THE INFINITE LIMIT OF SEPARABLE PERMUTATIONS

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ABSTRACT. Let $P_n^{sep}$ denote the uniform probability measure on the set of separable permutations in $S_n$. Let $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ with an appropriate metric and denote by $S(\mathbb{N}, \mathbb{N}^*)$ the compact metric space consisting of functions $\sigma = \{\sigma_i\}_{i=1}^{\infty}$ from $\mathbb{N}$ to $\mathbb{N}^*$ which are injections when restricted to $\sigma^{-1}(\mathbb{N})$; that is, if $\sigma_i = \sigma_j$, $i \neq j$, then $\sigma_i = \infty$. Extending permutations $\sigma \in S_n$ by defining $\sigma_j = j$, for $j > n$, we have $S_n \subset S(\mathbb{N}, \mathbb{N}^*)$. We show that $\{P_n^{sep}\}_{n=1}^{\infty}$ converges weakly on $S(\mathbb{N}, \mathbb{N}^*)$ to a limiting distribution of regenerative type, which we calculate explicitly.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $S_n$ denote the permutations of $[n] := \{1, \cdots, n\}$. Given $\sigma \in S_k$ and $\tau \in S_l$, the direct sum of $\sigma$ and $\tau$ is the permutation in $S_{k+l}$ given by

$$(\sigma \oplus \tau)(i) = \begin{cases} 
\sigma(i), & i = 1, \cdots, k; \\
\tau(i-k) + k, & i = k + 1, \cdots k + l,
\end{cases}$$

and the skew sum $\sigma \ominus \tau$ is the permutation in $S_{k+l}$ given by

$$(\sigma \ominus \tau)(i) = \begin{cases} 
\sigma(i) + l, & i = 1, \cdots, k; \\
\tau(i-k), & i = k + 1, \cdots k + l.
\end{cases}$$

A permutation is indecomposable if it cannot be represented as the direct sum of two nonempty permutations and is skew indecomposable if it cannot be written as the skew sum of two nonempty permutations. A permutation is separable if it can be obtained from the singleton permutation by iterating direct sums and skew sums. Equivalently, a permutation is separable if it can be successively decomposed and skew decomposed until all of the

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indecomposable and skew indecomposable pieces of the permutation are singletons. (For example, using one-line notation, consider the separable permutation \(\sigma = 4352167\). It can be decomposed into \(43521 \oplus 12\). Then \(43521\) can be skew decomposed into \(213 \ominus 21\) and \(12\) can be decomposed into \(1 \oplus 1\). Now \(213\) can be decomposed into \(21 \oplus 1\) and \(21\) can be skew decomposed into \(1 \ominus 1\). Finally, again \(21\) can be skew decomposed into \(1 \ominus 1\).) It is well-known [1] that a permutation is separable if and only it avoids the patterns 2413 and 3142.

Let \(\text{SEP}(n)\) denote the set of separable permutations in \(S_n\), and let \(P_{\text{sep}}^n\) denote the uniform measure on \(\text{SEP}(n)\). In this paper we investigate the weak limiting behavior of the probability measures \(\{P_{\text{sep}}^n\}_{n=1}^\infty\) as \(n \to \infty\).

In the limit we obtain a probability measure not on the set of permutations of \(\mathbb{N}\), but on a more general structure that we now describe. We note that [3] considered similar types of limits for permutations avoiding a pattern of length three.

Let \(\mathbb{N}^* = \mathbb{N} \cup \{\infty\}\) with the metric \(d_{\mathbb{N}^*}(i,j) = \sum_{k=i}^{j-1} 2^{-k}\), for \(1 \leq i < j \leq \infty\). Denote by \(S(\mathbb{N}, \mathbb{N}^*)\) the set of functions \(\sigma = \{\sigma_i\}_{i=1}^\infty\) from \(\mathbb{N}\) to \(\mathbb{N}^*\) which are injections when restricted to \(\sigma^{-1}(\mathbb{N})\); that is, if \(\sigma_i = \sigma_j, i \neq j\), then \(\sigma_i = \infty\). The space \(S(\mathbb{N}, \mathbb{N}^*)\) can be identified with the countably infinite product \(\mathbb{N}^* \times \mathbb{N}^* \cdots\). Since \(\mathbb{N}^*\) is a compact metric space, it follows that \(S(\mathbb{N}, \mathbb{N}^*)\) is also a compact metric space with the metric \(D(\sigma, \tau) := \sum_{i=1}^\infty \frac{d_{\mathbb{N}^*}(\sigma_i, \tau_i)}{2^i}\). Any permutation \(\sigma \in S_n\) may be identified uniquely with an element of \(S(\mathbb{N}, \mathbb{N}^*)\) by defining \(\sigma_j = j\), for \(j > n\). Consequently, if \(\mu_n\) is a probability measure on \(S_n\), for each \(n \in \mathbb{N}\), then \(\{\mu_n\}_{n=1}^\infty\) may be considered as a sequence of probability measures on the compact metric space \(S(\mathbb{N}, \mathbb{N}^*)\).

We now set the stage in order to describe our convergence result. A finite set of consecutive integers in \(\mathbb{N}\) will be called a block. For \(a, b \in \mathbb{R}\) with \(a \leq b\), we will denote the block \(\{a, \cdots, b\}\) by \([a, b]\). The limiting distribution of \(\{P_{\text{sep}}^n\}_{n=1}^\infty\) that we will obtain on \(S(\mathbb{N}, \mathbb{N}^*)\) has a regenerative structure. In order to describe this regenerative structure, we need to consider permutations of blocks \(I\) not as functions with a domain, but rather just as images. We will call such an object a permutation image of \(I\). Thus, for
example, if $I = [3, 5]$, then there are six permutation images of $I$; namely $(3 \ 4 \ 5), (3 \ 5 \ 4), (4 \ 3 \ 5), (4 \ 5 \ 3), (5 \ 3 \ 4), (5 \ 4 \ 3)$. We will denote a generic permutation image of $I$ by $\sigma_I^{\text{im}}$. We define a separable permutation image $\sigma_I^{\text{im}}$ in the obvious way, analogous to our original definition of a separable permutation, or equivalently, as a permutation image that avoids the patterns $2413$ and $3142$. Similarly, we define indecomposable and skew indecomposable separable permutation images. For example, the permutation images $(4 \ 2 \ 3 \ 6 \ 5)$ and $(5 \ 4 \ 6 \ 2 \ 3)$ of the block $[2, 6]$ are both separable, the former one being skew indecomposable and the latter one begin indecomposable.

We also define $\infty^{(j)}$ to be the $j$-fold image of $\infty$: $\infty^{(j)} = (\underbrace{\infty \cdots \infty}_j), j \in \mathbb{N}$; we call this a block of infinities. We can now concatenate these simple permutation images of blocks with each other and with blocks of infinities to build more complicated permutation images with infinities. For example, if $I_1 = [3, 5]$ and $I_2 = [20, 23]$, and if the permutation images $\sigma_I^{\text{im}}$, $i = 1, 2$, are given by $\sigma_{I_1}^{\text{im}} = (5 \ 3 \ 4)$ and $\sigma_{I_2}^{\text{im}} = (22 \ 20 \ 21 \ 23)$, then

$$\infty^{(2)} \ast \sigma_{I_1}^{\text{im}} \ast \infty^{(1)} \ast \sigma_{I_2}^{\text{im}} := (\infty, \infty, 5, 3, 4, \infty, 22, 20, 21, 23).$$

Let $s_n = |\text{SEP}(n)|, n \geq 1$, denote the number of separable permutations in $S_n$. Let

$$s(x) = \sum_{n=1}^{\infty} s_n x^n$$

denote the generating function of $\{s_n\}_{n=1}^{\infty}$. For a separable permutation, define the length of the first indecomposable block and the length of the first skew indecomposable block respectively by

$$|B_{1,-}^{+n}(\sigma)| = \min\{k : \sigma([k]) = [n]\}, \sigma \in \text{SEP}(n);$$

$$|B_{1,-}^{-n}(\sigma)| = \min\{k : \sigma([k]) = [n] - [n - k]\}, \sigma \in \text{SEP}(n).$$

Let $B_{1,+n}^{+n}(\sigma)$ denote the corresponding permutation image of the block $[1, |B_{1,+n}^{+n}(\sigma)|]$ (the first $|B_{1,+n}^{+n}(\sigma)|$ entries in $\sigma$), and let $B_{1,-n}^{-n}(\sigma)$ denote the corresponding permutation image of the block $[n - |B_{1,-n}^{-n}(\sigma)| + 1, n]$ (the first $|B_{1,-n}^{-n}(\sigma)|$ entries in $\sigma$). Note that, by construction, the permutation
image $B^+_1(n)(\sigma)$ is indecomposable and the permutation image $B^+_1(n)(\sigma)$ is skew decomposable.

By the definition of separable permutations, for each $\sigma \in \text{SEP}(n)$, with $n \geq 2$, exactly one out of $|B^+_1(n)(\sigma)|$ and $|B^-_1(n)(\sigma)|$ is equal to $n$, and by symmetry,

$$\{|\sigma \in \text{SEP}(n) : |B^+_1(n)(\sigma)| = n\} = |\{|\sigma \in \text{SEP}(n) : |B^-_1(n)(\sigma)| = n\}| = \frac{1}{2}s_n, \ n \geq 2.$$  

That is, half of the permutations in $\text{SEP}(n)$, $n \geq 2$, are indecomposable and half are skew indecomposable. Partitioning $\text{SEP}(n)$ by $\{|B^+_1(n)| = j\}_{j=1}^n$ (or alternatively, by $\{|B^-_1(n)| = j\}_{j=1}^n$), and using the concatenating structure of separable permutations, it follows that

$$s_n = s_1s_{n-1} + \frac{1}{2}\sum_{j=2}^{n-1} s_js_{n-j} + \frac{1}{2}s_n, \ n \geq 2.$$  

From this it is easy to show that

$$s(x) = \frac{1}{2}(1 - x - \sqrt{x^2 - 6x + 1}), \ \text{for} \ |x| < 3 - 2\sqrt{2}.$$  

From the above formula for the generating function, one can show that

$$s_n \sim \frac{1}{2\sqrt{\pi n^3}}(3 - 2\sqrt{2})^{-n+\frac{1}{2}}.$$  

(See [2, p. 474-475].) Our $s_n$ is equal to their $D_{n-1}$, and $D_n$ is known at the $n$th Schröder number (or “big” Schröder number.) From (1.5), it follows that (1.4) also holds for $|x| = 3 - 2\sqrt{2}$; we have

$$s(3 - 2\sqrt{2}) = \sqrt{2} - 1.$$  

In (1.7)-(1.11) below, we define the five types of random variables that will be used to describe the limiting distribution of $\{P_{n}^{\text{sep}}\}_{n=1}^\infty$:

\begin{equation}
\text{(1.7)} \\
\chi_{0,1}^{(n)}_{\infty} \text{ are IID and distributed according to the Bernoulli distribution with parameter } \frac{1}{2}:
\end{equation}

$$P(\chi_{0,1}^{(n)} = 0) = P(\chi_{0,1}^{(n)} = 1) = \frac{1}{2}.$$
\( \{N^{(n)}\}_{n=1}^{\infty} \) are IID and distributed according to the following distribution:

\[
P(N^{(n)} = j) = \begin{cases} 
\frac{\sqrt{2}}{2}, & j = 0; \\
\sqrt{2}(3 - 2\sqrt{2})^j, & j = 1, 2, \ldots .
\end{cases}
\]

Remark. For later use in Proposition 1, we note that \( EN^{(n)} = (\frac{1}{2})^2 \).

\( \{\{R_{m}^{(n)}\}_{m=1}^{\infty}\}_{n=1}^{\infty} \) are IID and distributed according to the following distribution:

\[
P(R_{m}^{(n)} = k) = \frac{8k(3 - 2\sqrt{2})^k}{\sqrt{2} - 1}, k = 1, 2, \ldots .
\]

\( \{L^{(n)}\}_{n=1}^{\infty} \) are IID and distributed according to the following distribution:

\[
P(L^{(n)} = j) = \begin{cases} 
\frac{2(3 - 2\sqrt{2})}{2 - \sqrt{2}}, & j = 1 \\
\frac{s_j(3 - 2\sqrt{2})^j}{2 - \sqrt{2}}, & j = 2, 3, \ldots .
\end{cases}
\]

Remark. It follows from (1.6) and the fact that \( s_1 = 1 \) that the distributions in (1.9) and (1.10) are indeed probability distributions.

\( \Pi^{\text{sep}}(I) \) is a uniformly random, indecomposable, separable permutation image of the finite block \( I \subset \mathbb{N} \), and \( \{\Pi^{\text{sep}}(I) : I \subset \mathbb{N} \text{ a finite block}\} \) are independent.

All the random variables in (1.7)-(1.11) are assumed to be mutually independent.

In the theorem below, we present a rather involved formula for the \( S(\mathbb{N}, \mathbb{N}^*) \)-valued random variable whose distribution is the limiting distribution of \( \{P_{n}^{\text{sep}}\}_{n=1}^{\infty} \). It is worthwhile to begin with a more verbal and informal description.

The random variable is formed by regenerative concatenation. Its first piece is obtained via the random variables \( \chi_{0,1}^{(1)}, N^{(1)}, \{R_{m}^{(1)}\}_{m=1}^{\infty}, L^{(1)} \) and the random variable \( \Pi^{\text{sep}}(I_1) \) with random block \( I_1 \) as specified below. To construct this piece, first we use the random variables \( N^{(1)} \) and \( \{R_{m}^{(1)}\}_{m=1}^{\infty} \).
and discard the block \([1, \sum_{m=1}^{N(1)} R^{(1)}_m]\); these numbers will not appear anywhere in the range of the \(S(N, N^*)\)-valued random variable. (Note though, that it is possible for this block to be empty since \(N^{(1)}\) can be equal to 0.) Then with the addition of the random variables \(\chi^{(1)}_{0,1}\) and \(L^{(1)}\), we build a block as follows. If \(\chi^{(1)}_{0,1} = 0\), then we lay down the block \(\infty^{(L^{(1)})}\), that is, a row of infinities of length \(L^{(1)}\). However, if \(\chi_{0,1} = 1\), then we lay down the block \(\Pi^{\text{sep}}(I_1)\), a uniformly random, indecomposable, separable permutation image of the random block \(I_1 := [1 + \sum_{m=1}^{N^{(1)}} R^{(1)}_m, L^{(1)} + \sum_{m=1}^{N^{(1)}} R^{(1)}_m]\) of length \(L^{(1)}\).

The second piece is obtained using the random variables \(\chi^{(2)}_{0,1}, N^{(2)}\), \(\{R^{(2)}_m\}_{m=1}^{\infty}\), \(L^{(2)}\) and the random variable \(\Pi^{\text{sep}}(I_2)\) with random block \(I_2\) as specified below. From the first piece, the block of numbers \([1, \sum_{m=1}^{N^{(1)}} R^{(1)}_m]\) was discarded. Furthermore, if \(\chi^{(1)}_{0,1}\) was equal to 1, then the block of numbers \(I_1 = [1 + \sum_{m=1}^{N^{(1)}} R^{(1)}_m, L^{(1)} + \sum_{m=1}^{N^{(1)}} R^{(1)}_m]\) was used, in which case these numbers are also no longer available. We now discard a block of length \(\sum_{m=1}^{N^{(2)}} R^{(2)}_m\), starting with the smallest number available. That is, we remove the block

\[
[\chi^{(1)}_{0,1} L^{(1)} + \sum_{m=1}^{N^{(1)}} R^{(1)}_m + 1, \chi^{(1)}_{0,1} L^{(1)} + \sum_{m=1}^{N^{(1)}} R^{(1)}_m + \sum_{m=1}^{N^{(2)}} R^{(2)}_m].
\]

Then with the addition of the random variables \(\chi^{(2)}_{0,1}\) and \(L^{(2)}\), we build a block as follows. If \(\chi^{(2)}_{0,1} = 0\), then we lay down the block \(\infty^{(L^{(2)})}\), that is, a row of infinities of length \(L^{(2)}\). However, if \(\chi^{(2)}_{0,1} = 1\), then we lay down the block \(\Pi^{\text{sep}}(I_2)\), a uniformly random, indecomposable, separable permutation image of the random block

\[
I_2 := [1 + \chi^{(1)}_{0,1} L^{(1)} + \sum_{m=1}^{N^{(1)}} R^{(1)}_m + \sum_{m=1}^{N^{(2)}} R^{(2)}_m, L^{(2)} + \chi^{(1)}_{0,1} L^{(1)} + \sum_{m=1}^{N^{(1)}} R^{(1)}_m + \sum_{m=1}^{N^{(2)}} R^{(2)}_m]\]

of length \(L^{(2)}\). The construction of the random variable continues in this regenerative fashion.

We now state the theorem For convenience, we set

\[\chi^{(0)}_{0,1} = L^{(0)} = 0.\]
Theorem 1. The distributions \( \{ P_{n}^{sep} \}_{n=1}^{\infty} \), considered on the space \( S(\mathbb{N}, \mathbb{N}^*) \), converge weakly to the distribution of the random variable

\[
\begin{align*}
&\star_{n=1}^{\infty} (\chi_{0,1}^{(n)} \Pi^{sep}(I_n) + (1 - \chi_{0,1}^{(n)}) \infty^{(L^{(n)})}) := \\
&(\chi_{0,1}^{(1)} \Pi^{sep}(I_1) + (1 - \chi_{0,1}^{(1)}) \infty^{(L^{(1)})}) \star (\chi_{0,1}^{(2)} \Pi^{sep}(I_2) + (1 - \chi_{0,1}^{(2)}) \infty^{(L^{(2)})}) \star \cdots,
\end{align*}
\]

with

\[
I_n = \left[ 1 + \sum_{k=1}^{n} \chi_{0,1}^{(k)} L^{(k-1)} + \sum_{k=1}^{n} \sum_{m=1}^{N^{(k)}} R^{(k)}_{m}, L^{(n)} + \sum_{k=1}^{n} \chi_{0,1}^{(k)} L^{(k-1)} + \sum_{k=1}^{n} \sum_{m=1}^{N^{(k)}} R^{(k)}_{m} \right],
\]

where the random variables involved here are as in (1.7)-(1.11).

Note from the theorem that in the first \( n \) pieces of the concatenation defining the limiting random variable, the numbers from 1 up to \( \sum_{k=1}^{n} \chi_{0,1}^{(k)} L^{(k-1)} + \sum_{k=1}^{n} \sum_{m=1}^{N^{(k)}} R^{(k)}_{m} \) have been involved. Of these numbers, \( \sum_{k=1}^{n} \sum_{m=1}^{N^{(k)}} R^{(k)}_{m} \) of them have been discarded and don’t appear in the limiting random variable, and \( \sum_{k=1}^{n} \chi_{0,1}^{(k)} L^{(k)} \) of them do appear in the limiting random variable. Thus, the ratio of the quantity of numbers appearing to the quantity of numbers appearing or discarded in the first \( n \) pieces of the concatenation is given by

\[
\frac{\sum_{k=1}^{n} \chi_{0,1}^{(k)} L^{(k)}}{\sum_{k=1}^{n} \chi_{0,1}^{(k)} L^{(k-1)} + \sum_{k=1}^{n} \sum_{m=1}^{N^{(k)}} R^{(k)}_{m}}.
\]

Note from (2.11) and (1.5) that the random variables \( L^{(k)} \) and \( R^{(k)}_{m} \) do not have a finite first moment, so the law of large numbers does not hold for them. Actually, they are in the domain of attraction of stable laws with parameter \( \frac{1}{2} \). We have the following proposition.

Proposition 1. i.

\[
\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n} L^{(k)} \underset{dist}{=} Z_L,
\]

where \( Z_L \) is the one-sided stable distribution with stability parameter \( \frac{1}{2} \) and with characteristic function

\[
\phi_{Z_L}(t) = E \exp(-itZ_L) = \exp\left( -\left( \frac{1}{2} \right)^{3/2} |t|^{1/2} \left( 1 + i \text{sgn}(t) \right) \right).
\]
ii. 
\[
\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n} R_{m}^{(k)} \overset{\text{dist}}{=} Z_{R},
\]
where \(Z_{R}\) is the one-sided stable distribution with stability parameter \(\frac{1}{2}\) and with characteristic function
\[
\phi_{Z_{R}}(t) = E \exp(-itZ_{R}) = \exp \left( - \left( \frac{1}{2} \right)^{\frac{1}{2}} |t|^{\frac{1}{2}} (1 + i \text{sgn}(t)) \right).
\]

iii. 
\[
\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n} \chi_{0,1}^{(k)} L^{(k)} \overset{\text{dist}}{=} \lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n} \sum_{m=1}^{N^{(k)}} R_{m}^{(k)} = Z,
\]
where \(Z\) is the one-sided stable distribution with stability parameter \(\frac{1}{2}\) and with characteristic function
\[
\phi_{Z}(t) = E \exp(-itZ) = \exp \left( - \left( \frac{1}{2} \right)^{\frac{1}{2}} |t|^{\frac{1}{2}} (1 + i \text{sgn}(t)) \right).
\]

This gives us the following corollary.

**Corollary 1.** The ratio of the quantity of numbers appearing to the quantity of numbers appearing or discarded in the first \(n\) pieces of the concatenation of the limiting random variable in Theorem 1 satisfies
\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} \chi_{0,1}^{(k)} L^{(k)}}{\sum_{k=1}^{n} \chi_{0,1}^{(k)} L^{(k)} + \sum_{k=1}^{n} \sum_{m=1}^{N^{(k)}} R_{m}^{(k)}} \overset{\text{dist}}{=} \frac{Z_1}{Z_1 + Z_2},
\]
where \(Z_1\) and \(Z_2\) are IID random variables distributed as \(Z\) in part (iii) of Proposition 1.

In section 2 we proof some preliminary results that will be used in the proof of Theorem 1, in section 3 we prove Theorem 1 and in section 4 we proof Proposition 1.

2. Preliminary results concerning separable permutations and Schröder numbers

From (1.2), we have
\[
(2.1) \quad P_{n}^{\text{SEP}}(|B_{1}^{\uparrow,n}| = n) = P_{n}^{\text{SEP}}(|B_{1}^{\uparrow,n}| = n) = \frac{1}{2}, \quad n \geq 2,
\]
and from (1.3) and the text immediately preceding it, we have

\( P_n(|B_1^+| = j \mid |B_1^−| < n) = P_n(|B_1^−| = j \mid |B_1^−| < n) \)

\( = \begin{cases} \frac{s_n}{s_n-j}, j = 1; \\
\frac{s_{n-1}}{s_n-j}, j = 2, \cdots, n-1. \end{cases} \)

From (2.1) and (2.2), it follows that the distributions of \( |B_1^+| \) and \( |B_1^−| \) under \( P_{n}^{\text{SEP}} \) are identical; thus, in the sequel we will sometimes use the notation \( |B_1^±,n| \) in formulas that hold for both \( |B_1^+| \) and \( |B_1^−| \). It also follows from (2.1) and (1.3) that

\( P_n(n - |B_1^−| = k \mid |B_1^−| < n) = \begin{cases} \frac{s_k}{s_n}, k = 1, 2, \cdots, n-2; \\
\frac{s_{n-1}}{s_n}, k = n-1. \end{cases} \)

(Formula (2.3) also holds with \( |B_1^+| \) in place of \( |B_1^−| \), but it won’t be needed.)

From (1.5), (2.2) and (2.3), we have

\( \lim_{n \to \infty} P_n(|B_1^±| = j \mid |B_1^±| < n) = \begin{cases} 2(3 - 2\sqrt{2}), j = 1; \\
s_j(3 - 2\sqrt{2})^j, j = 2, 3, \cdots. \end{cases} \)

and

\( \lim_{n \to \infty} P_n(n - |B_1^−| = k \mid |B_1^−| < n) = s_k(3 - 2\sqrt{2})^k, k = 1, 2, \cdots. \)

Using (1.6), the sum on the right hand side of (2.4) is given by

\( 2(3 - 2\sqrt{2}) + \sum_{j=2}^{\infty} s_j(3 - 2\sqrt{2})^j = 3 - 2\sqrt{2} + s(3 - 2\sqrt{2}) = 2 - \sqrt{2}, \)

and the sum on the right hand side of (2.5) is given by

\( \sum_{k=1}^{\infty} s_k(3 - 2\sqrt{2})^k = s(3 - 2\sqrt{2}) = \sqrt{2} - 1. \)

From (2.4) and (2.6) it follows that the random variables \( \{ |B_1^±| \}_{n=1}^{\infty} \) under the measures \( P_n(\cdot \mid |B_1^±| < n) \), considered on the space \( \mathbb{N}^* \), converge weakly
to the random variable $\tilde{L}$ with distribution

$$P(\tilde{L} = j) = \begin{cases} 2(3 - 2\sqrt{2}); & j = 1; \\ s_j(3 - 2\sqrt{2}), & j = 2, 3, \ldots; \\ \sqrt{2} - 1, & j = \infty. \end{cases}$$

From (2.5) and (2.7) it follows that the random variables $\{n - |B_1^{-n}|\}_{n=1}^{\infty}$ under the measures $P_n(\cdot | |B_1^{-n}| < n)$, considered on the space $\mathbb{N}^*$, converge weakly to the random variable $\tilde{R}$ with distribution

$$P(\tilde{R} = j) = \begin{cases} \frac{s_j(3 - 2\sqrt{2})}{\sqrt{2} - 1} k, & k = 1, 2, \ldots; \\ 2 - \sqrt{2}, & j = \infty. \end{cases}$$

Since $(2 - \sqrt{2}) + (\sqrt{2} - 1) = 1$, it follows that on the space $\mathbb{N}^* \times \mathbb{N}^*$, the random vectors $\{(|B_1^{-n}|, n - |B_1^{-n}|)\}_{n=1}^{\infty}$ under the measures $P_n(\cdot | |B_1^{-n}| < n)$ converge to the random vector $(\tilde{L}, \tilde{R})$ with distribution

$$P((\tilde{L}, \tilde{R}) = (j, \infty)) = \begin{cases} 2(3 - 2\sqrt{2}), & j = 1; \\ s_j(3 - 2\sqrt{2}), & j = 2, 3, \ldots; \\ \frac{s_j(3 - 2\sqrt{2})}{\sqrt{2} - 1} k, & k = 1, 2, \ldots. \end{cases}$$

Note in particular that $P((\tilde{L}, \tilde{R}) = (\infty, \infty)) = 0$.

Denote respectively by $R$ and $L$ random variables whose distributions are those of $R$ conditioned on $\{\tilde{R} < \infty\}$ and $L$ conditioned on $\{\tilde{L} < \infty\}$. That is,

$$P(R = k) = \frac{s_k(3 - 2\sqrt{2})}{\sqrt{2} - 1} k, \quad k = 1, 2, \ldots;$$

$$P(L = j) = \begin{cases} 2(3 - 2\sqrt{2}), & j = 1; \\ s_j(3 - 2\sqrt{2}), & j = 2, 3, \ldots. \end{cases}$$

Note that $R$ and $L$ are the distributions respectively of $R_m^{(n)}$ and $L^{(n)}$ in (1.9) and (1.10).
3. Proof of Theorem 1

To write down a complete and entirely rigorous proof of the theorem is extremely tedious and may well obscure the relative simplicity of the ideas behind the proof. Thus, we will give a somewhat informal proof, with quite a bit of verbal explanation, relating at times simultaneously to the situation for large $n$ and the situation in the limit after $n \to \infty$. At the end of this proof, we then prove completely rigorously a particular case of the proof. From this, it will be clear that one can precede similarly to obtain the entire proof.

Consider a permutation $\sigma$ under $P_n^{\text{sep}}$, for $n$ very large. That is, $\sigma$ is a uniformly distributed separable permutation in $S_n$. By (2.1), with probability $\frac{1}{2}$, one has $|B_1^+, n(\sigma)| < n$ and $|B_1^-, n(\sigma)| = n$, and with probability $\frac{1}{2}$ one has $|B_1^+, n(\sigma)| = n$ and $|B_1^-, n(\sigma)| < n$. Consider first the former case. Then the distribution of $|B_1^+, n(\sigma)|$ (under $P_n(\cdot \mid |B_1^+, n| < n)$) is given by the distribution in (2.2). From (2.4) and (2.6), it follows that as $n \to \infty$, $|B_1^+, n(\sigma)|$ will run off to infinity with probability approaching $\sqrt{2} - 1$, while with probability approaching $2 - \sqrt{2}$, it will converge to a limit which is distributed as $L$ in (2.11). Now consider the latter case. The distribution of $|B_1^-, n(\sigma)|$ (under $P_n(\cdot \mid |B_1^-| < n)$) is also given by the distribution in (2.2). Thus, as with $|B_1^+, n(\sigma)|$, $|B_1^- n(\sigma)|$ will run off to infinity with probability approaching $\sqrt{2} - 1$, while with probability approaching $2 - \sqrt{2}$, it will converge to a limit which is distributed as $L$ in (2.11).

We denote the four mutually exclusive cases above as follows:

$$(+ F) : \quad |B_1^+, n(\sigma)| < n \quad \text{and} \quad |B_1^+, n(\sigma)| \text{ does not run off to infinity;}$$

$$(- F) : \quad |B_1^-, n(\sigma)| < n \quad \text{and} \quad |B_1^-, n(\sigma)| \text{ does not run off to infinity;}$$

$$(+ I) : \quad |B_1^+, n(\sigma)| < n \quad \text{and} \quad |B_1^+, n(\sigma)| \text{ runs off to infinity;}$$

$$(- I) : \quad |B_1^-, n(\sigma)| < n \quad \text{and} \quad |B_1^-, n(\sigma)| \text{ runs off to infinity.}$$

The probabilities for these four cases are respectively

$$(3.2) \quad p_{(+ F)} = \frac{1}{2}(2 - \sqrt{2}); \quad p_{(- F)} = \frac{1}{2}(2 - \sqrt{2}); \quad p_{(+ I)} = \frac{1}{2}(\sqrt{2} - 1); \quad p_{(- I)} = \frac{1}{2}(\sqrt{2} - 1).$$
The notation (+) will denote the union of the two cases (+F) and (+I), and the notation (−) will denote the union of the two cases (−F) and (−I). Note that the cases (+) are the cases in which the permutation \( \sigma \) is skew indecomposable and the cases (−) are the cases in which it is indecomposable. The notation (F) will denote the union of the two cases (+F) and (−F).

If (F) occurs, then the first piece of the concatenation in the statement of the theorem is obtained immediately. Indeed, recalling the definition of \( B_{1}^{n}(\sigma) \) after (1.1), and recalling (2.4), (2.11) and (1.10), we see that in the case (+F), this piece is a uniformly random indecomposable separable permutation image \( \Pi_{\text{sep}}(I_{1}) \), where \( I_{1} = [1, L^{(1)}] \), while in the case (−F) it is the block of infinities \( \infty(L^{(1)}) \), where \( L^{(1)} \) is as in the statement of the theorem. Setting this piece aside now, in the case (+F), this essentially returns us to the situation we started from, except that now we are looking at uniformly random separable permutation images of \([L^{(1)} + 1, n]\), for large \( n \), while in the case (−F), this returns us exactly to the situation we started from, and we again look at uniformly random separable permutations of \([n]\), for large \( n \). We emphasize that if (F) occurs, then we obtain the first piece of the concatenation and with regard to the rest of the permutation, we are again presented with the four possible cases in (3.1) with the four corresponding probabilities in (3.2).

Now consider the case (+I). Then for large \( n \), \( m := |B_{1}^{n}(\sigma)| \) will be very large itself. Recalling the definition of \( B_{1}^{n}(\sigma) \) after (1.1), we see that \( B_{1}^{n}(\sigma) \) is a uniformly distributed indecomposable separable permutation in \( S_{m} \) with \( m \) large. This returns us to the two cases in (−) with corresponding probabilities \( 2p(−F) \) and \( 2p(−I) \). Note that here the first piece of the concatenation has not yet been obtained.

Now consider the case (−I). Since we are assuming that \( |B_{1}^{-n}(\sigma)| \) is running off to infinity, it follows from the paragraph in which (2.10) appears that \( n − |B_{1}^{-n}(\sigma)| \) will converge to a limit which is distributed as \( R \) in (2.11). By the definition of \( |B_{1}^{-n}(\sigma)| \), the initial block of numbers \([1, n − |B_{1}^{-n}(\sigma)|]\) will appear in the final \( n − |B_{1}^{-n}(\sigma)| \) positions of \( \sigma \). Thus, in the limit as \( n \to \infty \), the first \( R^{(1)} \) numbers in the permutation get swept out to infinity.
and will not appear in the limiting object, where \( R^{(1)} \) is as in the statement of the theorem. Also, recalling the definition of \( B_{1}^{-,n}(\sigma) \) after (1.1), we see that \( B_{1}^{-,n}(\sigma) \) is a uniformly distributed skew indecomposable permutation image of the block \([n-|B_{1}^{-,n}(\sigma)|+1,n]\). Since \( n-|B_{1}^{-,n}(\sigma)| \) converges weakly to \( R^{(1)} \), this essentially returns us to the two cases (+), with corresponding probabilities \( 2p_{+,F} \) and \( 2p_{+,I} \), except that now we are looking at uniformly random indecomposable separable permutation images of \([R^{(1)}+1,n]\), for large \( n \). Note that here the first piece of the concatenation has not yet been obtained.

We summarize the mechanism we have discovered above in the following four statements:

1. One begins from one of the four states in (3.1) with corresponding probabilities in (3.2). The first piece of the concatenation is obtained when a state in (F) is first reached. If (+F) is reached, then the first part of the concatenation is an indecomposable separable permutation image whose length is distributed as \( L \), and if (−F) is reached, then the first part of the concatenation is a block of infinities whose length is distributed as \( L \).

2. From the state (+I), one moves to the state (−F) with probability \( 2p_{-,F} \), in which case the first piece of the concatenation is obtained and it is a block of infinities whose length is distributed as \( L \), while one moves to the state (−I) with probability \( 2p_{-,I} \).

3. From the state (−I), one moves to the state (+F) with probability \( 2p_{+,F} \), in which case the first piece of the concatenation is obtained and it is an indecomposable permutation image whose length is distributed as \( L \), while one moves to the state (+I) with probability \( 2p_{+,I} \).

4. Every arrival at the state (−I) causes the \( r \) smallest numbers currently available (that is, that have not yet been involved in the construction) to be discarded; they will never be used in the construction. Here \( r \) has the distribution of \( R \).

Since \( p_{+,F} = p_{-,F} \) and \( p_{+,I} = p_{-,I} \), it is clear that with probability \( \frac{1}{2} \) the first piece of the concatenation will be an indecomposable separable permutation image, and with probability \( \frac{1}{2} \) it will be a block of infinities. These
two possibilities are represented in the statement of the theorem through the random variable \( \chi_{0,1}^{(1)} \).

We now show that the number of arrivals at state \((-I)\) prior to the first arrival at a state in \((F)\) has the distribution of \( N^{(1)} \) as in (1.8). Let \( A \) denote the number of such arrivals. For \( k \geq 1 \), in order to have \( A \geq k \), either the first state visited is \((-I)\) and then \( k - 1 \) times in a row one moves from \((-I)\) to \((+I)\) and back to \((-I)\), or the first state visited is \((+I)\), the next one is \((-I)\) and then \( k - 1 \) times in a row one moves from \((-I)\) to \((+I)\) and back to \((-I)\). The probability of the former scenario is \( \frac{1}{2} \left( \sqrt{2} - 1 \right) \left( \sqrt{2} - 1 \right)^{k-1} \) while the probability of the latter scenario is \( \frac{1}{2} \left( \sqrt{2} - 1 \right) \left( \sqrt{2} - 1 \right)^{k-1} \). Adding these, one obtains \( P(A \geq k) = \frac{\sqrt{2}}{2} \left( \sqrt{2} - 1 \right) (\sqrt{2} - 1)^{2k-2} \). After a bit of arithmetic, one finds that

\[
P(A = k) = P(A \geq k) - P(A \geq k + 1) = \frac{\sqrt{2}}{2} (3 - 2\sqrt{2})^k, \quad k \geq 1.
\]

Since \( \sum_{k=1}^{\infty} \frac{\sqrt{2}}{2} (3 - 2\sqrt{2})^k = 1 - \frac{\sqrt{2}}{2} \), one obtains \( P(A = 0) = \frac{\sqrt{2}}{2} \). Thus the number of arrivals at state \((-I)\) prior to the first arrival at a state in \((F)\) indeed has the distribution of \( N^{(1)} \).

In light of the above, we see that by the time the first piece of the concatenation is constructed, a certain block of numbers, beginning from 1, will have been removed from consideration, and the length of that block is distributed as \( \sum_{m=1}^{N^{(1)}} R^{(1)}_m \). Thus, in the case that the first piece of the concatenation is a permutation image, it will be an indecomposable separable permutation image of the random block \( I_1 = [1 + \sum_{m=1}^{N^{(1)}} R^{(1)}_m, L^{(1)} + \sum_{m=1}^{N^{(1)}} R^{(1)}_m] \).

After the first piece of the concatenation is obtained, be it a random permutation image or a block of infinities, we begin again with the same mechanism, however now the first number available for use is \( 1 + \chi_{0,1}^{(1)} L^{(1)} + \sum_{m=1}^{N^{(1)}} R^{(1)}_m \). In light of the regenerative nature described above, this completes the proof of the theorem.

As promised at the beginning of the proof, we now give a completely rigorous derivation for one case, the case corresponding to arriving first at \((-I)\) and then going to \((+F)\). Because the first state is assumed to be in \((-)\), we are assuming that \( |B_1^{-n}|(\sigma) < n \). Let \( n_1(\sigma) = |B_1^{-n}|(\sigma) \). Let \( \tau = \tau(\sigma) \)
denote the permutation image of the block \([n-n_1(\sigma)+1, n]\) corresponding to the last \(n_1(\sigma)\) entries of \(\sigma\). Note that \(\tau\) is a skew indecomposable separable permutation image. In fact, conditioned on \(n-n_1(\sigma) = n-n_1\), \(\tau\) is a uniformly distributed skew indecomposable separable permutation image of the block \([n-n_1+1, n]\).

Since \(\tau\) is a skew indecomposable separable permutation image, we have \(|B_1^{+n-n_1(\sigma)}|(\tau) < n-n_1(\sigma)\). Here we have taken the liberty to extend in the obvious way the domain of definition of \(|B_1^{+k}|(\cdot)\), \(k \geq 1\), which has been defined on separable permutations in \(S_k\), to include separable permutation images of blocks of length \(k\). Let \(n'_1(\tau) = |B_1^{+n-n_1(\sigma)}|(\tau)\), and let \(\tau' = \tau'(\tau)\) denote the first \(n'_1(\tau)\) entries of \(\tau\). Note that \(\tau'\) is an indecomposable separable permutation image of the block \([n-n_1(\sigma)+1, n-n_1(\sigma)+n'_1(\tau)]\). In fact, conditioned on \(n_1(\sigma) = n_1\) and \(n'_1(\tau) = n'_1\), \(\tau'\) is a uniformly distributed indecomposable separable permutation image of the block \([n-n_1+1, n-n_1+n'_1]\).

For \(j, k \geq 1\) and for \(n\) sufficiently large to accommodate the \(j\) and \(k\), we have

\[
\text{(3.3)} \quad P_n^{\text{sep}}(n-n_1(\sigma) = k, n'_1(\tau) = j, \tau' = \tau_0) =
\]

\[
P_n^{\text{sep}}(\tau' = \tau_0 \mid n'_1(\tau) = j, n-n_1(\sigma) = k) \times
\]

\[
P_n^{\text{sep}}(n'_1(\tau) = j \mid n-n_1(\sigma) = k) P_n^{\text{sep}}(n-n_1(\sigma) = k),
\]

for \(\tau_0\) an indecomposable separable permutation image of \([k+1, k+j]\).

Now

\[
\text{(3.4)} \quad P_n^{\text{sep}}(\tau' = \tau_0 \mid n'_1(\tau) = j, n-n_1(\sigma) = k) = \begin{cases} 1, & j = 1; \\ \frac{2}{s_j}, & j \geq 2, \end{cases}
\]

since there are \(\frac{s_j}{2}\) indecomposable separable permutation images of \([k+1, k+j]\), for \(j \geq 2\). Also,

\[
\text{(3.5)} \quad P_n^{\text{sep}}(n'_1(\tau) = j \mid n-n_1(\sigma) = k) = P_n^{\text{sep}}(|B_1^{+k}|(\tau) = j \mid |B_1^{+k}|(\tau) < k).
\]
(The conditioning on \(\{|B_{1}^{+,k}|(\tau) < k\}\) comes in because \(\tau\) is skew indecomposable.) From (2.4) we have

\[
(3.6) \lim_{k \to \infty} P_{n}^{\text{sep}}(|B_{1}^{+,k}|(\tau) = j | |B_{1}^{+,k}|(\tau) < k) = \begin{cases} 2(3 - 2\sqrt{2}), & j = 1; \\ s_{j}(3 - 2\sqrt{2})^{j}, & j = 2, 3, \ldots. \end{cases}
\]

Also, from (2.1) and (2.5), we have

\[
(3.7) \lim_{n \to \infty} P_{n}^{\text{sep}}(n - n_{1}(\sigma) = k) = \lim_{n \to \infty} P_{n}^{\text{sep}}(n - |B_{1}^{-,n}|(\sigma) = k) = \frac{1}{2}s_{k}(3 - 2\sqrt{2})^{k}.
\]

From (3.3)-(3.7) we obtain

\[
(3.8) \lim_{n \to \infty} P_{n}^{\text{sep}}(n - n_{1}(\sigma) = k, n'_{1}(\tau) = j, \tau' = \tau_{0}) = \begin{cases} \left(\frac{1}{2}s_{k}(3 - 2\sqrt{2})^{k}\right)\left(2(3 - 2\sqrt{2})\right), & j = 1; k = 1, 2, \ldots; \\ \left(\frac{1}{2}s_{k}(3 - 2\sqrt{2})^{k}\right)\left(s_{j}(3 - 2\sqrt{2})^{j}\right)\frac{2}{s_{j}}, & j = 2, 3, \ldots; k = 1, 2, \ldots. \end{cases}
\]

We rewrite the right hand side above as

\[
(3.9) \begin{cases} \left(\frac{1}{2}(\sqrt{2} - 1)\right)\left(\frac{s_{k}(3 - 2\sqrt{2})^{k}}{\sqrt{2} - 1}\right)\left(2 - \sqrt{2}\right)\frac{2(3 - 2\sqrt{2})}{2 - \sqrt{2}}, & j = 1; k = 1, 2, \ldots; \\ \left(\frac{1}{2}(\sqrt{2} - 1)\right)\left(\frac{s_{k}(3 - 2\sqrt{2})^{k}}{\sqrt{2} - 1}\right)\left(2 - \sqrt{2}\right)\frac{s_{j}(3 - 2\sqrt{2})^{j}}{2 - \sqrt{2}}\frac{2}{s_{j}}, & j = 2, 3, \ldots; k = 1, 2, \ldots. \end{cases}
\]

The term \(\frac{1}{2}(\sqrt{2} - 1)\) is \(p(-1)\), the term \(2 - \sqrt{2}\) is \(2p(+)\), the terms \(\frac{s_{k}(3 - 2\sqrt{2})^{k}}{\sqrt{2} - 1}, k = 1, 2, \ldots\), give the distribution of \(R\) as in (2.11), the terms \(\frac{2(3 - 2\sqrt{2})}{2 - \sqrt{2}}, j = 1; \frac{s_{j}(3 - 2\sqrt{2})^{j}}{2 - \sqrt{2}}\), \(j = 2, 3, \ldots\) give the distribution of \(L\) as in (2.11), and the term \(\frac{2}{s_{j}}, j \geq 2\) indicates choosing uniformly from the indecomposable separable permutation images of the block \([k + 1, k + j]\). \(\square\)

### 4. Proof of Proposition 1

**Proof of (i).** We need to show that \(E \exp(-it\sum_{k=1}^{n}L^{(k)})\) converges to the characteristic function appearing in part (i). We consider \(t\) to be fixed for the proof. We have

\[
(4.1) E \exp(-it\frac{\sum_{k=1}^{n}L^{(k)}}{n^{2}}) = \left(E \exp(-it\frac{L}{n^{2}})\right)^{n}.
\]
From (2.11) and (1.4) along with the fact that (1.5) guarantees that (1.4) holds with \( x \) replaced by complex \( z \) satisfying \( |z| = 3 - 2\sqrt{2} \), we have
\[
E \exp\left(\frac{-it}{n^2} L\right) = \frac{2(3 - 2\sqrt{2})}{2 - \sqrt{2}} \exp\left(-\frac{it}{n^2}\right) + \sum_{j=2}^{\infty} \frac{s_j(3 - 2\sqrt{2})^j}{2 - \sqrt{2}} \exp\left(-\frac{itj}{n^2}\right) =
\]
\[
\frac{2(3 - 2\sqrt{2})}{2 - \sqrt{2}} \exp\left(-\frac{it}{n^2}\right) + \frac{1}{2 - \sqrt{2}} s((3 - 2\sqrt{2})e^{-\frac{it}{n^2}}) = \frac{2 - \sqrt{2}}{2} \exp\left(-\frac{it}{n^2}\right) +
\]
\[
\frac{2 + \sqrt{2}}{4} \left(1 - (3 - 2\sqrt{2})e^{-\frac{it}{n^2}} - \sqrt{(3 - 2\sqrt{2})^2 e^{-\frac{2it}{n^2}} - 6(3 - 2\sqrt{2})e^{-\frac{it}{n^2}} + 1}\right).
\]
Noting that \( 3 - 2\sqrt{2} \) is a root of \( z^2 - 6z + 1 \), we have after expanding in a power series and doing some arithmetic,
\[
(3 - 2\sqrt{2})e^{-\frac{2it}{n^2}} - 6(3 - 2\sqrt{2})e^{-\frac{it}{n^2}} + 1 = (18\sqrt{2} - 25)\frac{t^2}{n^4} + i(12\sqrt{2} - 16)\frac{t}{n^2} + O\left(\frac{1}{n^6}\right).
\]

For \( n \) sufficiently large, this number lies in the right half plane. Thus we interpret the square root above as \( \sqrt{z} = |z|^\frac{1}{2} \exp\left(\frac{i}{2} \text{Arg}(z)\right) \), with \( \text{Arg}(z) \in (-\pi, \pi) \). Thus, from (4.3) we have
\[
\sqrt{(3 - 2\sqrt{2})^2 e^{-\frac{2it}{n^2}} - 6(3 - 2\sqrt{2})e^{-\frac{it}{n^2}} + 1} = \frac{2(3\sqrt{2} - 4)\frac{1}{2} |t| \frac{1}{2} e^{i \text{sgn}(t) \frac{\pi}{4}} + O\left(\frac{1}{n^2}\right)}{n}.
\]

Thus,
\[
1 - (3 - 2\sqrt{2})e^{-\frac{it}{n^2}} - \sqrt{(3 - 2\sqrt{2})^2 e^{-\frac{2it}{n^2}} - 6(3 - 2\sqrt{2})e^{-\frac{it}{n^2}} + 1} =
\]
\[
2\sqrt{2} - 2 - \frac{2(3\sqrt{2} - 4)\frac{1}{2} |t| \frac{1}{2} e^{i \text{sgn}(t) \frac{\pi}{4}} + O\left(\frac{1}{n^2}\right)}{n},
\]

and from (4.2),
\[
E \exp\left(-\frac{it}{n^2} L\right) = 1 - \left(\frac{1}{2}\right)^\frac{1}{2} \frac{1}{n} e^{i \text{sgn}(t) \frac{\pi}{4}} + O\left(\frac{1}{n^2}\right),
\]

(4.5)

(\text{where we’ve used the fact that} \( \frac{2 + \sqrt{2}}{4}(3\sqrt{2} - 4)^\frac{1}{2} = (\frac{1}{2})^4 \)). From (4.1) and (4.5), we obtain
\[
\lim_{n \to \infty} E \exp\left(-it \frac{\sum_{k=1}^{n} L^{(k)}}{n^2}\right) = \lim_{n \to \infty} \left(1 - \left(\frac{1}{2}\right)^\frac{1}{2} \frac{1}{n} e^{i \text{sgn}(t) \frac{\pi}{4}} + O\left(\frac{1}{n^2}\right)^\frac{1}{2}\right)^n =
\]
\[
\exp\left(-\left(\frac{1}{2}\right)^\frac{1}{2} \frac{1}{n} e^{i \text{sgn}(t) \frac{\pi}{4}}\right) = \exp\left(-\left(\frac{1}{2}\right)^\frac{1}{2} \frac{1}{n} \left(1 + i \text{sgn}(t)\right)\right).
\]
Proof of (ii). The proof is very similar to that of part (i), so we leave it to the reader. However, for use in part (iii), we note that similar to (4.5), we have

\[ E \exp(-\frac{it}{n^2} R) = 1 - 2^{\frac{j}{2}} \left| \frac{t}{n} \right|^j e^{i \text{sgn}(t) \frac{\pi}{4}} + O\left( \frac{1}{n^2} \right). \]

Proof of part (iii). We will prove the result concerning \( \frac{1}{n^2} \sum_{k=1}^{n} \sum_{m=1}^{N(k)} R_m^{(k)} \).

The proof for \( \frac{1}{n^2} \sum_{k=1}^{n} \chi_{0,1}^{(k)} L^{(k)} \) is very similar. What we will show is that the characteristic functions of the random variables \( \frac{1}{n^2} \sum_{k=1}^{n} \sum_{m=1}^{N(k)} R_m^{(k)} \) converge to \( \phi_Z(t\text{EN}^{(1)}) \), where \( \phi_Z \) is as in part (ii). As noted after (1.8), \( \text{EN}^{(1)} = \left( \frac{1}{2} \right)^2 \). For the case of \( \frac{1}{n^2} \sum_{k=1}^{n} \chi_{0,1}^{(k)} L^{(k)} \), one shows that the characteristic functions of these random variables converge to \( \phi_{Z_L}(t\chi_{0,1}^{(1)}) = \phi_{Z_L} \left( \frac{1}{2} t \right) \). It turns out that both \( \phi_{Z_R}(\left( \frac{1}{2} \right)^2 t) \) and \( \phi_{Z_L}(\frac{1}{2} t) \) are equal to the characteristic function in part (iii).

We consider \( t \) fixed for the proof. We have

\[ E \exp\left(-\frac{it}{n^2} \sum_{k=1}^{n} \sum_{m=1}^{N(k)} R_m^{(k)} \right) = \left( E \exp\left(-\frac{it}{n^2} \sum_{m=1}^{N(1)} R_m^{(1)} \right) \right)^n, \]

and from (1.8) and conditioning,

\[ E \exp\left(-\frac{it}{n^2} \sum_{m=1}^{N(1)} R_m^{(1)} \right) = \sqrt{2} \frac{2}{2} + \sum_{j=1}^{\infty} \sqrt{2} (3 - 2\sqrt{2})^j \left( E \exp(-\frac{it}{n^2} R) \right)^j. \]

Using (4.7) and noting that \( (1 - z)^j = 1 - jz + R_2(z) \), with \( |R_2(z)| \leq \frac{j(j-1)}{2} |z|^2 \), we have

\[ \frac{\sqrt{2}}{2} + \sum_{j=1}^{\infty} \sqrt{2} (3 - 2\sqrt{2})^j \left( E \exp(-\frac{it}{n^2} R) \right)^j = \]

\[ \frac{\sqrt{2}}{2} + \sum_{j=1}^{\infty} \sqrt{2} (3 - 2\sqrt{2})^j \left( 1 - 2^{\frac{j}{2}} \left| \frac{t}{n} \right|^j e^{i \text{sgn}(t) \frac{\pi}{4}} + O\left( \frac{1}{n^2} \right) \right)^j = \]

\[ \frac{\sqrt{2}}{2} + \sum_{j=1}^{\infty} \sqrt{2} (3 - 2\sqrt{2})^j \left( 1 - 2^{\frac{j}{2}} \left| \frac{t}{n} \right|^j e^{i \text{sgn}(t) \frac{\pi}{4}} j + O\left( \frac{1}{n^2} \right) \right) = \]

\[ 1 - 2^{\frac{j}{2}} \left| \frac{t}{n} \right|^j e^{i \text{sgn}(t) \frac{\pi}{4}} \text{EN} + O\left( \frac{1}{n^2} \right). \]
From (4.8), (4.9), (4.10) and the fact that $EN = (\frac{1}{2})^2$, we obtain

$$\lim_{n \to \infty} E \exp \left( -\frac{it}{n^2} \sum_{k=1}^{n} \sum_{m=1}^{N^{(k)}} R_{m}^{(k)} \right) = \exp \left( -\left( \frac{1}{2} \right)^2 |t|^\frac{3}{2} \left( 1 + i \sgn(t) \right) \right).$$

□

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