2)\\
&\le\exp(\lambda C_{\lambda,\kappa,M}\mu_{n,\kappa})\times\\
&E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k-1}(\eta_j-\eta_{j-1})1_{\{\eta_j-\eta_{j-1}>M(L_n\epsilon_n)^2\}});\eta_k<\tau_{A_n}<\eta_{k+1}),\\
&\text{for } \lambda>0 \text{ and } n\ge n_\lambda.
\end{aligned}
$$

Since $E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k-1}(\eta_j-\eta_{j-1})1_{\{\eta_j-\eta_{j-1}>M(L_n\epsilon_n)^2\}})|\eta_l<\tau_{A_n}<\eta_{l+1})$ is independent of $l\ge k-1$, it follows from (6.18) that

$$
\begin{aligned}
&E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k}(\eta_j-\eta_{j-1})1_{\{\eta_j-\eta_{j-1}>M(L_n\epsilon_n)^2\}})|\eta_k<\tau_{A_n}<\eta_{k+1})\\
&\le\exp(\lambda C_{\lambda,\kappa,M}\mu_{n,\kappa})\times\\
&E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k-1}(\eta_j-\eta_{j-1})1_{\{\eta_j-\eta_{j-1}>M(L_n\epsilon_n)^2\}})|\eta_{k-1}<\tau_{A_n}<\eta_k),
\end{aligned}
$$

and thus by induction we conclude that

$$
\begin{aligned}
&E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k}(\eta_j-\eta_{j-1})1_{\{\eta_j-\eta_{j-1}>M(L_n\epsilon_n)^2\}})|\eta_k<\tau_{A_n}<\eta_{k+1}) \qquad (6.19)\\
&\le\exp(k\lambda C_{\lambda,\kappa,M}\mu_{n,\kappa}),\ \text{for } \lambda>0 \text{ and } n\ge n_\lambda,\ \text{where } \lim_{M\to\infty}C_{\lambda,\kappa,M}=0.
\end{aligned}
$$

Now (6.16) and (6.19) give

$$
\begin{aligned}
&P_{m_{1,n}}(\sum_{j=1}^{k}(\eta_j-\eta_{j-1})1_{\{\eta_j-\eta_{j-1}>M(L_n\epsilon_n)^2\}}>\delta t|\eta_k<\tau_{A_n}<\eta_{k+1})\\
&\le\exp(-\lambda(\delta-C_{\lambda,\kappa,M})t), \qquad (6.20)\\
&\text{for } \lambda>0,\ n\ge n_\lambda,\ \text{and } k\le\frac{t}{\mu_{n,\kappa}},\ \text{where } \lim_{M\to\infty}C_{\lambda,\kappa,M}=0.
\end{aligned}
$$

Finally we turn to the first term on the right hand side of (6.13). Let $x\in D^c_{n,l}\cap\partial A^R_{n,\kappa}$. By Lemma 1-ii, $\frac{\eta_1}{(L_n\epsilon_n)^2}$ under $P_x(\cdot\ |\eta_1<\tau_{A_n})$ is stochastically dominated by a geometric random variable with parameter p_κ, and by Lemma 1-iii, $\mu_{n,\kappa,x} = E_x(\eta_1|\eta_1 < \tau_{A_n}) \le (1+\gamma_n(l,\kappa))\mu_{n,\kappa}$, where $\lim_{\kappa\to 0}\lim_{l\to\infty}\lim_{n\to\infty}\gamma_n(l,\kappa) = 0$. Thus, since $E_x(\exp(\lambda\eta_1)|\eta_1 < \tau_{A_n}) = 1 + \sum_{j=1}^{\infty} E_x(\eta_1^j|\eta_1 < \tau_{A_n})\lambda^j$, it follows that

(6.21)

$$E_x(\exp(\lambda\eta_1)|\eta_1 < \tau_{A_n}) \le \psi_{n,l,\kappa}(\lambda) \equiv 1 + \mu_{n,\kappa}(1+\gamma_n(l,\kappa))\lambda + \sum_{j=2}^{\infty} b_{j,\kappa}((L_n\epsilon_n)^2\lambda)^j,$$

for $x \in D_{n,l}^c \cap \partial A_{n,\kappa}^R$,

where $b_{j,\kappa}$ is the j-th moment of a geometric random variable with parameter p_κ. Applying Chebyshev's inequality to the first term on the right hand side of (6.13), we have for $\lambda > 0$

(6.22)

$$P_{m_{1,n}}(\sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{D_{n,l}^c}(X(\eta_{j-1})) > (1-3\delta)t|\eta_k < \tau_{A_n} < \eta_{k+1})$$

$$\le \exp(-\lambda(1-3\delta)t)E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{D_{n,l}^c}(X(\eta_{j-1})))|\eta_k < \tau_{A_n} < \eta_{k+1}).$$

Using (6.21) for the inequality below, we estimate the right hand side of (6.22) by writing

(6.23)

$$E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{D_{n,l}^c}(X(\eta_{j-1}))); \eta_k < \tau_{A_n} < \eta_{k+1})$$

$$= E_{m_{1,n}}\exp(\lambda\sum_{j=1}^{k-1}(\eta_j - \eta_{j-1})1_{D_{n,l}^c}(X(\eta_{j-1})))1_{\{\eta_{k-1}<\tau_{A_n}\}}\times$$

$$(1_{D_{n,l}}(X(\eta_{k-1}))P_{X(\eta_{k-1})}(\eta_1 < \tau_{A_n} < \eta_2)+$$

$$1_{D_{n,l}^c}(X(\eta_{k-1}))E_{X(\eta_{k-1})}\exp(\lambda\eta_1)1_{\{\eta_1<\tau_{A_n}<\eta_2\}})$$

$$= E_{m_{1,n}}\exp(\lambda\sum_{j=1}^{k-1}(\eta_j - \eta_{j-1}))1_{D_{n,l}^c}(X(\eta_{j-1})))1_{\{\eta_{k-1}<\tau_{A_n}\}}\times$$

$$\Big(1_{D_{n,l}}(X(\eta_{k-1})) + 1_{D_{n,l}^c}(X(\eta_{k-1}))E_{X(\eta_{k-1})}(\exp(\lambda\eta_1)|\eta_1 < \tau_{A_n} < \eta_2)\Big)\times$$

$$P_{X(\eta_{k-1})}(\eta_1 < \tau_{A_n} < \eta_2)$$

$$\le \psi_{n,l,\kappa}(\lambda)E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k-1}(\eta_j - \eta_{j-1})1_{D_{n,l}^c}(X(\eta_{j-1}))); \eta_k < \tau_{A_n} < \eta_{k+1}).$$

Since $E_{m_{1,n}}(\exp(\lambda\sum_{j=1}^{k-1}(\eta_j - \eta_{j-1})1_{D_{n,l}^c}(X(\eta_{j-1})))|\eta_l < \tau_{A_n} < \eta_{l+1})$ is independent of $l \geq k-1$, it follows from (6.23) that

$$E_{m_{1,n}}(\exp(\lambda \sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{D^c_{n,l}}(X(\eta_{j-1})))|\eta_k < \tau_{A_n} < \eta_{k+1})$$
$$\leq \psi_{n,l,\kappa}(\lambda) E_{m_{1,n}}(\exp(\lambda \sum_{j=1}^{k-1}(\eta_j - \eta_{j-1})1_{D^c_{n,l}}(X(\eta_{j-1})))|\eta_{k-1} < \tau_{A_n} < \eta_k),$$

and thus by induction we conclude that

$$E_{m_{1,n}}(\exp(\lambda \sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{D^c_{n,l}}(X(\eta_{j-1})))|\eta_k < \tau_{A_n} < \eta_{k+1}) \leq \psi^k_{n,l,\kappa}(\lambda). \tag{6.24}$$

Let

$$\Lambda_{n,l,\kappa}(\lambda) = \log \psi_{n,l,\kappa}(\lambda) \tag{6.25}$$

and let

$$I_{n,l,\kappa}(y) = \sup_{\lambda > 0}(\lambda y - \Lambda_{n,l,\kappa}(\lambda)) \tag{6.26}$$

denote the Fenchel-Legendre transform of the convex function $\Lambda_{n,l,\kappa}$.

From (6.22), (6.24), (6.25) and the fact that

$$\rho^k_{n,\kappa} = \exp(-k|\log \rho_{n,\kappa}|) = \exp(-t(1-3\delta)\frac{|\log \rho_{n,\kappa}|}{\frac{t(1-3\delta)}{k}}),$$

we obtain

$$\begin{aligned} &\rho^k_{n,\kappa} P_{m_{1,n}}(\sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{D^c_{n,l}}(X(\eta_{j-1})) > (1-3\delta)t|\eta_k < \tau_{A_n} < \eta_{k+1}) \\ &\leq \exp(-t(1-3\delta)\left(\frac{|\log \rho_{n,\kappa}| + \lambda\frac{t(1-3\delta)}{k} - \Lambda_{n,l,\kappa}(\lambda)}{\frac{t(1-3\delta)}{k}}\right)). \end{aligned} \tag{6.27}$$

From (6.26) and the fact that $\lambda > 0$ in (6.27) is arbitrary, it follows that

$$\rho^k_{n,\kappa} P_{m_{1,n}}(\sum_{j=1}^{k}(\eta_j - \eta_{j-1})1_{D^c_{n,l}}(X(\eta_{j-1})) > (1-3\delta)t|\eta_k < \tau_{A_n} < \eta_{k+1})$$
$$\leq \exp(-t(1-3\delta)\left(\inf_{y>0}\frac{|\log \rho_{n,\kappa}| + I_{n,l,\kappa}(y)}{y}\right)),$$

and thus

$$\sum_{\beta\frac{t}{\mu_{n,\kappa}}<k\le\frac{t}{\mu_{n,\kappa}}} \rho^k_{n,\kappa}\times$$

$$P_{m_{1,n}}(\sum_{j=1}^{k}(\eta_j-\eta_{j-1})1_{D^c_{n,l}}(X(\eta_{j-1}))>(1-3\delta)t|\eta_k<\tau_{A_n}<\eta_{k+1}) \tag{6.28}$$
$$\le \frac{t}{\mu_{n,\kappa}}\exp(-t(1-3\delta)\left(\inf_{y>0}\frac{|\log\rho_{n,\kappa}|+I_{n,l,\kappa}(y)}{y}\right)).$$

From (6.2), (6.11), (6.13)-(6.15), (6.20) and (6.28), we conclude that

(6.29)
$$\sum_{\beta\frac{t}{\mu_{n,\kappa}}\le k\le\frac{t}{\mu_{n,\kappa}}} P_{m_{1,n}}(\tau_{A_n}>t,\eta_k<\tau_{A_n}<\eta_{k+1})\le\exp(-c_n\delta t)+\exp(-t\beta\zeta_n(q,l,\kappa))$$
$$+\exp(-\lambda(\delta-C_{\lambda,\kappa,M})t)+\frac{t}{\mu_{n,\kappa}}\exp(-t(1-3\delta)\left(\inf_{y>0}\frac{|\log\rho_{n,\kappa}|+I_{n,l,\kappa}(y)}{y}\right)),$$

if $\frac{qM}{C_0}\le\delta,$ for $n\ge n_\lambda$ and $t\ge t_n,$ where $\lim_{n\to\infty}c_n=\infty,$
$\lim_{n\to\infty}\zeta_n(q,l,\kappa)=\infty,$ and $\lim_{M\to\infty}C_{\lambda,\kappa,M}=0.$

Recall from the remark following Proposition 2 that $\lim_{n\to\infty}\frac{|\log\rho_{n,\kappa}|}{\mu_{n,\kappa}}$ does not depend on $\kappa\in(0,\frac{1}{2}]$. Thus in the sequel, we will sometimes suppress the κ appearing in this expression. We will show that

$$\liminf_{\kappa\to0}\liminf_{l\to\infty}\liminf_{n\to\infty}\inf_{y>0}\frac{|\log\rho_{n,\kappa}|+I_{n,l,\kappa}(y)}{y}=\lim_{n\to\infty}\frac{|\log\rho_n|}{\mu_n}. \tag{6.30}$$

Before proving (6.30), we complete the proof of the proposition. From (6.1), (6.3), (6.10) and (6.29), we have

(6.31)
$$\limsup_{t\to\infty}\frac{1}{t}\log P_{m_{1,n}}(\tau_{A_n}>t)$$
$$\le-\min(\frac{|\log\rho_{n,\kappa}|}{\mu_{n,\kappa}},\ \lambda(1-\beta C_{\lambda,\kappa}),\ c_n\delta,\beta\zeta_n(q,l,\kappa),\ \lambda(\delta-C_{\lambda,\kappa,M}),$$
$$(1-3\delta)\inf_{y>0}\frac{|\log\rho_{n,\kappa}|+I_{n,l,\kappa}(y)}{y}),\ \text{for any}\ \ \lambda,l,M>0,\ q,\beta\in(0,1)$$
$$\text{and}\ \delta\in(0,\frac{1}{3})\ \text{satisfying}\ \frac{qM}{C_0}\le\delta,\ \text{and sufficiently large}\ n\ \text{depending on}\ \lambda.$$

For fixed $\delta\in(0,\frac{1}{3})$, choose $\lambda\ge\frac{2}{\delta}\lim_{n\to\infty}\frac{|\log\rho_n|}{\mu_n}$, then choose $\beta\le\frac{1}{2C_{\lambda,\kappa}}$ and M sufficiently large so that $C_{\lambda,\kappa,M}\le\frac{1}{2}\delta$ (which is possible because $\lim_{M\to\infty}C_{\lambda,\kappa,M}=$

0), and then choose q sufficiently small so that $\frac{qM}{C_0} \le \delta$. Now let $n \to \infty$, then let $l \to \infty$ and then let $\kappa \to 0$. Since $\lim_{n\to\infty} \zeta_n(q,l,\kappa) = \lim_{n\to\infty} c_n = \infty$, it follows from (6.30) and (6.31) that

$$\limsup_{n\to\infty} \limsup_{t\to\infty} \frac{1}{t} \log P_{m_{1,n}}(\tau_{A_n} > t) \le -(1-3\delta) \lim_{n\to\infty} \frac{|\log \rho_n|}{\mu_n}. \tag{6.32}$$

The proposition now follows from (6.32), (1.2) and the fact that $\delta > 0$ is arbitrary.

We now turn to the proof of (6.30). We recall the following well-known fact concerning Fenchel-Legendre transforms, which follows without much difficulty from (6.26): the function $I_{n,l,\kappa}$ is nonnegative and strictly convex and satisfies

$$I_{n,l,\kappa}(y) = \nu y - \Lambda_{n,l,\kappa}(\nu) \ \text{ if } \ \Lambda'_{n,l,\kappa}(\nu) = y. \tag{6.33}$$

We will show momentarily that for sufficiently large n, $\inf_{y>0} \frac{|\log \rho_{n,\kappa}| + I_{n,l,\kappa}(y)}{y}$ is attained at some $y \in (0,\infty)$. Differentiating, we find that the minimum must occur at a y which satisfies $yI'_{n,l,\kappa}(y) - I_{n,l,\kappa}(y) = |\log \rho_{n,\kappa}|$. Let $H_{n,l,\kappa}(y) = yI'_{n,l,\kappa}(y) - I_{n,l,\kappa}(y)$. Then $H'_{n,l,\kappa}(y) = yI''_{n,l,\kappa}(y)$. Since $I_{n,l,\kappa}$ is strictly convex, we conclude that the minimum is attained at the unique root of the equation $H_{n,l,\kappa}(y) = |\log \rho_{n,\kappa}|$. Denote this root by $y^*_{n,l,\kappa}$. Recalling (6.21) and (6.25), define

$$\mu_{n,l,\kappa} = \Lambda'_{n,l,\kappa}(0) = \mu_{n,\kappa}(1 + \gamma_n(l,\kappa)). \tag{6.34}$$

Then, from (6.33), $I_{n,l,\kappa}(\mu_{n,l,\kappa}) = 0$ and it follows that

$$\frac{|\log \rho_{n,\kappa}|}{y^*_{n,l,\kappa}} \le \inf_{y>0} \frac{|\log \rho_{n,\kappa}| + I_{n,l,\kappa}(y)}{y} \le \frac{|\log \rho_{n,\kappa}|}{\mu_{n,l,\kappa}}. \tag{6.35}$$

We now return to show that the infimum is indeed attained. Using the right hand inequality in (6.35), which did not depend on the existence of an attained infimum, along with (6.34) and the assumption in the proposition that $\lim_{n\to\infty} \frac{|\log \rho_{n,\kappa}|}{\mu_{n,\kappa}} \in (0,\infty)$, we conclude that

$$\sup_{n\ge1} \inf_{y>0} \frac{|\log \rho_{n,\kappa}| + I_{n,l,\kappa}(y)}{y} < \infty. \tag{6.36}$$

By Lemma 1-ii, $\lim_{n\to\infty}\sup\{\lambda : \Lambda_{n,l,\kappa}(\lambda) < \infty\} = \infty$. Using this with (6.25) and (6.26), it follows easily that $\lim_{n\to\infty}\liminf_{y\to\infty}\frac{I_{n,l,\kappa}(y)}{y} = \infty$. From this and (6.36) we conclude that the infimum is attained.

In light of (6.35), in order to complete the proof of (6.30) we need to analyze $y^*_{n,l,\kappa} - \mu_{n,l,\kappa}$. Using (6.33), we can define $y = y_{n,l,\kappa}(\nu)$ by

$$I_{n,l,\kappa}(y_{n,l,\kappa}(\nu)) = \nu y_{n,l,\kappa}(\nu) - \Lambda_{n,l,\kappa}(\nu) \text{ and } \Lambda'_{n,l,\kappa}(\nu) = y_{n,l,\kappa}(\nu). \tag{6.37}$$

Differentiating the first equation in (6.37) and then using the second one gives

$$I'_{n,l,\kappa}(y_{n,l,\kappa}(\nu))y'_{n,l,\kappa}(\nu) = y_{n,l,\kappa}(\nu) + \nu y'_{n,l,\kappa}(\nu) - \Lambda'_{n,l,\kappa}(\nu) = \nu y'_{n,l,\kappa}(\nu). \tag{6.38}$$

Differentiating the second equation in (6.37) gives

$$y'_{n,l,\kappa}(\nu) = \Lambda''_{n,l,\kappa}(\nu). \tag{6.39}$$

Since $\Lambda_{n,l,\kappa}$ is strictly convex, $y'_{n,l,\kappa}(\nu) \neq 0$. Therefore, we conclude from (6.38) and (6.39) that

$$I'_{n,l,\kappa}(y_{n,l,\kappa}(\nu)) = \nu. \tag{6.40}$$

Differentiating (6.40) gives $I''_{n,l,\kappa}(y_{n,l,\kappa}(\nu))y'_{n,l,\kappa}(\nu) = 1$ and thus from (6.39)

$$I''_{n,l,\kappa}(y_{n,l,\kappa}(\nu)) = \frac{1}{\Lambda''_{n,l,\kappa}(\nu)}. \tag{6.41}$$

We can now analyze $y^*_{n,l,\kappa} - \mu_{n,l,\kappa}$. Define $\nu^*_{n,l,\kappa}$ by $y_{n,l,\kappa}(\nu^*_{n,l,\kappa}) = y^*_{n,l,\kappa}$ and note that $y_{n,l,\kappa}(0) = \mu_{n,l,\kappa}$. Since $H_{n,l,\kappa}(\mu_{n,l,\kappa}) = 0$ and $H_{n,l,\kappa}(y^*_{n,l,\kappa}) = |\log\rho_{n,\kappa}|$, we have

$$|\log\rho_{n,\kappa}| = \int_{\mu_{n,l,\kappa}}^{y^*_{n,l,\kappa}} H'_{n,l,\kappa}(y)dy = \int_0^{\nu^*_{n,l,\kappa}} H'_{n,l,\kappa}(y_{n,l,\kappa}(\nu))y'_{n,l,\kappa}(\nu)d\nu. \tag{6.42}$$

Recalling that $H'_{n,l,\kappa}(y) = yI''_{n,l,\kappa}(y)$ and using (6.39) and (6.41), we conclude from (6.42) that

$$|\log\rho_{n,\kappa}| = \int_0^{\nu^*_{n,l,\kappa}} y_{n,l,\kappa}(\nu)d\nu \geq \nu^*_{n,l,\kappa}\mu_{n,l,\kappa}. \tag{6.43}$$

Now (6.39) and (6.43) yield

$$y^*_{n,l,\kappa}-\mu_{n,l,\kappa}=y_{n,l,\kappa}(\nu^*_{n,l,\kappa})-y_{n,l,\kappa}(0)\le\nu^*_{n,l,\kappa}\sup_{\nu\in[0,\nu^*_{n,l,\kappa}]}y'_{n,l,\kappa}(\nu)$$
$$\le\frac{|\log\rho_{n,\kappa}|}{\mu_{n,l,\kappa}}\sup_{\nu\in[0,\frac{|\log\rho_{n,\kappa}|}{\mu_{n,l,\kappa}}]}\Lambda''_{n,l,\kappa}(\nu). \tag{6.44}$$

From (6.34) and the assumption that $\lim_{n\to\infty}\frac{|\log\rho_{n,\kappa}|}{\mu_{n,\kappa}}\in(0,\infty)$, we can choose S_κ so that

$$S_\kappa\ge\sup_{n,l\ge1}\frac{|\log\rho_{n,\kappa}|}{\mu_{n,l,\kappa}}. \tag{6.45}$$

Using (6.21) and (6.25), a direct calculation reveals that

$$\sup_{\nu\in[0,S_\kappa]}\Lambda''_{n,l,\kappa}(\nu)\le c_\kappa(L_n\epsilon_n)^4,\ \text{for some } c_\kappa\in(0,\infty)\text{ and } n\ge n_\kappa. \tag{6.46}$$

From (6.44)-(6.46), (6.34) and Lemma 1-i, we conclude that

$$\lim_{n\to\infty}\frac{y^*_{n,l,\kappa}-\mu_{n,l,\kappa}}{\mu_{n,l,\kappa}}=0. \tag{6.47}$$

From (6.34), (6.35) and (6.47), we obtain

$$\liminf_{n\to\infty}\inf_{y>0}\frac{|\log\rho_{n,\kappa}|+I_{n,l,\kappa}(y)}{y}=(1+\gamma(l,\kappa))^{-1}\lim_{n\to\infty}\frac{|\log\rho_{n,\kappa}|}{\mu_{n,\kappa}}$$
$$=(1+\gamma(l,\kappa))^{-1}\lim_{n\to\infty}\frac{|\log\rho_n|}{\mu_n},\ \text{where } \gamma(l,\kappa)=\lim_{n\to\infty}\gamma_n(l,\kappa). \tag{6.48}$$

Letting $l\to\infty$ and $\kappa\to\infty$ in (6.48) and recalling from Lemma 1-iii that $\lim_{\kappa\to\infty}\lim_{l\to\infty}\gamma(l,\kappa)=0$, we obtain (6.30). This completes the proof of Proposition 3. □

7. Proof of Lemmas 1-3. In this section, $P^{\mathcal{W}}_z$ will denote standard Wiener measure for a free (non-reflected) $d-$dimensional Brownian motion starting from $z\in R^d$.

Proof of Lemma 1. *i.* From the definitions it follows that

$$P_{m_{1,n}}(\eta_1\ge(L_n\epsilon_n)^2|\eta_1<\tau_{A_n})\ge P_{m_{2,n}}(\eta_1\ge(L_n\epsilon_n)^2). \tag{7.1}$$

Let $\tau_s=\inf\{t\ge0:|X(t)|=s\}$ and let $x_n\in R^d$ satisfy $|x_n|=L_n$. Since $dist(\partial A^R_{n,\kappa},\partial A^L_n)=(1-\kappa)L_n\epsilon_n\ge\frac12L_n\epsilon_n$, it follows from geometric considerations, Brownian scaling and (7.1) that

$$P_{m_{1,n}}(\eta_1\ge(L_n\epsilon_n)^2|\eta_1<\tau_{A_n})\ge P^{\mathcal{W}}_0(\tau_{\frac12L_n\epsilon_n}\ge(L_n\epsilon)^2)=P^{\mathcal{W}}_0(\tau_{\frac12}\ge1)\ge C_0,$$

where $C_0 > 0$ is independent of n and κ. This proves part (i).

ii. By the strong Markov property it is enough to show that

$$P_x(\eta_1 \leq (L_n\epsilon_n)^2|\eta_1 < \tau_{A_n}) \geq p_\kappa, \text{ for all } x \in \Omega - A_n, \tag{7.2}$$

for some $\rho_\kappa > 0$. We will show that for some $c, c_\kappa > 0$

$$P_x(\sigma_1 \leq \frac{1}{2}(L_n\epsilon_n)^2|\sigma_1 < \tau_{A_n}) \geq c, \text{ for all } x \in A_n^L - A_n \tag{7.3}$$

and

$$P_z(\tau_{A^R_{n,\kappa}} \leq \frac{1}{2}(L_n\epsilon_n)^2) \geq c_\kappa, \text{ for all } z \in \Omega - A^R_{n,\kappa}, \tag{7.4}$$

where $\tau_{A^R_{n,\kappa}} = \{\inf t \geq 0 : X(t) \in A^R_{n,\kappa}\}$. Letting $\rho_\kappa = cc_\kappa$, (7.2) now follows from (7.3) and (7.4) along with the strong Markov property and the definitions of σ_1 and η_1.

Consider first (7.3). Let τ_s be as in the proof of part (i). By Brownian scaling, (7.3) is equivalent to

$$P^{\mathcal{W}}_x(\tau_{L_n} \leq \frac{1}{2}L_n^2 \,|\tau_{L_n} < \tau_1) \geq c, \text{ for } 1 < |x| < L_n. \tag{7.5}$$

Let

$$h(x) = \begin{cases} \frac{|x|^{2-d}-1}{(L_n)^{2-d}-1}, \text{ if } d \geq 3; \\ \frac{\log|x|}{\log L_n}, \text{ if } d = 2. \end{cases}$$

Then $h(x) = P^{\mathcal{W}}_x(\tau_{L_n} < \tau_1)$, for $1 < |x| < L_n$, and under $P^{\mathcal{W}}_x(\cdot\,|\tau_{L_n} < \tau_1)$, $X(\cdot)$ is a diffusion generated by the h-transformed operator $\frac{1}{2}\Delta^h \equiv \frac{1}{2}\Delta + \frac{\nabla h}{h}\nabla$. Since everything is radial, the problem is one-dimensional. We will write $h(r)$ for $h(|x|)$ with $|x| = r$. The generator of the radial part of Brownian motion is $\frac{1}{2}\frac{d^2}{dr^2} + \frac{d-1}{2r}\frac{d}{dr}$, while the generator of the radial part of the h-transformed process is $\frac{1}{2}\frac{d^2}{dr^2} + \frac{d-1}{2r}\frac{d}{dr} + \frac{h'(r)}{h(r)}\frac{d}{dr}$. Since $\frac{h'(r)}{h(r)}$ is positive, it follows from the Ikeda-Watanable comparison theorem that τ_{L_n} is stochastically smaller under $P_x(\cdot\,|\tau_{L_n} < \tau_1)$ than under $P_x(\cdot)$. Consequently, (7.5) will follow if we show that

$$P^{\mathcal{W}}_x(\tau_{L_n} \leq \frac{1}{2}L_n^2) \geq c, \text{ for } 1 < |x| < L_n.$$

But this last inequality holds by Brownian scaling.

We now turn to (7.4). It follows from the spacing assumption that $\sup_{z\in\Omega-A_n} dist(z,\{x_j^{(n)}\}_{j=1}^n) \sim n^{-\frac{1}{d}}$. Thus, by (3.1), we can find a $c_0>0$ such that $dist(z,\{x_j^{(n)}\}_{j=1}^n) \le c_0 L_n\epsilon_n$, for all $z\in\Omega$. Let $j_n(z)$ be such that $dist(z, x_{j_n(z)}^{(n)}) \le c_0 L_n\epsilon_n$. We have

$$P_z(\tau_{A_{n,\kappa}^R} \le \frac{1}{2}(L_n\epsilon_n)^2) \ge P_z(X(\frac{1}{2}(L_n\epsilon_n)^2) \in B_{R_{n,\kappa}\epsilon_n}(x_{j_n(z)}^{(n)})). \tag{7.6}$$

Letting $\tau_\Omega = \inf\{t\ge 0 : X(t)\in\partial\Omega\}$, we have

$$P_z(X(\frac{1}{2}(L_n\epsilon_n)^2) \in B_{R_{n,\kappa}\epsilon_n}(x_{j_n(z)}^{(n)})) \ge P_z^{\mathcal{W}}(X(\frac{1}{2}(L_n\epsilon_n)^2) \in B_{R_{n,\kappa}\epsilon_n}(x_{j_n(z)}^{(n)})) - P_z^{\mathcal{W}}(\tau_\Omega < \frac{1}{2}(L_n\epsilon_n)^2). \tag{7.7}$$

It follows from the Gaussian density that

$$\begin{aligned} &P_z^{\mathcal{W}}(X(\frac{1}{2}(L_n\epsilon_n)^2) \in B_{R_{n,\kappa}\epsilon_n}(x_{j_n(z)}^{(n)})) \\ &= \int_{B_{R_{n,\kappa}\epsilon_n}(x_{j_n(z)}^{(n)})} (\pi(L_n\epsilon_n)^2)^{-\frac{d}{2}} \exp(-\frac{|y-z|^2}{(L_n\epsilon_n)^2})dy \ge \\ &(\pi(L_n\epsilon_n)^2)^{-\frac{d}{2}} \exp(-\frac{((c_0+\kappa)L_n\epsilon_n)^2}{(L_n\epsilon_n)^2})(R_{n,\kappa}\epsilon_n)^d\omega_d \\ &= \pi^{-\frac{d}{2}}\kappa^d\omega_d \exp(-(c_0+\kappa)^2). \end{aligned} \tag{7.8}$$

As is well known ([9, Theorem 2.2.2-ii])

$$P_z^{\mathcal{W}}(\sup_{s\in[0,t]}(|X(s)-z|>\lambda) \le c_1\exp(-c_2\frac{\lambda^2}{t}), \tag{7.9}$$

for constants $c_1,c_2>0$ independent of t and λ. Thus, for $l>0$,

$$P_z^{\mathcal{W}}(\tau_\Omega < \frac{1}{2}(L_n\epsilon_n)^2) \le c_1\exp(-2c_2l^2), \text{ for all } z\in\Omega \text{ satisfying } dist(z,\partial\Omega) > lL_n\epsilon_n. \tag{7.10}$$

From (7.7), (7.8) and (7.10), it follows that for sufficiently large l there exists a constant $c_{l,\kappa}>0$ such that

$$P_z(X(\frac{1}{2}(L_n\epsilon_n)^2) \in B_{R_{n,\kappa}\epsilon_n}(x_{j_n(z)}^{(n)})) \ge c_{l,\kappa}, \text{ for all } n \text{ and all } z\in\Omega-A_{n,\kappa}^R \text{ which satisfy } dist(z,\Omega) > lL_n\epsilon_n. \tag{7.11}$$

We now show that there exists a $C_{l,\kappa} > 0$ such that

$$P_z(X(\frac{1}{2}(L_n\epsilon_n)^2) \in B_{R_{n,\kappa}\epsilon_n}(x^{(n)}_{j_n(z)})) \geq C_{l,\kappa}, \text{for all } n \text{ and all } z \in \Omega - A^R_{n,\kappa} \text{ which satisfy } dist(z,\Omega) \leq lL_n\epsilon_n. \tag{7.12}$$

This will complete the proof since (7.4) follows from (7.6), (7.11) and (7.12). Clearly it suffices to prove (7.12) for sufficiently large n (depending on l). Let $\{U_k\}_{k=1}^K$ be open sets in R^d satisfying $\partial\Omega \subset \cup_{k=1}^K U_k$. For each k, let $(\phi_1^{(k)}, ..., \phi_{d-1}^{(k)})$ be a diffeomorphism from $U_k \cap \partial\Omega$ to R^{d-1}. For $r > 0$, let $\Omega_r = \{x \in \Omega : dist(x, \partial\Omega) < r\}$. Define $\phi_d^{(k)}(x) = dist(x, \partial\Omega)$ and let $\phi^{(k)} = (\phi_1^{(k)}, ..., \phi_d^{(k)})$. Let $R^d_+ = \{x \in R^d : x_d > 0\}$. By the compactness of $\partial\Omega$, it follows that for some $r_0 > 0$, each of the maps $\phi^{(k)} : U_k \cap \bar{\Omega}_{r_0} \to \bar{R}^d_+$ is a diffeomorphism which takes $U_k \cap \partial\Omega$ into ∂R^d_+. It follows from the construction that

$$\nabla\phi_i^{(k)} \cdot N = 0 \text{ on } U_k \cap \partial\Omega, \text{ for } i = 1, ..., d-1, \tag{7.13}$$

where N is the unit outward normal to Ω at $\partial\Omega$.

Let f be a smooth function defined on $\bar{R}^d_+$ which satisfies $\frac{\partial f}{\partial x_d} = 0$ on ∂R^d_+. Then it follows from (7.13) that the function $f(\phi^{(k)})$ satisfies $\nabla(f(\phi^{(k)})) \cdot N = 0$ on $U_k \cap \partial\Omega$. Thus, from the martingale approach to reflected diffusions [13], we conclude that $f(\phi^{(k)}(X(t))) - \int_0^t \frac{1}{2}\Delta(f(\phi^{(k)}))(X(s))ds$ is a local martingale up until the time that $X(t)$ exits $U_k \cap \bar{\Omega}$. A direct calculation shows that $\Delta(f(\phi^{(k)})) = L^{(k)}f$, where $L^{(k)} = \sum_{i,j=1}^d a_{i,j}^{(k)} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i^{(k)} \frac{\partial}{\partial x_i}$ with $a_{i,j}^{(k)} = \nabla\phi_i^{(k)}\nabla\phi_j^{(k)}$ and $b_i^{(k)} = \Delta\phi_i^{(k)}$. Thus, letting $Y^{(k)}(t) = \phi^{(k)}(X(t))$, it follows that $f(Y^{(k)}(t)) - \int_0^t (L^{(k)}f)(Y^{(k)}(s))ds$ is a local martingale until $Y^{(k)}(t)$ exits $\bar{V}_k$, where $V_k = \phi^{(k)}(U_k) \cap R^d_+$. From the martingale characterization of reflected diffusions, we conclude that $Y^{(k)}(t)$ is a normally reflected $L^{(k)}$-diffusion until exiting $\bar{V}_k$. Now let $W_k = (\bar{V}_k \cup (-V_k))^{int}$ and let $\hat{L}^{(k)}$ be the extension of L to W_k obtained by continuing the coefficients symmetrically about ∂R^d_+. Let $\hat{Y}^{(k)}(t)$ denote the non-reflected diffusion process in W_k corresponding to $\hat{L}^{(k)}$. We will denote probabilities for the $Y^{(k)}$ process by $Q^{(k)}_\cdot$ and for the $\hat{Y}^{(k)}$ process by $\hat{Q}^{(k)}_\cdot$.

Clearly there exists a $c > 0$ such that for any ball $\mathcal{B} \subset U_k \cap \Omega$, there exists a ball $B \subset \phi^{(k)}(\mathcal{B}) \subset V_k$ satisfying $|B| \geq c|\mathcal{B}|$. Let τ_k be the exit time of $Y^{(k)}$ from

$\bar{V}_k$ and let $\hat{\tau}_k$ be the exit time of $\hat{Y}^{(k)}$ from W_k. Using symmetry considerations for the equality below, we have for $x \in U_k \cap \Omega$ and $y = \phi(x) \in V_k$,

$$
\begin{aligned}
P_x(X(t) \in \mathcal{B}) &\geq Q_y^{(k)}(Y^{(k)}(t \wedge \tau_k) \in B) = \hat{Q}_y^{(k)}(\hat{Y}^{(k)}(t \wedge \hat{\tau}_k) \in B \cup (-B)) \\
&\geq \hat{Q}_y^{(k)}(\hat{Y}^{(k)}(t \wedge \hat{\tau}_k) \in B).
\end{aligned} \tag{7.14}
$$

We use (7.14) to prove (7.12). Let $\mathcal{B} = B_{R_{n,\kappa}\epsilon_n}(x^{(n)}_{j_n(z)})$ and $t = \frac{1}{2}(L_n\epsilon_n)^2$. Since $|B| \geq c|\mathcal{B}|$, we have $|B| \geq C(R_{n,\kappa}\epsilon_n)^d$, for some $C > 0$. The diffusion measures $Q_\cdot^{(k)}$ correspond to uniformly elliptic operators $L^{(k)}$; thus, their transition kernels are bounded from above and below for small time by Gaussian kernels. In particular then, inequalities of the type in (7.8) and (7.9) hold for the process $\hat{Y}^{(k)}$. Using this along with the fact that $\hat{\tau}_k$ is independent of n whereas $\lim_{n\to\infty} \frac{1}{2}(L_n\epsilon_n)^2 = 0$ now shows that for sufficiently large n the right hand side of (7.14) is bounded away from 0 by a constant $C_{l,\kappa}$.

iii. By the definitions, we have

$$
\begin{cases}
\mu_{n,\kappa,x} = E_x(\sigma_1|\sigma_1 < \tau_{A_n}) + E_x E_{X(\sigma_1)}(\eta_1|\sigma_1 < \tau_{A_n}), \text{ for } x \in \partial A^R_{n,\kappa}, \\
\mu_{n,\kappa} = E_{m_{1,n}}(\sigma_1|\sigma_1 < \tau_{A_n}) + E_{m_{2,n}}\eta_1.
\end{cases} \tag{7.16}
$$

By symmetry,

$$
E_x(\sigma_1|\sigma_1 < \tau_{A_n}) = E_{m_{1,n}}(\sigma_1|\sigma_1 < \tau_{A_n}), \quad \text{for all } x \in \partial A^R_{n,\kappa}. \tag{7.17}
$$

Thus, to prove part (iii), we need to compare $E_x E_{X(\sigma_1)}(\eta_1|\sigma_1 < \tau_{A_n})$ with $E_{m_{2,n}}\eta_1$. We introduce the following notation. For $z \in A^L_n$, there exists a unique index $j_n(z)$ such that $z \in B_{L_n\epsilon_n}(x^{(n)}_{j_n(z)})$. Let $U^{(n)}_j$ denote the uniform probability measure on $\partial B_{L_n\epsilon_n}(x^{(n)}_j)$. We will prove the following two estimates:

$$
\begin{aligned}
&E_x E_{X(\sigma_1)}(\eta_1|\sigma_1 < \tau_{A_n}) \leq (1 + \nu_n(\kappa)) E_{U^{(n)}_{j_n(x)}}\eta_1, \text{ for all } x \in \partial A^R_{n,\kappa}, \\
&\text{where } \lim_{\kappa\to 0}\lim_{n\to\infty} \nu_n(\kappa) = 0,
\end{aligned} \tag{7.18}
$$

and

(7.19)
$$
\begin{aligned}
&E_{U^{(n)}_{j_n(x)}}\eta_1 \leq (1 + b_n(l,\kappa)) E_{m_{2,n}}\eta_1, \text{ for } x \in \partial A^R_{n,\kappa} \text{ satisfying } dist(x, \partial\Omega) > lL_n\epsilon_n, \\
&\text{where } \lim_{l\to\infty}\lim_{n\to\infty} b_n(l,\kappa) = 0.
\end{aligned}
$$

Part (iii) now follows from (7.16)-(7.19).

We first prove (7.18). For $z \in A_n^L$, define $\theta(z)$ to be the angle made by the segment from $x^{(n)}_{j_n(z)}$ to z; that is,

$$\theta(z) = \frac{z - x^{(n)}_{j_n(z)}}{|z - x^{(n)}_{j_n(z)}|} \in S^{d-1}.$$

Recall the definition of τ_s after (7.1). By Brownian scaling, $P_x(\theta(X(\sigma_1)) \in \cdot|\sigma_1 < \tau_{A_n})$, for $x \in \partial A^R_{n,\kappa}$, is distributed like the distribution of the angular part of a Brownian motion at time $\tau_{\frac{1}{\kappa}}$ starting from the point on the unit circle with angular part $\theta(x)$, and conditioned on $\tau_{\frac{1}{\kappa}} < \tau_{\frac{1}{L_n\kappa}}$. Using the skew product representation for the angular part of a Brownian motion along with the fact that the density of the distribution of $\frac{X(t)}{|X(t)|}$ on S^{d-1} under $P^{\mathcal{W}}_{\cdot}$ converges pointwise and uniformly to the uniform density as $t \to \infty$, it follows that $P_x(X(\sigma_1) \in \cdot) = (1 - \hat{\nu}_n(\kappa))U^{(n)}_{j_n(x)} + \hat{\nu}_n(\kappa)V$, where V is some distribution on $\partial B_{L_n\epsilon_n}(x^{(n)}_{j_n(x)})$ (whose dependence on the parameters we suppress) and where $\lim_{\kappa\to 0}\lim_{n\to\infty}\hat{\nu}_n(\kappa) = 0$. Now (7.18) follows from this along with the fact that

(7.20)
$$\frac{1}{C}E_{z_1}\eta_1 \le E_{z_2}\eta_1 \le CE_{z_1}\eta_1, \text{ for all } z_1, z_2 \in \partial A_n^L, \text{ where } C \text{ is independent of } n \text{ and } \kappa.$$

To prove (7.20), let $\tau_{A_n^L} = \inf\{t \ge 0 : X(t) \in \partial A_n^L\}$. Recalling that $\tau_{A^R_{n,\kappa}} = \inf\{t \ge 0 : X(t) \in \partial A^R_{n,\kappa}\}$ and that $\kappa \in (0, \frac{1}{2}]$, the strong Markov property gives $E_z\eta_1 = E_z\tau_{A^R_{n,\frac{1}{2}}} + E_xE_{X(\tau_{A^R_{n,\frac{1}{2}}})}\tau_{A^R_{n,\kappa}}$, for $z \in \partial A_n^L$. Thus, since $P_x(\tau_{A^R_{n,\kappa}} < \tau_{A_n^L})$ is the same for all $x \in \partial A^R_{n,\frac{1}{2}}$, another application of the strong Markov property shows that in order to prove (7.20) it is enough to show that

$$\text{(7.21)} \qquad \frac{1}{C}E_{z_1}\tau_{A^R_{n,\frac{1}{2}}} \le E_{z_2}\tau_{A^R_{n,\frac{1}{2}}} \le CE_{z_1}\tau_{A^R_{n,\frac{1}{2}}}, \text{ for all } z_1, z_2 \in \partial A_n^L,$$
where C is independent of n.

Now, on the one hand, (7.4) with $\kappa = \frac{1}{2}$ shows that $\frac{\tau_{A^R_{n,\frac{1}{2}}}}{(L_n\epsilon_n)^2}$ under P_z is stochastically dominated by a geometric random variable with parameter independent of n and independent of $z \in \partial A_n^L$. On the other hand, since $dist(z, \partial A^R_{n,\frac{1}{2}}) = \frac{1}{2}L_n\epsilon_n$, for

$z \in \partial A_n^L$, it follows from Brownian scaling that there exists $c > 0$ independent of n and $z \in \partial A_n^L$ such that $P_z(\tau_{A^R_{n,\frac{1}{2}}} \geq (L_n\epsilon_n)^2) \geq c$. Now (7.21) follows from these facts.

We now turn to the proof of (7.19). Recall that a lattice point contained in Ω is called a boundary lattice point if one of its nearest neighbor lattice points is not in Ω, and recall that boundary lattice points were excluded in the construction of the holes. Since the lattice spacing is on the order $n^{-\frac{1}{d}}$, and since $L_n\epsilon_n \sim n^{-\frac{1}{d}}$ by (3.1), it follows that there exists a $c_0 > 0$ such that for any $M > 0$, the sets $B_{ML_n\epsilon_n}(x_j^{(n)}) \cap (\Omega - A_n)$ with j satisfying $dist(x_j^{(n)}, \partial\Omega) > (M + c_0)L_n\epsilon_n$, look identical—that is, they are all translations of one another. Trivially, we have $dist(x_j^{(n)}, \partial\Omega) \geq dist(z, \partial\Omega) - L_n\epsilon_n$, for $z \in \partial B_{L_n\epsilon_n}(x_j^{(n)})$. In light of the above facts, the following assertion holds:

(7.22)
$$E_z(\eta_1; \eta_1 \leq M(L_n\epsilon_n)^2, \sup_{0\leq s\leq\eta_1} dist(z, X(s)) \leq ML_n\epsilon_n)$$
only depends on $\theta(z)$, for $z \in \partial A_n^L$ satisfying $dist(z, \partial\Omega) > (M + c_0 + 1)(L_n\epsilon_n)$.

From (7.9) we have

$$P_z(\sup_{0\leq s\leq M(L_n\epsilon_n)^2} dist(z, X(s)) > ML_n\epsilon_n) \leq c_1 \exp(-c_2M), \quad \text{for } z \in \Omega \text{ satisfying } dist(z, \partial\Omega) > ML_n\epsilon_n, \tag{7.23}$$

and from (7.23) we have

$$\begin{aligned} &E_z(\eta_1; \eta_1 \leq M(L_n\epsilon_n)^2, \sup_{0\leq s\leq\eta_1} dist(z, X(s)) > ML_n\epsilon_n) \\ &\leq M(L_n\epsilon_n)^2 c_1 \exp(-c_2M), \text{ for } z \in \partial A_n^L \text{ satisfying } dist(z, \partial\Omega) > ML_n\epsilon_n. \end{aligned} \tag{7.24}$$

By part (ii), $\frac{\eta_1}{(L_n\epsilon_n)^2}$ under $P_z(\cdot\ )$ with $z \in \partial A_n^L$ is dominated stochastically by a geometric random variable with parameter p_κ; thus,

(7.25)
$$E_z(\eta_1; \eta_1 > M(L_n\epsilon_n)^2) \leq (L_n\epsilon_n)^2 \sum_{k=M}^{\infty} kp_\kappa(1 - p_\kappa)^k \equiv (L_n\epsilon_n)^2 c_{\kappa,M}, \text{ for } z \in \partial A_n^L.$$

From (7.24) and (7.25) we conclude that

(7.26)
$$E_z\eta_1 - E_z(\eta_1; \eta_1 \leq M(L_n\epsilon_n)^2, \sup_{0\leq s\leq\eta_1} dist(z, X(s)) \leq ML_n\epsilon_n) \leq (L_n\epsilon_n)^2 a_{\kappa,M},$$
for $z \in \partial A_n^L$ satisfying $dist(z, \partial\Omega) > ML_n\epsilon_n$, where $\lim_{M\to\infty} a_{\kappa,M} = 0$.

Since $dist(z,\partial\Omega) > (M+c_0+1)L_n\epsilon_n$ if $dist(x_j^{(n)},\partial\Omega) > (M+c_0+2)L_n\epsilon_n$ and $j_n(z) = j$,, it follows from (7.22) and (7.26) that

$$\begin{aligned}&|E_{U_i^{(n)}}\eta_1 - E_{U_j^{(n)}}\eta_1| \le (L_n\epsilon_n)^2 a_{\kappa,M},\\ &\text{if } dist(x_i^{(n)},\partial\Omega), dist(x_j^{(n)},\partial\Omega) > (M+c_0+2)L_n\epsilon_n.\end{aligned} \tag{7.27}$$

If $x \in \partial A^R_{n,\kappa}$ satisfies $dist(x,\partial\Omega) > (M+c_0+3)L_n\epsilon_n$, then $dist(x^{(n)}_{j_n(x)},\partial\Omega) > (M+c_0+2)L_n\epsilon_n$. Thus, it follows from (7.27) that

(7.28)

$$\begin{aligned}&|E_{U^{(n)}_{j_n(x)}}\eta_1 - E_{U_j^{(n)}}\eta_1| \le (L_n\epsilon_n)^2 a_{\kappa,M}, \text{ if } dist(x_j^{(n)},\partial\Omega) > (M+c_0+2)L_n\epsilon_n\\ &\text{and if } x \in \partial A^R_{n,\kappa} \text{ satisfies } dist(x,\partial\Omega) > (M+c_0+3)L_n\epsilon_n.\end{aligned}$$

Let

$$J_{n,l} = \{j \in \{1,...,n\} : dist(x_j^{(n)},\partial\Omega) > lL_n\epsilon_n\}.$$

From the fact that $\lim_{n\to\infty} L_n\epsilon_n = 0$, it is clear that

$$\lim_{n\to\infty}\frac{|J_{n,l}|}{n} = 1, \text{ for all } l. \tag{7.29}$$

By Theorem 2-i, $m_{2,n}$ is distributed uniformly on ∂A^L_n. (Indeed, the measures in (1.9) are uniform if the $\{R_j\}_{j=1}^{N_0}$ and $\{L_j\}_{j=1}^{N_0}$ appearing there don't depend on j, and in our context N_0 is equal to n and R_j and L_j are equal to $R_{n,\kappa}\epsilon_n$ and $L_n\epsilon_n$ respectively for all j.) Thus $m_{2,n} = \frac{1}{n}\sum_{j=1}^n U_{n,j}$. Using this with (7.28) and (7.29), we have

$$\begin{aligned}&E_{m_{2,n}}\eta_1 \ge \frac{|J_{n,M+c_0+2}|}{n}(E_{U^{(n)}_{j_n(x)}}\eta_1 - (L_n\epsilon_n)^2 a_{\kappa,M}), \text{ if } x \in \partial A^R_{n,\kappa} \text{ satisfies}\\ &dist(x,\partial\Omega) > (M+c_0+3)L_n\epsilon_n.\end{aligned} \tag{7.30}$$

Since $dist(z,\partial A^R_{n,\kappa}) = (L_n - R_{n,\kappa})\epsilon_n = (1-\kappa)L_n\epsilon_n \ge \frac{1}{2}L_n\epsilon_n$, for $z \in \partial A^L_n$, it follows from Brownian scaling that there exists a $c > 0$, independent of κ, such that $P_z(\eta_1 \ge (L_n\epsilon_n)^2) \ge c$, for all n and all $z \in \partial A^L_n$. Thus,

$$E_{m_{2,n}}\eta_1 \ge c(L_n\epsilon_n)^2. \tag{7.31}$$

From (7.30) and (7.31) we conclude that

$$\begin{aligned}&E_{U^{(n)}_{j_n(x)}} \le (\frac{n}{J_{n,M+c_0+2}} + \frac{a_{\kappa,M}}{c})E_{m_{2,n}}\eta_1,\\ &\text{for } x \in \partial A^R_{n,\kappa} \text{ satisfying } dist(x,\partial\Omega) > (M+c_0+3)L_n\epsilon_n.\end{aligned} \tag{7.32}$$

Recalling from (7.26) that $\lim_{M\to\infty} a_{\kappa,M} = 0$, (7.19) now follows from (7.29) and (7.32).

$\square$

Proof of Lemma 2. The distribution of τ_{A_n} under $P_x(\cdot\, |\tau_{A_n} < \eta_1)$ with $x \in \partial A^R_{n,\kappa}$ is the distribution of the hitting time of $\partial B_{\epsilon_n}(0)$ by a d-dimensional Brownian motion starting from a point on $\partial B_{R_{n,\kappa}\epsilon_n}(0)$ and conditioned to hit $\partial B_{\epsilon_n}(0)$ before hitting $\partial B_{L_n\epsilon_n}(0)$. This conditioned Brownian motion corresponds to the h-transformed infinitesimal generator $\frac{1}{2}\Delta^{h_n}$ defined by $\frac{1}{2}\Delta^{h_n} f = \frac{1}{2}\frac{1}{h_n}\Delta(fh_n)$, where

$$h_n(x) = \begin{cases} \frac{|x|^{2-d}-(L_n\epsilon_n)^{2-d}}{(\epsilon_n)^{2-d}-(L_n\epsilon_n)^{2-d}}, & \text{if } d \geq 3 \\ \frac{\log|x|-\log L_n\epsilon_n}{\log\epsilon_n-\log L_n\epsilon_n}, & \text{if } d = 2 \end{cases}$$

is the probability that a Brownian motion starting from x with $|x| \in [\epsilon_n, L_n\epsilon_n]$ will hit $\partial B_{\epsilon_n}(0)$ before hitting $\partial B_{L_n\epsilon_n}(0)$. Similar to (1.2), we have $P_x(\tau_{A_n} > t\, |\tau_{A_n} < \eta_1) \leq \exp(-c_n t)$ for large t and for c_n less than the principal eigenvalue for $-\frac{1}{2}\Delta^h$ on $B_{L_n\epsilon_n}(0) - \bar{B}_{\epsilon_n}(0)$ with the Dirichlet boundary condition on $\partial B_{L_n\epsilon_n}(0)$. The principal eigenvalue is invariant under h-transforms [9, Theorem 4.3.3-iv]; thus, the principal eigenvalue mentioned above coincides with that of $-\frac{1}{2}\Delta$ on $B_{L_n\epsilon_n}(0) - \bar{B}_{\epsilon_n}(0)$ with the Dirichlet boundary condition on the entire boundary. Since the volume of the annular domain $B_{L_n\epsilon_n}(0) - \bar{B}_{\epsilon_n}(0)$ shrinks to 0 as $n \to \infty$, the principal eigenvalue approaches ∞ as $n \to \infty$.

$\square$

Proof of Lemma 3. We will assume that the statement in the lemma is not true and come to a contradiction. If the lemma is not true, then we may assume without loss of generality that there exist $q_0, l_0, \kappa_0, c_0 > 0$ and a k_n such that

(7.33)
$$P_{m_{1,n}}\Big(\frac{1}{k}\sum_{j=0}^{k-1} 1_{D_{n,l_0}}(X(\eta_j)) \geq q_0 |\eta_k < \tau_{A_n} < \eta_{k+1}\Big) \geq \exp(-c_0\mu_{n,\kappa_0}k), \text{ for } \ k \geq k_n.$$

(We say "without loss of generality" because the precise negation of the lemma is that for infinitely many n, the inequality in (7.33) holds for infinitely many k.) We will use (7.33) to show that there exist $q_1, c_1 > 0$ such that

$$\liminf_{t\to\infty} \frac{1}{t} \log P_{m_{1,n}}\Big(\frac{1}{t}\int_0^t 1_{D_{n,l_0+\frac{3}{2}}}(X(s))ds \geq q_1\Big) \geq -c_1, \text{ for all } n. \tag{7.34}$$

Before proving that (7.34) follows from (7.33), we show how (7.34) leads to a contradiction. Let $L_t(\cdot) = \frac{1}{t}\int_0^t 1_{\{\cdot\}}(X(s))ds$ denote the occupation measure for the path $X(\cdot)$ up to time t. Let $\mathcal{P}(\bar{\Omega})$ denote the space of probability measures on $\bar{\Omega}$ and let $C_n = \{\nu \in \mathcal{P}(\bar{\Omega}) : \nu(D_{n,l_0+\frac{3}{2}}) \geq q_1\}$. Then C_n is compact in the weak topology on $\mathcal{P}(\bar{\Omega})$ since it is closed and $\bar{\Omega}$ is compact. Thus, by the Donsker Varadhan large deviations theory [3, Theorems 1 and 5],

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t} \log P_{m_{1,n}}\Big(\frac{1}{t}\int_0^t 1_{D_{n,l_0+\frac{3}{2}}}(X(s))ds \geq q_1\Big) \\ &= \limsup_{t\to\infty} \frac{1}{t} \log P_{m_{1,n}}(L_t \in C_n) \leq -\inf_{\nu\in C_n} I(\nu), \end{aligned} \tag{7.35}$$

where the $I-$function is given by

(7.36)
$$I(\nu) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla f|^2(x)dx, & \text{if } (\frac{d\nu}{dx})^{\frac{1}{2}} = f; \\ \infty, & \text{if } \nu \text{ is not absolutely continuous w.r.t. Lebesgue measure on } \bar{\Omega}. \end{cases}$$

From (7.34) and (7.35), it follows that for each n there exists a $\nu_n \in C_n$ such that $I(\nu_n) \leq c_1$. Since $\mathcal{P}(\bar{\Omega})$ is compact, there exists a subsequence of $\{\nu_n\}$ which converges to some $\nu_0 \in \mathcal{P}(\bar{\Omega})$. It is easy to see from the alternative variational definition of I [3, 1.12] that it is lower semicontinuous; thus,

$$I(\nu_0) \leq c_1. \tag{7.37}$$

Note that the nested sequence $\{D_{n,l_0+\frac{3}{2}}\}_{n=1}^\infty$ of closed sets satisfies $\cap_{n=1}^\infty D_{n,l_0+\frac{3}{2}} = \partial\Omega$. It then follows easily from weak convergence that $\nu_0(\partial\Omega) \geq q_1$, which shows that ν_0 is not absolutely continuous with respect to Lebesgue measure. This fact in conjunction with (7.36) and (7.37) gives a contradiction.

We now return to prove that (7.34) follows from (7.33). Let $\delta > 0$. For any

$q > 0$, we have from (4.3)

(7.38)
$$P_{m_{1,n}}(\frac{1}{t}\int_0^t 1_{D_{n,l_0+\frac{3}{2}}}(X(s))ds \geq q)$$
$$\geq P_{m_{1,n}}(\frac{1}{t}\int_0^t 1_{D_{n,l_0+\frac{3}{2}}}(X(s))ds \geq q \mid \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n})P_{m_{1,n}}(\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n})$$
$$= \rho_{n,\kappa_0}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]} P_{m_{1,n}}(\frac{1}{t}\int_0^t 1_{D_{n,l_0+\frac{3}{2}}}(X(s))ds \geq q \mid \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n}).$$

From the geometry and from the fact that $\kappa_0 \in (0, \frac{1}{2}]$, we have

$$1_{D_{n,l_0}}(X(\eta_{j-1}))\int_{\eta_{j-1}}^{\sigma_j} 1_{D_{n,l_0+\frac{3}{2}}}(X(s))ds = 1_{D_{n,l_0}}(X(\eta_{j-1}))(\sigma_j - \eta_{j-1}). \tag{7.39}$$

Recall the definition of $B_{n,l,q,k}$ in (6.12). Defining $\eta_0 = 0$ for convenience and using (7.39), we have

(7.40)
$$P_{m_{1,n}}(\frac{1}{t}\int_0^t 1_{D_{n,l_0+\frac{3}{2}}}(X(s))ds \geq q \mid \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n})$$
$$\geq P_{m_{1,n}}(\frac{1}{t}\int_0^{\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]}} 1_{D_{n,l_0+\frac{3}{2}}}(X(s))ds \geq q,\ \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} \leq t, B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]} | \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n})$$
$$\geq P_{m_{1,n}}(\frac{1}{t}\sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}\int_{\eta_{j-1}}^{\sigma_j} 1_{D_{n,l_0+\frac{3}{2}}}(X(s))ds \geq q,\ \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} \leq t, B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]} | \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n})$$
$$\geq P_{m_{1,n}}(\frac{1}{t}\sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}(\sigma_j - \eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1})) \geq q,\ \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} \leq t, B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]} |$$
$$\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n}).$$

Applying the universal inequality
$$P_{m_{1,n}}(E_1 \cap E_2 | \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n})$$
$$\geq P_{m_{1,n}}(E_1 | \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n}) + P_{m_{1,n}}(E_2 | \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n}) - 1$$

to the events $E_1 = B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]} \cap \{\frac{1}{t}\sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}(\sigma_j - \eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1})) \geq q\}$ and $E_2 = \{\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} \leq t\}$, we have

(7.41)
$$P_{m_{1,n}}(\frac{1}{t}\sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}(\sigma_j - \eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1})) \geq q,\ \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} \leq t, B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]} | \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n})$$
$$\geq P_{m_{1,n}}(B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]}, \frac{1}{t}\sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}(\sigma_j - \eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1})) \geq q | \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n})$$
$$+ P_{m_{1,n}}(\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} \leq t | \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n}) - 1.$$

For the second probability on the right hand side of (7.41) we use Chebyshev's inequality and (6.9) to obtain the estimate

$$P_{m_{1,n}}(\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} \le t|\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n}) = 1 - P_{m_{1,n}}(\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} > t|\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n})$$

$$\ge 1 - \exp(-\lambda t)E_{m_{1,n}}(\exp(\lambda \sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]} (\eta_j - \eta_{j-1}))|\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n}) \tag{7.42}$$

$$\ge 1 - \exp(-\lambda t(1 - C_{\lambda,\kappa_0}[\frac{\delta t}{\mu_{n,\kappa_0}}]\frac{\mu_{n,\kappa_0}}{t})), \text{ for } \lambda > 0 \text{ and } n \ge n_\lambda.$$

For the first probability on he right hand side of (7.41), we use (7.33) along with the fact that

$$P_{m_{1,n}}(\frac{1}{k}\sum_{j=0}^{k-1} 1_{D_{n,l_0}}(X(\eta_j)) \ge q_0|\eta_k < \tau_{A_n} < \eta_{k+1})$$

$$= P_{m_{1,n}}(\frac{1}{k}\sum_{j=0}^{k-1} 1_{D_{n,l_0}}(X(\eta_j)) \ge q_0|\eta_k < \tau_{A_n})$$

to obtain the estimate

(7.43)

$$P_{m_{1,n}}(B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]}, \frac{1}{t}\sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]} (\sigma_j - \eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1})) \ge q|\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n})$$

$$= P_{m_{1,n}}(\frac{1}{t}\sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]} (\sigma_j - \eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1})) \ge q|B_{n,l_0,q_0,\ [\frac{\delta t}{\mu_{n,\kappa_0}}]}, \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n})\times$$

$$P_{m_{1,n}}(B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]}|\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n}) \ge \exp(-c_0\mu_{n,\kappa_0}[\frac{\delta t}{\mu_{n,\kappa_0}}])\times$$

$$P_{m_{1,n}}(\frac{1}{t}\sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]} (\sigma_j - \eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1})) \ge q|B_{n,l_0,q_0,\ [\frac{\delta t}{\mu_{n,\kappa_0}}]}, \eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]} < \tau_{A_n}).$$

Let

$$\hat{B}_{n,l,q,k} = \{\{x_j\}_{j=0}^{k-1} : \sum_{j=0}^{k-1} 1_{D_{n,l}}(x_j) \ge q\}.$$

Then we can write the probability on the right hand side of (7.43) as

(7.44)

$$P_{m_{1,n}}\Big(\frac{1}{t}\sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}(\sigma_j-\eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1}))\ge q|B_{n,l_0,q_0,\,[\frac{\delta t}{\mu_{n,\kappa_0}}]},\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]}<\tau_{A_n}\Big)$$

$$=\int_{\hat B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]}}P_{m_{1,n}}\Big(\frac{1}{t}\sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}(\sigma_j-\eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1}))\ge q|$$

$$B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]},\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]}<\tau_{A_n},X(\eta_{j-1})=x_{j-1},j=1,...,[\frac{\delta t}{\mu_{n,\kappa_0}}]\Big)d\Gamma,$$

where Γ is a probability measure on $\hat B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]}$. ($\Gamma$ depends on all the parameters, but we suppress this dependence.) Note that it is the conditioning on the event $B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]}$ that causes Γ to be supported on $\hat B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]}$. Once we've conditioned on the locations $\{X(\eta_{j-1})\}_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}$, the random variables $\{(\sigma_j-\eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1}))\}_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}$ become independent regardless of the rest of the conditioning. Since $\{x_{j-1}\}_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}\in\hat B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]}$, we have $(\sigma_j-\eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1}))$ $=\sigma_j-\eta_{j-1}$ for at least $[q_0[\frac{\delta t}{\mu_{n,\kappa_0}}]]$ of the indices j between 1 and $[\frac{\delta t}{\mu_{n,\kappa_0}}]$, almost surely. From Brownian scaling and the definitions, it follows that under the above conditional probability the random variable $\frac{\sigma_j-\eta_{j-1}}{(L_n\epsilon_n)^2}$ is distributed as follows. For $x\in R^d$ with $|x|=1$, let $Q_x^{(n)}$ denote the distribution of the exit time from $B_1(0)-\bar B_{\frac{1}{L_n}}(0)$ of a Brownian motion starting from $\kappa_0(1,0,...,0)$ and conditioned to exit the above region at x. Then $\frac{\sigma_j-\eta_{j-1}}{(L_n\epsilon_n)^2}$ is distributed like $\int_{\partial B_1(0)}Q_x^{(n)}m_j(dx)$, where $m_j(dx)$ is a probability measure on $\partial B_1(0)$ whose dependence on $\{x_{i-1}\}_{i=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}$ has been suppressed. Clearly we can find a distribution Q which is supported on $(0,\infty)$ and which is stochastically dominated by $Q_x^{(n)}$ for all $x\in\partial B_1(0)$ and all large n. Thus, letting $\{W_j\}_{j=1}^\infty$ be an IID Q-distributed sequence of random variables on some probability space with probability measure $\mathcal{P}$, and letting $S_k=\sum_{j=1}^k W_j$, it follows from the above discussion that

$$
\begin{aligned}
&P_{m_{1,n}}(\frac{1}{t}\sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}(\sigma_j-\eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1}))\ge q| \\
&B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]},\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]}<\tau_{A_n},X(\eta_{j-1})=x_{j-1},j=1,...,[\frac{\delta t}{\mu_{n,\kappa_0}}]) \\
&\ge\mathcal{P}(\frac{S_{[q_0[\frac{\delta t}{\mu_{n,\kappa_0}}]]}}{t}\ge\frac{q}{(L_n\epsilon_n)^2}),\ \text{for } \{x_{j-1}\}_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}\in\hat{B}_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]}.
\end{aligned}
\tag{7.45}
$$

By Lemma 1-ii, $\frac{q}{(L_n\epsilon_n)^2}\le\frac{q(1-p_{\kappa_0})}{p_{\kappa_0}\mu_{n,\kappa_0}}$ and by the strong law of large numbers

$$
\lim_{t\to\infty}\frac{S_{[q_0[\frac{\delta t}{\mu_{n,\kappa_0}}]]}}{t}=\frac{q_0\delta\mathcal{E}W_1}{\mu_{n,\kappa_0}},\ \text{a.s. }\mathcal{P}.
$$

Thus, choosing $q_1=\frac{q_0\delta p_{\kappa_0}\mathcal{E}W_1}{2(1-p_{\kappa_0})}$, if follows from (7.45) that there exists a t_0 such that

$$
\begin{aligned}
&P_{m_{1,n}}(\frac{1}{t}\sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}(\sigma_j-\eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1}))\ge q_1| \\
&B_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]},\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]}<\tau_{A_n},X(\eta_{j-1})=x_{j-1},j=1,...,[\frac{\delta t}{\mu_{n,\kappa_0}}])\ge\frac{1}{2}, \\
&\text{for } \{x_{j-1}\}_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}\in\hat{B}_{n,l_0,q_0,[\frac{\delta t}{\mu_{n,\kappa_0}}]},\ \text{if } t\ge t_0.
\end{aligned}
\tag{7.46}
$$

From (7.44) and (7.46) we conclude that

(7.47)

$$
P_{m_{1,n}}(\frac{1}{t}\sum_{j=1}^{[\frac{\delta t}{\mu_{n,\kappa_0}}]}(\sigma_j-\eta_{j-1})1_{D_{n,l_0}}(X(\eta_{j-1}))\ge q_1|B_{n,l_0,q_0,\ [\frac{\delta t}{\mu_{n,\kappa_0}}]},\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]}<\tau_{A_n})\ge\frac{1}{2},
$$

for sufficiently large t.

Now fix some $\lambda>0$. Keeping in mind the exponential term on the right hand side of (7.42) and on the right hand side of (7.43), note that we can choose $\delta_{\lambda,\kappa_0}>0$ so that $\lambda(1-C_{\lambda,\kappa_0}\delta_{\lambda,\kappa_0})>c_0\delta_{\lambda,\kappa_0}$. Thus, we conclude from (7.40)-(7.43) and (7.47) that

$$
\begin{aligned}
&P_{m_{1,n}}(\frac{1}{t}\int_0^t 1_{D_{n,l_0+\frac{3}{2}}}(X(s))ds\ge q_1\mid\eta_{[\frac{\delta t}{\mu_{n,\kappa_0}}]}<\tau_{A_n})\ge\frac{1}{4}\exp(-c_0\delta_{\lambda,\kappa_0}t), \\
&\text{for sufficiently large } t.
\end{aligned}
\tag{7.48}
$$

Now (7.38) and (7.48) yield

(7.49)

$$
P_{m_{1,n}}(\frac{1}{t}\int_0^t 1_{D_{n,l_0+\frac{3}{2}}}(X(s))ds\ge q_1)\ge\frac{1}{4}\rho_{n,\kappa_0}^{[\frac{\delta_{\lambda,\kappa_0}t}{\mu_{n,\kappa_0}}]}\exp(-c_0\delta_{\lambda,\kappa_0}t),\ \text{for sufficiently large } t.
$$

Recall that the statement of the lemma includes the assumption that $\lim_{n\to\infty}\frac{|\log\rho_{n,\kappa_0}|}{\mu_{n,\kappa_0}} \in (0,\infty)$. Thus, from (7.49) we conclude that (7.34) holds with $c_1 = \delta_{\lambda,\kappa_0}(c_0 + \lim_{n\to\infty}\frac{|\log\rho_{n,\kappa_0}|}{\mu_{n,\kappa_0}}) < \infty$. □

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