Absolute Continuity/Singularity and Relative Entropy Properties for Probability Measures Induced by Diffusions on Infinite Time Intervals

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June 16, 2009

Abstract

In this paper we study mutual absolute continuity, finiteness of relative entropy and the possibility of their equivalence for probability measures on $C([0,\infty);\mathbb{R}^d)$ induced by diffusion processes. We also determine explicit events which distinguish between two mutually singular measures in certain one-dimensional cases.

1. Introduction and Statement of Results

Consider the diffusion operator,

$$\mathrm{L}_{a,b} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

defined on $\mathbb{R}^d$, where $a(x) = \{a_{ij}(x)\}_{i,j=1}^d$ is locally Hölder continuous and positive definite, on $\mathbb{R}^d$ and $\{b_i\}_{i=1}^d$ is locally Hölder continuous on $\mathbb{R}^d$. We will assume that the martingale problem for $L_{a,b}$ is well-posed—that is, that the diffusion process corresponding to $L_{a,b}$ does not explode. The unique solution to the martingale problem for $L_{a,b}$ will be denoted by $\{P_x^{a,b}\}_{x\in\mathbb{R}^d}$, where x denotes the value of the corresponding diffusion process at time 0. Of course, $P_x^{a,b}$ is a probability measure on $C([0,\infty),\mathbb{R}^d)$, the space of continuous trajectories $X = X(t)$ from $[0,\infty)$ to $\mathbb{R}^d$ with the topology of uniform convergence on bounded time intervals, and it is supported on paths satisfying $X(0) = x$. Denote the restriction of $P_x^{a,b}$ to $C([0,t],\mathbb{R}^d)$ by $P_{x;t}^{a,b}$. It follows from the Girsanov formula that $P_{x;t}^{a,b}$ and $P_{x;t}^{a,\hat{b}}$ are mutually absolutely continuous and that

$$\frac{dP_{x;t}^{a,\hat{b}}}{dP_{x;t}^{a,b}}(X(\cdot)) = \exp(\int_0^t a^{-1}(\hat{b}-b)(X(s))d\bar{X}(s) - \frac{1}{2}\int_0^t \langle \hat{b}-b, a^{-1}(\hat{b}-b)\rangle(X(s))ds),$$

where $\bar{X}(t) = X(t) - \int_0^t b(X(s))ds$.

Given two probability measures ν and μ, recall that the relative entropy $H(\nu;\mu)$ of ν with respect to μ is defined by $H(\nu;\mu) = \int \frac{d\nu}{d\mu}\log\frac{d\nu}{d\mu}d\mu \le \infty$, if ν is absolutely continuous with respect to μ, and is defined to be ∞ otherwise. A straightforward calculation reveals that

$$H(P_{x;t}^{a,\hat{b}}; P_{x;t}^{a,b}) = \frac{1}{2}\mathbb{E}_x^{a,\hat{b}}\int_0^t \langle \hat{b}-b, a^{-1}(\hat{b}-b)\rangle(X(s))ds.$$

It follows from this that if one restricts to the class of *bounded* drifts, then the relative entropy is always finite. Thus, for bounded drifts, mutual absolute continuity and finiteness of the relative entropies are equivalent when the measures are restricted to $C([0,t],\mathbb{R}^d)$.

There has been a lot of work concerning questions of absolute continuity for various types of stochastic processes on finite time intervals; see for example [5] and [6]. However, in the case of an infinite time horizon, it seems that very little work has been done. In this paper, we study mutual absolute continuity, finiteness of the relative entropy and the possibility of their equivalence for probability measures $P_x^{a,b}$ and $P_x^{a,\hat{b}}$ on $C([0,\infty),\mathbb{R}^d)$. If one observes a diffusion process $X(t)$, $0 \le t < \infty$, and wants to test the hypothesis that the process corresponds to $P_x^{a,\hat{b}}$ against the hypothesis that it corresponds to $P_x^{a,b}$, then observing one realization of the process on the infinite time interval will almost surely allow for the identification of the process if and only if the two measures are mutually singular. For certain pairs of mutually singular measures corresponding to one dimensional diffusions, we will determine explicit events which distinguish between the measures. The particular interest in entropy in this context comes from Stein's lemma applied to the Neyman-Pearson hypothesis testing. Let $\mathcal{F}_t = \sigma(X(s), 0 \le s \le t)$

and let $A_t \in \mathcal{F}_t$ be an acceptable region for the hypothesis $P_x^{a,\hat{b}}$ when observing the process up to time t. Let the probabilities of error be $\alpha_t = P_x^{a,\hat{b}}(A_t^c)$ and $\beta_t = P_x^{a,b}(A_t)$. For $\epsilon \in (0, \frac{1}{2})$, let $\beta_t^\epsilon \equiv \inf_{A_t \in \mathcal{F}_t, \alpha_t < \epsilon} \beta_t$. Then Stein's lemma [1] states that $\lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{1}{t} \log \beta_t^\epsilon = -H(P_x^{a,\hat{b}}, P_x^{a,b})$.

As usual, $\mu \perp \nu$ will denote that μ and ν are mutually singular, $\mu \ll \nu$ will denote that μ is absolutely continuous with respect to ν, and $\mu \sim \nu$ will denote that μ and ν are mutually absolutely continuous. We begin with the following basic criteria.

Theorem 1.
i. $P_x^{a,b} \perp P_x^{a,\hat{b}}$ *if and only if*

$$\int_0^\infty \langle \hat{b} - b, a^{-1}(\hat{b} - b)\rangle(X(s))ds = \infty \text{ a.s. } [P_x^{a,b}] \text{ or a.s. } [P_x^{a,\hat{b}}];$$

ii. $P_x^{\hat{b}} \ll P_x^b$ *if and only if*

$$\int_0^\infty \langle \hat{b} - b, a^{-1}(\hat{b} - b)\rangle(X(s))ds < \infty \text{ a.s. } [P_x^{a,\hat{b}}];$$

iii.

$$H(P_x^{a,\hat{b}}; P_x^{a,b}) = \frac{1}{2}\mathbb{E}_x^{a,\hat{b}} \int_0^\infty \langle \hat{b} - b, a^{-1}(\hat{b} - b)\rangle(X(s))ds. \tag{1.1}$$

We note three useful corollaries of Theorem 1 whose proofs will be given in section 2.

Corollary 1. *If* $P_x^{a,\hat{b}} \ll P_x^{a,b}(P_x^{a,\hat{b}} \perp P_x^{a,b})$ *for some* $x \in \mathbb{R}^d$, *then the same holds true for all* $x \in \mathbb{R}^d$.

Corollary 2. *If the invariant* σ*-fields for the diffusion processes corresponding to* $\mathrm{L}_{a,b}$ *and* $\mathrm{L}_{a,\hat{b}}$ *are trivial, then either* $P_x^{a,\hat{b}} \sim P_x^{a,b}$ *or* $P_x^{a,b} \perp P_x^{a,\hat{b}}$.

Corollary 3. *Assume that the diffusion process corresponding to either* $\mathrm{L}_{a,b}$ *or* $\mathrm{L}_{a,\hat{b}}$ *is recurrent. Then* $P_x^{a,b} \perp P_x^{a,\hat{b}}$.

Remark. Although Corollary 3 might seem "obvious", we note that even in the positive recurrent case, for dimension $d \geq 2$ the result does not follow immediately from ergodic considerations because there are infinitely many drifts corresponding to each invariant probablity measure [8].

In light of Corollary 3, in the sequel we will work only with transient diffusion processes. The transient diffusion process corresponding to $\mathrm{L}_{a,b}$ possesses a positive Green's functions, which will be denoted by $G^{a,b}(x, y)$. Recall that $G^{a,b}(x, y)$

satisfies $\mathbb{E}_x^{a,b}\int_0^\infty f(X(s))ds = \int_{\mathbb{R}^d} G^{a,b}(x,y)f(y)dy$, for $f\geq 0$, and the expression above is finite for all compactly supported f. Thus, it follows from (1.1) that

$$H(P_x^{a,\hat{b}}, P_x^{a,b}) = \frac{1}{2}\int_{\mathbb{R}^d}\langle \hat{b}-b, a^{-1}(\hat{b}-b)\rangle(y)G^{a,\hat{b}}(x,y)dy. \tag{1.2}$$

Whereas (1.2) gives a reasonably simple analytic formula for the relative entropy, in general there is no known analytic formula for the absolutely continuity/singularity dichotomy; that is, there is no analytic formula which is equivalent to condition (i) or (ii) of Theorem 1. (The exception to this, is in the one-dimensional case, as will be seen in Theorem 4 below.) Recalling that on finite time intervals, mutual absolute continuity and finiteness of the relative entropies were equivalent for the class of bounded drifts, we ask whether one can specify a nice class of diffusions for which mutual absolute continuity and finiteness of the relative entropies are equivalent? For such a class of diffusions, (1.2) would then give an analytic characterization of mutual absolute continuity. Before continuing, we note an example from a completely different context where such a phenomenom occurs. Let $\{P_n\}_{n=1}^\infty$ and $\{Q_n\}_{n=1}^\infty$ be independent sequences of Bernoulli measures on $\{0,1\}$, and let $P=\Pi_{n=1}^\infty P_n$ and $Q=\Pi_{n=1}^\infty Q_n$. If one restricts to the class of measures for which there exists an $\epsilon>0$ such that $\epsilon\leq P_n(0), Q_n(0)\leq 1-\epsilon$, for all n, then mutual absolute continuity of P and Q is equivalent to the finiteness of the relative entropies—the condition is $\sum_{n=1}^\infty (P_n(0)-Q_n(0))^2<\infty$. Without this restriction, the finiteness of the above sum still characterizes finite relative entropy, but it is possible to have mutual absolute continuity even if this sum is infinite (see [3, exercises 4.3.7 and 4.3.9]). Returning to our context, we will see that such an equivalence indeed holds for the class of Fuchsian diffusions, which we now define.

Definition 1. The operator $\mathrm{L}_{a,b}=\frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}$ on $\mathbb{R}^d$ is called Fuchsian if there exist constants $K_1,K_2>0$ such that
i.$K_1|v|^2\leq \sum_{i,j=1}^d a_{ij}(x)v_iv_j\leq K_2|v|^2$, for $x\in\mathbb{R}^d$;
ii. $|b(x)|\leq \frac{K_2}{1+|x|}$ for $x\in\mathbb{R}^d$.

We will prove the following theorem.

Theorem 2. *Assume the* $\mathrm{L}_{a,b}$ *and* $\mathrm{L}_{a,\hat{b}}$ *are Fuchsian. Then either*
i. $P_x^{a,b}\perp P_x^{a,\hat{b}}$
or
ii. $P_x^{a,b}\sim P_x^{a,\hat{b}}$ *and* $\sup_{x\in\mathbb{R}^d}H(P_x^{a,b};P_x^{a,\hat{b}})<\infty$, $\sup_{x\in\mathbb{R}^d}H(P_x^{a,\hat{b}};P_x^{a,b})<\infty$.

Remark. The total variation norm d_{TV} for probability measures μ and ν on $(\Omega,\mathcal{A})$ is defined by $d_{TV}(\mu,\nu)=\sup_{A\in\mathcal{A}}|\mu(A)-\nu(A)|$. The relative entropy provides an upper bound on the total variation norm through the inequality $d_{TV}(\mu,\nu) \le (2\min(H(\mu;\nu),H(\nu;\mu))^{\frac12}$ [2, exercise 6.2.17] Thus for example, if $H(P_x^{a,b+b_1},P_x^{a,b}) < \infty$, then it follows from (1.1) and the above inequality that $d_{TV}(P_x^{a,b},P_x^{a,b+\epsilon b_1}) \le \epsilon(2H(P_x^{a,b+b_1};P_x^{a,b}))^{\frac12}$. In particular, $P_x^{a,b+\epsilon b_1}$ converges in total variation norm to $P_x^{a,b}$ as $\epsilon \to 0$, which allows one to conclude that $P_x^{a,b+\epsilon b_1}$-probabilities of events involving the entire infinite time interval of a path converge to the corresponding $P_x^{a,b}$-probability. Without the finite relative entropy, one only knows that $P_x^{a,b+\epsilon b_1}$ converges weakly to $P_x^{a,b}$, which doesn't give any information for events that depend on the entire infinite time interval. From Theorem 2 it follows that if $P_x^{a,b}$ and $P_x^{a,b+b_1}$ are mutually absolutely continuous measures coming from Fuchsian diffusions, then as $\epsilon \to 0$, $P_x^{a,b+\epsilon b_1}$ converges in the total variation norm to $P_x^{a,b}$.

Using Theorem 1, we can give a very simple criterion for the mutual absolute continuity or the mutual singularity of Wiener measure and the measure induced by another Fuchsian diffusion. The Green's function for Brownian motion ($\mathrm{L} = \frac12\Delta$) is given by $G(x,y) = c_d|y-x|^{2-d}$, for $d \ge 3$, where c_d is an appropriate positive constant. Thus, the following corollary is an immediate consequence of Theorem 1, Corollary 1 and (1.2).

Corollary 4. *Let $\mathcal{W}_x$ ($= P_x^{I,0}$) denote d-dimensional Wiener measure on paths starting from $x \in \mathbb{R}^d$, $d \ge 3$, and let b satisfy $|b(x)| \le \frac{K}{1+|x|}$, for some $K > 0$.*
i. If $\int_{\mathbb{R}^d} \frac{|b(y)|}{|y|^{d-2}}dy < \infty$, then $P_x^{I,b} \sim \mathcal{W}_x$ and
$\sup_{x\in\mathbb{R}^d} H(\mathcal{W}_x;P_x^{I,b})$, $\sup_{x\in\mathbb{R}^d} H(P_x^{I,b};\mathcal{W}_x) < \infty$;
ii. If $\int_{\mathbb{R}^d} \frac{|b(y)|}{|y|^{d-2}}dy = \infty$, then $P_x^{I,b} \perp \mathcal{W}_x$.

We now show that the Fuchsian condition in Theorem 2 and Corollary 4 is sharp by showing that for any prescribed growth rate that is larger than Fuchsian, one can find a drift b growing no faster than this prescribed rate and for which the above dichotomy does not hold for $\mathcal{W}_x$ and $P_x^{I,b}$.

Theorem 3. *Let ρ be a positive, nondecreasing function on $[0,\infty)$ satisfying $\lim_{t\to\infty}\rho(t) = \infty$. Then there exists a drift vector b such that*
i. $\sup_{x\in\mathbb{R}^d} \frac{|x||b(x)|}{\rho(x)} < \infty$;
ii. $\mathcal{W}_x \ll P_x^{I,b}$;
and
iii. $H(\mathcal{W}_x;P_x^{I,b}) = \infty$.

In the one-dimensional case, we can give an explict analytic criterion for absolute continuity. We will assume without loss of generality that $\lim_{t\to\infty} X(t) = \infty, a.s.\ [P_x^{a,b}]$.

Indeed, the measure in the general case is just a convex combination of the measures obtained by conditioning on $\{\lim_{t\to\infty} X(t) = \infty\}$ and on $\{\lim_{t\to\infty} X(t) = -\infty\}$. Each of these conditioned diffusions corresponds to an h-transformed operator of the same form as the original one and thus belongs to the class of diffusions under study. Under the above assumption, the invariant σ-field is always trivial, so by Corollary 2 it follows that the mutual absolute continuity/singularity dichotomy is in effect.

Theorem 4. *Let $d = 1$ and assume that $\{\lim_{t\to\infty} X(t) = \infty\}$ a.s. $[P_x^{a,b}]$ and a.s. $[P_x^{a,\hat{b}}]$.*
i. If

$$\int_0^\infty dz\ \exp(-2\int_0^z \frac{b}{a}(u)du) \int_0^z dy\ \frac{(\hat{b}-b)^2}{a^2}(y) \exp(2\int_0^y \frac{b}{a}(u)du) < \infty,$$

then $P_x^{a,b} \sim P_x^{a,\hat{b}}$;
ii. If

$$\int_0^\infty dz\ \exp(-2\int_0^z \frac{b}{a}(u)du) \int_0^z dy\ \frac{(\hat{b}-b)^2}{a^2}(y) \exp(2\int_0^y \frac{b}{a}(u)du) = \infty,$$

then $P_x^{a,b} \perp P_x^{a,\hat{b}}$.
Furthermore,

$$H(P_x^{a,b}; P_x^{a,\hat{b}}) = \int_x^\infty dz\ \exp(-2\int_0^z \frac{b}{a}(u)du) \int_{-\infty}^z dy\ \frac{(\hat{b}-b)^2}{a^2}(y) \exp(2\int_0^y \frac{b}{a}(u)du). \tag{1.3}$$

Remark 1. It can be shown that the Green's function for $L_{a,b}$ is given by $G^{a,b}(x,y) = \frac{2}{a(y)} \int_{x\wedge y}^\infty dz \exp(-2\int_y^z \frac{b}{a}(u)du)$. Changing the order of integration in (1.3) shows that that (1.3) agrees with (1.2). Note also that since the conditions $P_x^{a,b} \sim P_x^{a,\hat{b}}$ and $P_x^{a,b} \perp P_x^{a,\hat{b}}$ are symetric with respect to the two measures, it follows that the convergence or divergence of the integral in (i) and (ii) is not affected by interchanging the roles of b and $\hat{b}$.

Remark 2. Since the lower limit in the inside integral is 0 in parts (i) and (ii) of Theorem 4 while it is $-\infty$ in (1.3), it is possible to have $P_x^{a,b} \sim P_x^{a,\hat{b}}$ but $H(P_x^{a,\hat{b}}; P_x^{a,b}) = \infty$; see example 2 below.

Here are two examples to illustrate Theorem 4.

Example 1. Let b and $\hat{b}$ be continuous functions vanishing identically on $(-\infty, 0]$ and satisfying for $x \geq 2$:

$$b(x) = \frac{k-1}{2x^l}, \text{ where } l \in [-1,1) \text{ and } k > 1, \text{ or } l = 1 \text{ and } k > 2;$$
$$\hat{b}(x) = b(x) + \frac{c}{x^\eta \log^\beta x}, \text{ where } c, \beta \in R \text{ and } \eta > -1.$$

Then

$$P_x^{1,b} \sim P_x^{1,\hat{b}} \text{ if and only if } \eta > \frac{1+l}{2} \text{ or } \eta = \frac{1+l}{2} \text{ and } \beta > \frac{1}{2}.$$

Otherwise $P_x^{1,b} \perp P_x^{1,\hat{b}}$.

Proof. The dichotomy follows from Theorem 4. It suffices to prove the claim for $\eta = \frac{1+l}{2}$ as the other cases follow easily from Theorem 1 by comparison. We claim that for some $C > 0$, the integrand in parts (i) and (ii) of Theorem 4 satisfies

$$\exp(-2\int_0^z \frac{b}{a}(u)du)\int_0^z dy\, \frac{(\hat{b}-b)^2}{a^2}(y)\exp(2\int_0^y \frac{b}{a}(u)du) \sim \frac{C}{z\log^{2\beta} z}, \text{ as } z \to \infty. \tag{1.4}$$

From this it follows that (i) holds if $\beta > \frac{1}{2}$ and (ii) holds if $\beta \le \frac{1}{2}$. To show (1.4), one writes $z\log^{2\beta} z\exp(-2\int_0^z \frac{b}{a}(u)du)\int_0^z dy\, \frac{(\hat{b}-b)^2}{a^2}(y)\exp(2\int_0^y \frac{b}{a}(u)du)$ in the form $\frac{\int_0^z dy\, \frac{(\hat{b}-b)^2}{a^2}(y)\exp(2\int_0^y \frac{b}{a}(u)du)}{z^{-1}\log^{-2\beta} z\exp(-2\int_0^z \frac{b}{a}(u)du)}$ and applies L'Hôpital's rule. The calculation is left to the reader. □

Example 2.

(a) Let $b \equiv 1$ and let $\hat{b}$ be a continuous function satisfying $b(x) = 1$, for $x \ge 1$, and $b(x) = 0$, for $x \le 0$. Since the condition $(\hat{b} - b)(X(t)) = 0$, for sufficiently large t, holds a.s. $[P_x^{1,b}]$ and a.s. $[P_x^{1,\hat{b}}]$, it follows from Theorem 1 that $P_x^{1,b} \sim P_x^{1,\hat{b}}$. Using (1.3) one can shows easily that $H(P_x^{1,b}; P_x^{1,\hat{b}}) < \infty$, but that $H(P_x^{1,\hat{b}}; P_x^{1,b}) = \infty$. Alternatively, note from (1.1) that the equality $H(P_x^{1,\hat{b}}; P_x^{1,b}) = \infty$ follows from the fact that given two distinct points $x, y \in R$, the expected hitting time of y by a Brownian motion starting from x is infinite.

(b) Now let

$$b(x) = \begin{cases} \cos x, & x < 0 \\ 1, & x \ge 0 \end{cases}$$

and let $\widehat{b}$ be as in part (a). By reasoning similar to part (a), one can show that $P_x^{1,b} \sim P_x^{1,\widehat{b}}$ and that $H(P_x^{1,b}; P_x^{1,\widehat{b}}) = H(P_x^{1,\hat{b}}; P_x^{1,b}) = \infty$.

In the case that $P_x^{a,b} \perp P_x^{a,\hat{b}}$, how does one find an event that distinguishes between the two measures in the sense that $P_x^{a,b}(A) = 1$ and $P_x^{a,\hat{b}}(A) = 0$? Here is a construction that leads to a distinguishing event in the general d-dimensional case. For the sake of simplicity, assume that a, b and $\hat{b}$ are bounded. The Radon-Nikodym derivative $\frac{dP_{x;t}^{a,\hat{b}}}{dP_{x;t}^{a,b}}$ is a nonnegative $P_x^{a,b}$-martingale with expectation equal to 1. Thus, by the martingale convergence theorem and Fatou's lemma, $\frac{dP^{a,\hat{b}}_{x;t}}{dP^{a,b}_{x;t}}$ converges a.s. $[P^{a,b}_x]$. Let $A = \{\lim_{t\to\infty} \frac{dP^{a,\hat{b}}_{x;t}}{dP^{a,b}_{x;t}} = \infty\}$. Of course, $P^{a,b}_x(A) = 0$; however from the Lebesgue decomposition theorem one finds that $P^{a,\hat{b}}_x(A) = 1$ if $P^{a,\hat{b}}_x \perp P^{a,b}_x$ ([3, Theorem 4.3.3]). Although the event A distinguishes between the two measures, it is not very illuminating—the event doesn't allow for any intuitive understanding about the supports of the two measures because it involves the long time behavior of a term with a stochastic integral which is difficult to analyze.

Returning to the one dimensional case with the assumption that $\lim_{t\to\infty} X(t) = \infty$ a.s., we will find "illuminating" distinguishing sets for three increasingly-difficult-to-distinguish cases. We begin with the following simple case.

Proposition 1. *Let $a > 0$ be bounded and continuous, and for $\gamma > 0$ and $l \in (-1, 1)$, let $b_{l,\gamma}$ be continuous, vanish identically on $(-\infty, 0]$ and satisfy* $b_{l,\gamma}(x) = \begin{cases} \frac{\gamma}{x^l}, & l \neq 0 \\ \gamma, & l = 0 \end{cases}$ *for $x > 1$. Then*

$$\lim_{t\to\infty} \frac{X^{1+l}(t)}{t} = (1 + l)\gamma, \ a.s.\ [P^{a,b_{l,\gamma}}_x].$$

Thus, letting $A_{l,\gamma} = \{\lim_{t\to\infty} \frac{X^{1+l}(t)}{t} = (1 + l)\gamma\}$, it follows that

$$P^{a,b_{l,\gamma}}_x(A_{l_1,\gamma_1}) = \begin{cases} 1, & \text{if } l = l_1 \text{ and } \gamma = \gamma_1; \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 1 does not cover the case $l = 1$, which is much more delicate. Thus, consider now the case $a = 1$ and $b(x) = \frac{k-1}{2x}$, for $x > 0$, with $k > 2$. As is well-known, the measure $P^{1,\frac{k-1}{2x}}_x$, with $x > 0$, corresponds to a transient Bessel process on $(0, \infty)$ which never reaches 0. In particular, if k is integral, then the process is the absolute value of a k-dimensional Brownian motion. The law of the iterated logarithm states that $\limsup_{t\to\infty} \frac{|X(t)|}{(2t \log\log t)^{\frac{1}{2}}} = 1$ a.s. $[\mathcal{W}_x]$ $(= [P^{I,0}_x])$. This result continues to hold when $k > 2$ is nonintegral. Since the growth rate is the same for all values of k, a simple result in the spirit of Proposition 2 is not possible for Bessel processes.

Proposition 2. *For $\rho > 0$, let A^ρ_n be the event that after hitting $((n + 1)!)^\rho$ for the first time, a path downcrosses the interval $[(n!)^\rho, ((n + 1)!)^\rho]$ before hitting $((n + 2)!)^\rho$. Then for $k > 2$,*

$$P^{1,\frac{k-1}{2x}}_x(A^\rho_n \text{ i.o.}) = \begin{cases} 0, & \text{if } \rho > \frac{1}{k-2} \\ 1, & \text{if } \rho \le \frac{1}{k-2}. \end{cases}$$

Remark. Proposition 2 shows that one can distinguish between different Bessel processes by keeping track of the amount of backtracking they do. A different result related to the problem of finding distinguishing events for Bessel processes can be found in [9, exercise X-3.20]: If $X(t)$ is a d-dimensional Brownian motion, then $\lim_{t\to\infty} \frac{1}{\log t}\int_1^t |X(s)|^{-2}ds = \frac{1}{d-2}$, *a.s.*.

The above two propositions allowed us to find distinguishing sets for two mutually singular measures in certain cases when the two drifts $b, \hat{b}$ are such that $b - \hat{b}$ and b are on the same order as $x \to \infty$. If the order of $b - \hat{b}$ is smaller than that of b, the task of distinguishing between the two measures becomes more delicate. We now consider such a case—namely, Example 1 above when $l = \eta = 1$, $c > 0$ and $\beta \in (0, \frac{1}{2}]$. This last requirement guarantees that the processes are mutually singular. We have

$$\begin{aligned} &b \text{ and } \hat{b} \text{ are continuous, vanish identically on } x \le 0, \text{ and satisfy} \\ &b(x) = \frac{k-1}{2x}, \text{ for } x \ge 2, \text{ where } k \ge 1; \\ &\hat{b}(x) = \frac{k-1}{2x} + \frac{c}{x\log^b x}, \text{ for } x \ge 2, \text{ where } k > 1, \ c > 1 \text{ and } \beta \in (0, \frac{1}{2}]. \end{aligned} \tag{1.5}$$

One can check that the method of Proposition 2 fails here: for each value of ρ, either both processes almost surely perform infinitely many downcrossings or both processes almost surely perform only finitely many downcrossings.

Instead of just checking whether or not an infinite number of downcrossings are performed, we will count the number of downcrossings. Let $N_n^{(\rho)}$ denote the number of downcrossings a path makes of the interval $[(n!)^\rho, ((n+1)!)^\rho]$ after hitting $((n+1)!)^\rho$ for the first time and before hitting $((n+2)!)^\rho$. Let $n_0^{(\rho)}(x) = \min\{n \ge 1 : ((n+1)!)^\rho \ge x\}$. It is easy to show that under $P_x^{1,b}$ and under $P_x^{1,\hat{b}}$, the random variables $\{N_n^{(\rho)}\}_{n=n_0^{(\rho)}(x)}^{\infty}$ are independent and distributed according to geometric distributions: $P_x^{1,b}(N_n^{(\rho)} = j) = (p_n^{(\rho)}(b))^j(1 - p_n^{(\rho)}(b))$ and $P_x^{1,\hat{b}}(N_n^{(\rho)} = j) = (p_n^{(\rho)}(\hat{b}))^j(1 - p_n^{(\rho)}(\hat{b}))$, for $j = 0, 1, \ldots$, where

$$p_n^{(\rho)}(B) = \frac{\int_{((n+1)!)^\rho}^{((n+2)!)^\rho} \exp(-2\int_0^y B(s)ds)dy}{\int_{(n!)^\rho}^{((n+2)!)^\rho} \exp(-2\int_0^y B(s)ds)dy}. \tag{1.6}$$

(A proof of these facts will be given in the proof of Proposition 3.)

Proposition 3. *Consider the measures $P_x^{1,b}$ and $P_x^{1,\hat{b}}$, where b and $\hat{b}$ are given by* (1.5), *with*

$$\beta \in (0, \frac{1}{3}].$$

For $\rho > 0$, let $N_n^{(\rho)}$ denote the number of downcrossings of the interval $[(n!)^\rho, ((n+1)!)^\rho]$ made by a path after it hits $((n+1)!)^\rho$ for the first time and before it hits $((n+2)!)^\rho$, and let $p_n^{(\rho)}(b)$ and $p_n^{(\rho)}(\hat{b})$ be as in (1.6). Then $p_n^{(\rho)}(\hat{b}) < p_n^{(\rho)}(b)$, so $0 < \frac{p_n^{(\rho)}(\hat{b})}{p_n^{(\rho)}(b)} < 1$. Fix

$$\rho = \frac{1-2\beta}{k-2}$$

and let $n_0(x) = \min\{n \geq 1 : ((n+1)!)^{\frac{1-2\beta}{k-2}} \geq x\}$. Then

$$\prod_{n=n_0(x)}^{\infty} \left(\frac{1-p_n^{(\rho)}(\hat{b})}{1-p_n^{(\rho)}(b)}\right)\left(\frac{p_n^{(\rho)}(\hat{b})}{p_n^{(\rho)}(b)}\right)^{N_n^{(\rho)}} = \begin{cases} 0, \ a.s. \ [P_x^{1,b}] \\ \infty \ a.s. \ [P_x^{1,\hat{b}}]. \end{cases} \tag{1.7}$$

Remark 1. Unfortunately, this method works only for $\beta \in (0, \frac{1}{3}]$ and not for the entire interval $(0, \frac{1}{2}]$ where $P_x^{1.b}$ and $P_x^{1,\hat{b}}$ are mutually singular.

Remark 2. The intuition behind (1.7) is as follows. Since $\frac{p_n^{(\rho)}(\hat{b})}{p_n^{(\rho)}(b)} \in (0,1)$, it follows that the smaller the $\{N_n^{(\rho)}\}$ are, the larger the expression on the left hand side of (1.7) is. Note that since $\hat{b}$ is larger than b, the $\{N_n^{(\rho)}\}$ tend to be larger under $P_x^{1,\hat{b}}$ than under $P_x^{1,b}$.

We prove Theorem 1 and Corollaries 1-3 in section 2, Theorems 2 and 3 in section 3, Theorem 4 in section 4, and Propositions 1-3 in section 5.

2. Proofs of Theorem 1 and Corollaries 1-3

We will need two lemmas for the proof of Theorem 1.

Lemma 1. *Let $(\mathcal{M}, \mathcal{Y})$ be a measurable space and $\{\mathcal{Y}_n\}_{n\geq 0}$ a filtration, satisfying $\sigma(\bigcup_{n\geq 0} \mathcal{Y}_n) = \mathcal{Y}$. Let P and Q be probability measures on $(\mathcal{M}, \mathcal{Y})$ such that $Q|_{\mathcal{Y}_n} \ll P|_{\mathcal{Y}_n}$ for all $n \in \mathbb{N}$. Then*

1. *$Q \ll P$ iff $\limsup_{n\to\infty} \frac{dQ|_{\mathcal{Y}_n}}{dP|_{\mathcal{Y}_n}} < \infty$, a.s. $[Q]$;*
2. *$Q \perp P$ iff $\limsup_{n\to\infty} \frac{dQ|_{\mathcal{Y}_n}}{dP|_{\mathcal{Y}_n}} = \infty$, a.s. $[Q]$ iff $\lim_{n\to\infty} \frac{dQ|_{\mathcal{Y}_n}}{dP|_{\mathcal{Y}_n}} = 0$, a.s. $[P]$;*
3. *If $Q \ll P$,*
$$H(Q;P) = \lim_{n\to\infty} H(Q|_{\mathcal{Y}_n}; P|_{\mathcal{Y}_n}).$$

Proof. The Lebesgue decomposition of Q with respect to P is given by $Q = Q^c + Q^\perp$, where $Q^c \ll P$ and $Q^\perp \perp P$. This decomposition is unique. Since $Q|_{\mathcal{Y}_n} \ll P|_{\mathcal{Y}_n}$ for all $n \in \mathbb{N}$, then (see, for example, [3, Theorem 4.3.3])

$$Q^c(A) = \int_A \lim_{n\to\infty} \frac{dQ|_{\mathcal{Y}_n}}{dP|_{\mathcal{Y}_n}} dP; \tag{2.1}$$

$$Q^\perp(A) = Q(\{\limsup_{n\to\infty} \frac{dQ|_{\mathcal{Y}_n}}{dP|_{\mathcal{Y}_n}} = \infty\} \bigcap A), \ A \in \mathcal{Y}. \tag{2.2}$$

Therefore 1 and 2 follow immediately from (2.1) and (2.2).

For the last assertion, note that it is easy to see that

$$Z_n = \mathbb{E}^P[\frac{dQ}{dP}|\mathcal{Y}_n] = \frac{dQ|_{\mathcal{Y}_n}}{dP|_{\mathcal{Y}_n}}, \text{ a.s. } [P].$$

Clearly, $\{Z_n\}$ is a P-uniformly integrable martingale. Since $\{\mathcal{Y}_n\}$ generate $\mathcal{Y}$, $\lim_{n\to\infty} Z_n = \frac{dQ}{dP}$, a.s. $[P]$. The function $x\log x$ is bounded from below. Therefore, by Fatou's lemma

$$\liminf_{t\to\infty} H(Q|_{\mathcal{Y}_n}; P|_{\mathcal{Y}_n}) = \liminf_{n\to\infty} \mathbb{E}^P Z_n \log Z_n \geq \mathbb{E}^P \frac{dQ}{dP}\log\frac{dQ}{dP} = H(Q;P).$$

However, Jensen's inequality for conditional expectation implies

$$\mathbb{E}^P[\frac{dQ}{dP}\log\frac{dQ}{dP}|\mathcal{Y}_n] \geq Z_n \log Z_n, \text{ a.s. } [P|_{\mathcal{Y}_n}],$$

therefore $H(Q;P) \geq H(Q|_{\mathcal{Y}_n}; P|_{\mathcal{Y}_n})$, completing the proof. □

Lemma 2 (Exercise IV-3.26 [9]). *Let $M(t)$ be a continuous local-martingale with respect to $(\Omega, \mathcal{F}_t)$. $\langle M\rangle(t)$ is the quadratic variation process associated with $M(t)$, and $\langle M\rangle(\infty) = \lim_{t\to\infty}\langle M\rangle(t)$. If*

$$\mathcal{E}(M)(t) = \exp\left(M(t) - \frac{1}{2}\langle M\rangle(t)\right),$$

then

$$\{\lim_{t\to\infty}\mathcal{E}(M)(t) = 0\} = \{\langle M\rangle(\infty) = \infty\}, \text{ a.s.}$$

Proof. Since $\mathcal{E}(M)(t)$ is a non-negative continuous local martingale, it follows immediately from Fatou's lemma for conditional expectation that in fact $\mathcal{E}(M)(t)$ is a supermartingale. Hence it converges almost surely. As

$$\mathcal{E}(M) = \mathcal{E}(\frac{M}{2})^2 \exp(-\frac{1}{4}\langle M\rangle),$$

$$\{\langle M\rangle(\infty) = \infty\} \subseteq \{\lim_{t\to\infty}\mathcal{E}(M)(t) = 0\}, \text{ a.s.}$$

The reverse inclusion is achieved similarly from the identity
$\mathcal{E}(-M) = \mathcal{E}(M)^{-1}\exp(\langle M\rangle)$. □

Proof of Theorem 1. Fix $x \in \mathbb{R}^d$. Let $B_n = \{y \in \mathbb{R}^d : |y| < n\}$ and let $\tau_n = \inf\{t \geq 0 : X(t) \notin B_n\}$. We will denote by $P^{a,b}_{x;\tau_n}$ ($P^{a,\hat{b}}_{x;\tau_n}$) the restriction of $P^{a,b}_x$ ($P^{a,\hat{b}}_x$) to $\mathcal{F}_{\tau_n}$. Let

$$M(t) = \int_0^t \langle a^{-1}(\widehat{b} - b)(X(s)), dX(s)\rangle - \int_0^t \langle a^{-1}(\widehat{b} - b), b\rangle(X(s))ds. \quad (2.3)$$

From [7, Theorem 1.5.1], for fixed n, $M(\tau_n \wedge t)$ is a $P_x^{a,b}$-martingale, and

$$\langle M\rangle(t) = \int_0^t \langle \widehat{b} - b, a^{-1}(\widehat{b} - b)\rangle(X(s))ds$$

is the associated quadratic variation process.

By the Girsanov transformation, $P^{a,b}_{x;\tau_n\wedge n} \sim P^{a,\hat{b}}_{x;\tau_n\wedge n}$, and

$$\frac{dP^{a,\hat{b}}_{x;\tau_n\wedge n}}{dP^{a,b}_{x;\tau_n\wedge n}} = \left(\frac{dP^{a,b}_{x;\tau_n\wedge n}}{dP^{a,\hat{b}}_{x;\tau_n\wedge n}}\right)^{-1} = \frac{1}{\mathcal{E}(-M)(\tau_n \wedge n)}, \text{ a.s. } [P_x^{a,\hat{b}}],$$

which implies

$$\{\limsup_{n\to\infty} \frac{dP^{a,\hat{b}}_{x;\tau_n\wedge n}}{dP^{a,b}_{x;\tau_n\wedge n}} = \infty\} = \{\lim_{n\to\infty} \mathcal{E}(-M)(\tau_n \wedge n) = 0\}, \text{ a.s. } [P_x^{a,\hat{b}}].$$

According to Lemma 2, this condition is equivalent to the statement

$$\{\limsup_{n\to\infty} \frac{dP^{a,\hat{b}}_{x;\tau_n\wedge n}}{dP^{a,b}_{x;\tau_n\wedge n}} = \infty\} = \{\langle M\rangle(\infty) = \infty\}, \text{ a.s. } [P_x^{a,\hat{b}}].$$

Using this along with Lemma 1 proves parts (i) and (ii) of the theorem.

For part (iii), similarly to (2.3) we let

$$\widehat{M}(t) = \int_0^t \langle a^{-1}(\widehat{b} - b)(X(s)), dX(s)\rangle - \int_0^t \langle a^{-1}(\widehat{b} - b), \widehat{b}\rangle(X(s))ds.$$

Then $\widehat{M}(\tau_n \wedge n)$ is a $P_x^{a,\hat{b}}$-martingale, and it follows that

$$\begin{aligned} H(P^{a,\hat{b}}_{x;\tau_n\wedge n}; P^{a,b}_{x;\tau_n\wedge n}) &= \mathbb{E}_x^{a,\hat{b}} \log \frac{dP^{a,\hat{b}}_{x;\tau_n\wedge n}}{dP^{a,b}_{x;\tau_n\wedge n}} \\ &= \mathbb{E}_x^{a,\hat{b}} \left(\widehat{M}(\tau_n \wedge n) + \frac{1}{2}\int_0^{\tau_n\wedge n} \langle \widehat{b} - b, a^{-1}(\widehat{b} - b)\rangle(X(s))ds\right) \\ &= \frac{1}{2}\mathbb{E}_x^{a,\hat{b}} \int_0^{\tau_n\wedge n} \langle \widehat{b} - b, a^{-1}(\widehat{b} - b)\rangle(X(s))ds. \end{aligned}$$

Letting $n \to \infty$, it follows from the monotone convergence theorem and Lemma 1 that (1.1) holds. □

Proof of Corollary 1. Let $x, y \in \mathbb{R}^d$, let $D \subset \mathbb{R}^d$ be a bounded domain containing x and$y$ and with smooth a boundary, a let $\tau_D = \inf\{t \geq 0 : X(t) \notin D\}$. As is well know, both $P_x(X(\tau_D)\in\cdot)$ and $P_y(X(\tau_D)\in\cdot)$ are mutually absolutely continuous with respect to Lebesgue measure on ∂D. Thus, since

$$\int_0^{\tau_D}\langle\hat b-b,a^{-1}(\hat b-b)\rangle(X(s))ds<\infty \ a.s.\ [P_x^{a,b}] \text{ and } a.s.\ [P_x^{a,\hat b}],$$

it follows from the strong Markov property and Theorem 1-i that

$$P_x^{a,b}\perp P_x^{a,\hat b}\Leftrightarrow P_y^{a,b}\perp P_y^{a,\hat b}\Leftrightarrow \int_0^{\tau_D}\langle\hat b-b,a^{-1}(\hat b-b)\rangle(X(s))ds=\infty\ a.s.\ [P_z^{a,b}] \text{ or } a.s.\ [P_x^{a,\hat b}],$$

for almost all $z\in\partial D$. Similarly, it follows that

$$P_x^{a,\hat b}\ll P_x^{a,b}\Leftrightarrow P_y^{a,\hat b}\ll P_y^{a,b}\Leftrightarrow \int_0^{\tau_D}\langle\hat b-b,a^{-1}(\hat b-b)\rangle(X(s))ds<\infty\ a.s.\ [P_z^{a,\hat b}],$$

for almost all $z\in\partial D$. □

Proof of Corollary 2. The event

$$\int_0^{\infty}\langle\hat b-b,a^{-1}(\hat b-b)\rangle(X(s))ds=\infty$$

is invariant. Thus, if the invariant σ-field is trivial for both $P_x^{a,b}$ and $P_x^{a,\hat b}$, then then either the above event or its complement occurs $a.s.$ $[P_x^{a,b}]$ and $a.s.$ $[P_x^{a,\hat b}]$. The dichotomy in the corollary now follows from Theorem 1-i and ii. □

Proof of Corollary 3. Let $x_0\in\mathbb{R}^d$ and $r>0$ be such that $\langle\widehat b-b,a^{-1}(\widehat b-b)\rangle>\epsilon>0$ on $B_r(x_0)$. Define stopping times as follows:

$$\begin{aligned}\sigma_1&=\inf\{t\ge 0: X(t)\in B_{r/2}(x_0)\}\\ \tau_n&=\inf\{t\ge 0: X(\sigma_n+t)\notin B_r(x_0)\}\\ \sigma_n&=\sigma_1\circ\Theta_{\tau_n},\end{aligned}$$

where $\Theta_t\omega(\cdot)=\omega(t+\cdot)$ is the standard shift operator. The Hölder continuity conditions imposed on the coefficients a and b imply the existence of a unique solution to

$$\begin{cases}(\mathrm{L}-\epsilon)u=0 & \text{in } B_r(x_0),\\ u=1 & \text{on } \partial B_r(x_0).\end{cases}$$

By the Feynman-Kac formula and the bounded convergence theorem,

$$\begin{aligned}u(x)&=\mathbb{E}_x^{a,b}\exp(-\epsilon\tau_1)\\ &>\mathbb{E}_x^{a,b}\exp\left(-\int_0^{\tau_1}\langle\widehat b-b,a^{-1}(\widehat b-b)\rangle(X(s))ds\right),\ x\in B_{r/2}(x_0).\end{aligned}$$

By the strong maximum principle, [7, Theorem 3.2.6],

$$\sup_{x\in\ \partial B_{r/2}(x_0)} u(x) = 1-\delta,\ \text{for some}\ \ \delta\in(0,1).$$

Let

$$v(x) = \mathbb{E}_x^{a,b}\exp\left(-\int_0^\infty \langle \widehat{b}-b, a^{-1}(\widehat{b}-b)\rangle(X(s))ds\right),\ x\in D.$$

We will show that $v=0$. From this it will follow that $\int_0^\infty\langle\widehat{b}-b,a^{-1}(\widehat{b}-b)\rangle(X(s))ds=\infty$ a.s. $[P_x^{a,b}]$, and then by Theorem 1-i, we have $P_x^{a,b}\perp P_x^{a,\hat{b}}$.

For $N\in\mathbb{N}$, successive applications of the strong Markov property show that

$$\begin{aligned} v(x) &\leq \mathbb{E}_x^{a,b}\prod_{n=1}^N \exp\left(-\int_{\sigma_n}^{\tau_n}\langle\widehat{b}-b,a^{-1}(\widehat{b}-b)\rangle(X(s))ds\right) \\ &= \mathbb{E}_x^{a,b}\prod_{n=1}^N \mathbb{E}_{X(\sigma_n)}^{a,b}\exp\left(-\int_0^{\tau_1}\langle\widehat{b}-b,a^{-1}(\widehat{b}-b)\rangle(X(s))ds\right) \\ &\leq (1-\delta)^N \underset{N\to\infty}{\to} 0. \end{aligned}$$

□

3. Proofs of Theorems 2 and 3

Proof of Theorem 2. By [7, Theorem 8.3.1], the cone

$$\{u\in C^2(\mathbb{R}^d): \mathrm{L}^{a,b}\,u=0 \text{ in } \mathbb{R}^d \text{ and } u>0\}$$

is one dimensional. It follows from [7, Theorems 8.3.1 and 9.1.2], that the invariant σ-field, $\mathcal{I}$, is trivial with respect to $P_x^{a,b}$ and $P_x^{a,\widehat{b}}$. The event

$$\{\int_0^\infty\langle\widehat{b}-b,a^{-1}(\widehat{b}-b)\rangle(X(s))ds<\infty\}\in\mathcal{I}.$$

Thus, in light of Theorem 1, either $P_x^{a,b}\sim P_x^{a,\widehat{b}}$ or $P_x^{a,b}\perp P_x^{a,\widehat{b}}$. To prove the theorem, we will assume that $P_x^{a,b}\sim P_x^{a,\widehat{b}}$ and show that $\sup_{x\in\mathbb{R}^d}H(P_x^{a,b};P_x^{a,\widehat{b}})<\infty$. The same type of argument also gives $\sup_{x\in\mathbb{R}^d}H(P_x^{a,\widehat{b}};P_x^{a,b})<\infty$.

Since $P_x^{a,b}\sim P_x^{a,\widehat{b}}$, by Theorem 1 we have

$$\int_0^\infty\langle\widehat{b}-b,a^{-1}(\widehat{b}-b)\rangle(X(s))ds<\infty \text{ a.s. } [P_x^{a,b}],\ x\in\mathbb{R}^d. \tag{3.1}$$

Let $B_n = \{y \in \mathbb{R}^d : |y| < n\}$ and let $\tau_n = \inf\{t \geq 0 : X(t) \notin B_n\}$. The locally Hölder continuity of the coefficients ensures that there exits a sequence of functions $\{u_n\}_{n\in\mathbb{N}} \subset C^{2,\gamma}(\overline{B_n})$ which satisfy

$$\begin{cases} \left(\mathrm{L}_{a,b} - \langle \widehat{b} - b, a^{-1}(\widehat{b} - b)\rangle\right) u_n = 0 & \text{in } B_n \\ u_n = 1 & \text{on } \partial B_n \end{cases}$$

In particular, the Feynman-Kac formula shows that

$$u_n(x) = \mathbb{E}_x^{a,b} u_n(X(t \wedge \tau_n)) \exp\left(-\int_0^{t\wedge\tau_n} \langle \widehat{b} - b, a^{-1}(\widehat{b} - b)\rangle(X(s))ds\right),$$

and from bounded convergence we then obtain

$$u_n(x) = \mathbb{E}_x^{a,b} \exp\left(-\int_0^{\tau_n} \langle \widehat{b} - b, a^{-1}(\widehat{b} - b)\rangle(X(s))ds\right).$$

Consequently, let

$$u(x) = \lim_{n\to\infty} u_n(x) = \mathbb{E}_x^{a,b} \exp\left(-\int_0^{\infty} \langle \widehat{b} - b, a^{-1}(\widehat{b} - b)\rangle(X(s))ds\right). \tag{3.2}$$

Fix $m \in \mathbb{N}$. The global Schauder estimates ([4, Theorem 6.6]) yield

$$\|u_n\|_{2,\gamma;B_m} \leq c(\|u_n\|_{0;B_m} + 1) \leq 2c, \;\; n \geq m$$

where $c = c(d, \gamma, K_1, K_3(B_m)) > 0$, K_1 is as in Definition 1 and

$$K_3(B_m) = \sum_{i,j=1}^{d} \|a_{ij}\|_{0,\gamma;B_m} + \sum_{i=1}^{d} \|b_i\|_{0,\gamma;B_m} + \|\langle \widehat{b} - b, a^{-1}(\widehat{b} - b)\rangle\|_{0,\gamma;B_m} < \infty.$$

Hence, $\{u_n\}_{n\geq m}$ and the sequences of the partial derivatives up to the second order are bounded in the $\|\cdot\|_{0;B_m}$-norm and equicontinuous on B_m. By the Arzela-Ascoli theorem, we can extract a subsequence, converging to u in the $\|\cdot\|_{2;B_m}$-norm. Hence, $u \in C^2(\mathbb{R}^d)$, and $(\mathrm{L} - \langle \widehat{b} - b, a^{-1}(\widehat{b} - b)\rangle)u = 0$. By (3.1), $0 < u \leq 1$. From the Feynman-Kac formula, $u(X(t)) \exp\left(-\int_0^t \langle \widehat{b} - b, a^{-1}(\widehat{b} - b)\rangle(X(s))ds\right)$ is a bounded $P_x^{a,b}$-martingale. Therefore, in light of (3.1) and the martingale convergence theorem, $\lim_{t\to\infty} u(X(t))$ exists, a.s. $[P_x^{a,b}]$. Furthermore,

$$u(x) = \mathbb{E}_x^{a,b} u(X(t)) \exp\left(-\int_0^{t} \langle \widehat{b} - b, a^{-1}(\widehat{b} - b)\rangle(X(s))ds\right). \tag{3.3}$$

From the bounded convergence theorem, (3.2) and (3.3) we see that

$$u(x) = \mathbb{E}_x^{a,b} \exp\left(-\int_0^{\infty} \langle \widehat{b} - b, a^{-1}(\widehat{b} - b)\rangle(X(s))ds\right) =$$

$$\mathbb{E}_x^{a,b} \lim_{t\to\infty} u(X(t)) \exp\left(-\int_0^{\infty} \langle \widehat{b} - b, a^{-1}(\widehat{b} - b)\rangle(X(s))ds\right).$$

With (3.1) again, we conclude that

$$\lim_{t\to\infty} u(X(t)) = 1, \text{ a.s. } [P_x^{a,b}]. \tag{3.4}$$

Since u is bounded, $u(X(t))-\int_0^t(\langle\widehat{b}-b,a^{-1}(\widehat{b}-b)\rangle u)(X(s))ds$ is a $P_x^{a,b}$-martingale. Therefore,

$$u(x) = 1 - \mathbb{E}_x^{a,b}\int_0^\infty (\langle\widehat{b}-b,a^{-1}(\widehat{b}-b)\rangle u)(X(s))ds. \tag{3.5}$$

We intend to show now that u is bounded from below by a positive constant. For this, we apply the technique used in [7, Theorem 8.3.1]. For $R>0$, let

$$\mathrm{L}_{a,b}^R = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(Rx)\frac{\partial}{\partial x_i\partial x_j} + \sum_{i=1}^d Rb_i(Rx)\frac{\partial}{\partial x_i} - R^2(\langle\widehat{b}-b,a^{-1}(\widehat{b}-b)\rangle(Rx)).$$

Let $u_R(x)=u(Rx)$ and $D'=\{x\in\mathbb{R}^d : 1<|x|<2\}$. It is easy to see that

$$\begin{cases} \mathrm{L}_{a,b}^R\, u_R = 0 & \text{in } \mathbb{R}^d, \text{ and} \\ u_R > 0 & \text{in } \mathbb{R}^d. \end{cases}$$

Since $\mathrm{L}_{a,b}$ is Fuchsian, the coefficients of $\mathrm{L}_{a,b}^R$ are uniformly bounded in R and the diffusion matrix of $\mathrm{L}_{a,b}^R$ is uniformly elliptic in $\mathbb{R}^d$, uniformly in R. Thus, by Harnack's inequality there exists $c=c(K_2/K_1,d,D')>0$, independent of R, such that for all $R>0,\ x,y\in D'$, $u_R(x)\ge cu_R(y)$. This is equivalent to

$$u(x)\ge cu(y),\ \ R<|x|,|y|<2R \tag{3.6}$$

Since u is positive and continuous in $\mathbb{R}^d$, in order to prove that $\inf_{x\in\mathbb{R}^d}u(x)>0$, we have to show that $\liminf_{|x|\to\infty}u(x)>0$. By (3.4), we can fix $\omega_0\in\Omega$ such that

$$\lim_{t\to\infty} u(X(t,\omega_0)) = 1. \tag{3.7}$$

Let $\{y_n\}\subset\mathbb{R}^d$ satisfy $|y_n|\uparrow\infty$. The continuity of the paths implies the existence of a sequence $\{t_n\}\subset[0,\infty)$ such that $|X(t_n,\omega_0)|=y_n$. By (3.7), there exists $N\in\mathbb{N}$, such that for $n\ge N$, $u(X(t_n))>1/2$. By (3.6), $u(y_n)\ge c/2$, so $\inf_{n\ge N}u(y_n)\ge c/2$. Since $\{y_n\}$ is arbitrary, $\liminf_{|x|\to\infty}u(x)>0$. Letting $\epsilon=\inf_{x\in\mathbb{R}^d}u(x)>0$, we obtain from (3.5)

$$\begin{aligned} \mathbb{E}_x^{a,b}\int_0^\infty \langle\widehat{b}-b,a^{-1}(\widehat{b}-b)\rangle(X(s))ds &\le \frac{1}{\epsilon}\mathbb{E}_x^{a,b}\int_0^\infty \langle\widehat{b}-b,a^{-1}(\widehat{b}-b)\rangle u(X(s))ds \\ &= \frac{1-u(x)}{\epsilon} \end{aligned}$$

Therefore, by Theorem 1, $\sup_{x\in\mathbb{R}^d}H(P_x^{a,b};P_x^{a,\widehat{b}})<\frac{1}{\epsilon}$. □

Proof of Theorem 3. We will assume, without loss of generality that $x = 0$. We first construct the candidate for the drift coefficient $\widehat{b}$. Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$. Set $x_1 = e_1$, and define inductively

$$\begin{cases} R_n = 3^{-n}|x_n| \\ x_{n+1} \in \{te_1 : t > 4|x_n|,\ \rho(t/2) > 2^{d(n+1)}\}, n \in \mathbb{N}. \end{cases} \tag{3.8}$$

The balls $\{B_{R_n}(x_n)\}_{n\in\mathbb{N}}$ are disjoint, because

$$|x_{n+1}| - |x_n| > \frac{3}{4}|x_{n+1}| = \frac{3^{n+2}R_{n+1}}{4} > \frac{9}{8}(R_{n+1} + R_n).$$

Define

$$\widehat{b}(x) = \begin{cases} \frac{\rho(|x_n|-R_n)x}{|x|^2} & x \in B_{R_n/2}(x_n) \text{ for some } n \in \mathbb{N}, \\ 0 & x \notin \bigcup_{n\in\mathbb{N}} B_{R_n}(x_n). \end{cases}$$

Whenever $x \in B_{R_n/2}(x_n)$, by definition we have $|x||\widehat{b}(x)| \le \rho(|x_n| - R_n)$; thus

$$\sup_{x\in\mathbb{R}^d} \frac{|x||\widehat{b}(x)|}{\rho(|x|)} \le 1. \tag{3.9}$$

The monotonicity of ρ allows the extension of $\widehat{b}$ to a smooth function on $\mathbb{R}^d$, supported on $\bigcup_{n\in\mathbb{N}} B_{R_n}(x_n)$, while (3.9) remains true.

We now prove that $\mathcal{W}_0 \ll P_0^{I,\widehat{b}}$ and $H(\mathcal{W}_0; P_0^{I,\widehat{b}}) = \infty$. Below, C is a positive constant which may vary from line to line. Recall that $G(x, y) = C|y - x|^{2-d}$. Since

$$u_n(x) = \left(\frac{R_n}{|x - x_n|}\right)^{d-2}$$

satisfies

$$\begin{cases} \frac{1}{2}\triangle u_n = 0 & \text{in } \mathbb{R}^d \backslash \{x_n\}, \\ u_n > 0 & \text{in } \mathbb{R}^d \backslash \{x_n\}, \\ u_n = 1 & \text{on } \partial B_{R_n}(x_n) \text{ and } \lim_{|x|\to\infty} u_n(x) = 0 \end{cases}$$

we have

$$\mathcal{W}_0(\{X(\cdot) \text{ hits } B_{R_n}(x_n)\}) = u_n(0) = \left(\frac{R_n}{|x_n|}\right)^{d-2}.$$

Thus

$$\sum_{n=1}^{\infty} \mathcal{W}_0(\{X(\cdot) \text{ hits } B_{R_n}(x_n)\}) = \sum_{n=1}^{\infty} 3^{-(d-2)n} < \infty,$$

which, according to the Borel-Cantelli lemma, implies that the process $[\mathcal{W}_0]$-a.s. hits only finitely many balls. Therefore, since $\widehat{b}$ is locally bounded and the process is transient, we have

$$\int_0^\infty |\widehat{b}|^2(X(s))ds < \infty, \text{ a.s. } [\mathcal{W}_0].$$

Hence by Theorem 1 $\mathcal{W}_0 \ll P_0^{I,\widehat{b}}$.

In order to estimate the relative entropy we shall use the representation (1.2). Recalling the definition of $\widehat{b}$, and recalling that the Green's function for d-dimensional Brownian motion is given by $C|y-x|^{2-d}$, we have

$$\begin{aligned} H(\mathcal{W}_0; P_0^{I,\widehat{b}}) &= C\int_{\mathbb{R}^d} |\widehat{b}(y)|^2 |y|^{2-d} dy \\ &\geq C\sum_{n=1}^{\infty} |\widehat{b}|^2(x_n + R_n/2) \int_{B_{R_n/2}(x_n)} |y|^{2-d} dy \equiv (*) \end{aligned}$$

Let ω_d denote the volume of the unit ball in $\mathbb{R}^d$. Since $|y|^{2-d}$ is harmonic in $B_{R_n}(x_n)$, the mean value property implies

$$\begin{aligned} (*) &= C\omega_d \sum_{n=1}^{\infty} |\widehat{b}|^2(|x_n| + R_n/2)|x_n|^{2-d} R_n^d \\ &= C\omega_d \sum_{n=1}^{\infty} \rho^2(|x_n| - R_n)\left(\frac{|x_n|}{|x_n| + R_n/2}\right)^2 \left(\frac{R_n}{|x_n|}\right)^d \\ &\geq C\omega_d \sum_{n=1}^{\infty} \rho^2(|x_n|/2)\left(\frac{R_n}{x_n}\right)^d \\ &\geq C\omega_d \sum_{n=1}^{\infty} (4/3)^{dn} = \infty, \end{aligned}$$

where the last two lines follow from (3.8). □

4. Proof of Theorem 4

We begin with some preliminaries. For $x_1 < x_2$, let

$$\sigma_{x_1,x_2} = \inf\{t \geq 0 : X(t) = x_1 \text{ or } X(t) = x_2\},$$

and

$$\sigma_x = \inf\{t \geq 0 : X(t) = x\}.$$

Recall ([7, Theorem 5.1.1]) that the assumption $\lim_{t\to\infty} X(t) = \infty$, a.s. $[P_x^{a,b}]$, has an analytic equivalent. Namely, $\lim_{t\to\infty} X(t) = \infty$, a.s. $[P_x^{a,b}]$ if and only if

$$\begin{cases} \int_{-\infty}^{0} \exp\left(-2\int_0^y b/a(s)ds\right) dy = \infty, \text{ and} \\ \int_0^{\infty} \exp\left(-2\int_0^y b/a(s)ds\right) dy < \infty. \end{cases} \tag{4.1}$$

Let

$$v_{z_1,z_2}(x) = \frac{\int_{z_1}^{x} \exp\left(-2\int_0^y b/a(s)ds\right) dy}{\int_{z_1}^{z_2} \exp\left(-2\int_0^y b/a(s)ds\right) dy}. \tag{4.2}$$

Then

$$\begin{cases} \mathrm{L}_{a,b}\, v_{z_1,z_2} = 0 & \text{in } (z_1, z_2), \\ v_{z_1,z_2} > 0 & \text{in } (z_1, z_2), \\ v_{z_1,z_2}(z_1) = 0, & v_{z_1,z_2}(z_2) = 1. \end{cases}$$

As a result,

$$P_x^{a,b}(\sigma_{z_1} > \sigma_{z_2}) = \mathbb{E}_x^{a,b} v_{z_1,z_2}(X(\sigma_{z_1,z_2})) = v_{z_1,z_2}(x). \tag{4.3}$$

Since we assume no explosion, $\lim_{z_2\to\infty} \sigma_{z_2} = \infty$, a.s. $[P_x^{a,b}]$, which implies that

$$P_x^{a,b}(\{\sigma_{z_1} = \infty\}) = \lim_{z_2\to\infty} v_{z_1,z_2}(x) > 0, \tag{4.4}$$

by (4.1). Similarly, $\lim_{z_1\to-\infty} P_x^{a,b}(\{\sigma_{z_1} = \infty\}) = 1$, so with $\epsilon > 0$ given,

$$\exists r \in (-\infty, x), \quad \text{such that } P_x^{a,b}(\{\sigma_r = \infty\}) > 1 - \epsilon. \tag{4.5}$$

Proof of Theorem 4. We begin by showing that if

$$\int_0^\infty \exp(-2\int_0^y b/a(s)ds) \left(\int_0^y (\widehat{b} - b)^2/a^2(z) \exp(2\int_0^z b/a(s)ds)dz \right) dy = \infty, \tag{4.6}$$

then $P_x^{a,\widehat{b}} \perp P_x^{a,b}$.

Fix $y_0 \in \mathbb{R}$ and let

$$u(x) = 2\int_0^x \exp(-2\int_0^y b/a(s)ds) \left(\int_0^y (\widehat{b} - b)^2/a^2(z) \exp(2\int_0^z b/a(s)ds)dz \right) dy. \tag{4.7}$$

Note that u satisfies $\mathrm{L}_{a,b}\, u = \frac{(\widehat{b}-b)^2}{a}$ in $\mathbb{R}$. Fix $x \in \mathbb{R}$, let $\epsilon > 0$ be arbitrarily chosen and pick r satisfying (4.5). By choosing 0 properly, we may assume that

$$\inf_{z\in[r,\infty)} u(z) > 0. \tag{4.8}$$

Now,

$$\begin{cases} \left(\mathrm{L}_{a,b} - \frac{(\widehat{b}-b)^2}{au}\right) u = 0 & \text{in } (z,\infty); \\ u > 0 & \text{in } (z,\infty). \end{cases}$$

Let $w \in (x,\infty)$ be arbitrary. From the Feynman-Kac formula and the bounded convergence theorem we obtain

$$\begin{aligned} u(x) &= \mathbb{E}_x^{a,b} u(X(\sigma_{r,w})) \exp\left(-\int_0^{\sigma_{r,w}} \frac{(\widehat{b}-b)^2}{au}(X(s))ds\right) \\ &= u(r)\mathbb{E}_x^{a,b} \mathbf{1}_{\{\sigma_r<\sigma_w\}} \exp\left(-\int_0^{\sigma_r} \frac{(\widehat{b}-b)^2}{au}(X(s))ds\right) \\ &\quad + u(w)\mathbb{E}_x^{a,b} \mathbf{1}_{\{\sigma_r>\sigma_w\}} \exp\left(-\int_0^{\sigma_w} \frac{(\widehat{b}-b)^2}{au}(X(s))ds\right). \end{aligned}$$

Dividing by $u(w)$ gives

$$\begin{aligned} \frac{u(x)}{u(w)} &= \frac{u(r)}{u(w)}\mathbb{E}_x^{a,b} \mathbf{1}_{\{\sigma_r<\sigma_w\}} \exp\left(-\int_0^{\sigma_r} \frac{(\widehat{b}-b)^2}{au}(X(s))ds\right) \\ &+ \mathbb{E}_x^{a,b} \mathbf{1}_{\{\sigma_r>\sigma_w\}} \exp\left(-\int_0^{\sigma_w} \frac{(\widehat{b}-b)^2}{au}(X(s))ds\right). \end{aligned} \tag{4.9}$$

By (4.6) and (4.1) we have $\lim_{w\to\infty} u(w) = \infty$. Thus, letting $w \to \infty$ in (4.9) gives

$$0 = \mathbb{E}_x^{a,b} \mathbf{1}_{\{\sigma_r=\infty\}} \exp\left(-\int_0^{\infty} \frac{(\widehat{b}-b)^2}{au}(X(s))ds\right).$$

And by (4.8), in fact

$$0 = \mathbb{E}_x^{a,b} \mathbf{1}_{\{\sigma_r=\infty\}} \exp\left(-\int_0^{\infty} \frac{(\widehat{b}-b)^2}{a}(X(s))ds\right).$$

The choice of r then implies that

$$P_x^{a,b}(\{\int_0^\infty (\widehat{b}-b)^2/a(X(s))ds = \infty\}) > 1-\epsilon.$$

As ϵ is arbitrary, $\int_0^\infty (\widehat{b}-b)^2/a(X(s))ds = \infty$, a.s. $[P_x^{a,b}]$, and $P_x^{a,\widehat{b}} \perp P_x^{a,b}$ follows from Theorem 1-ii.

We now show that if

$$\int_0^\infty \exp(-2\int_0^y b/a(s)ds)\left(\int_0^y (\widehat{b}-b)^2/a^2(z)\exp(2\int_0^z b/a(s)ds)dz\right)dy < \infty, \tag{4.10}$$

then $P_x^{a,b} \ll P_x^{a,\widehat{b}}$ and $P_x^{a,\widehat{b}} \ll P_x^{a,b}$. Let $u(x)$ be as in (4.7) with $y_0 = 0$. It then follows from (4.10) that

$$M \equiv \lim_{x\to\infty} u(x) < \infty.$$

Fix any $x \in \mathbb{R}$ and pick r satisfying (4.5). Then

$$u(x) = \mathbb{E}_x^{a,b} u(X(\sigma_{r,w} \wedge t)) - \mathbb{E}_x^{a,b} \int_0^{\sigma_{r,w}\wedge t} (\widehat{b}-b)^2/a(X(s))ds.$$

Letting $t \to \infty$ and applying the bounded and monotone convergence theorems on the first and the second terms of the right-hand side, respectively, give

$$\begin{aligned} u(x) &= \mathbb{E}_x^{a,b} u(X(\sigma_{r,w})) - \mathbb{E}_x^{a,b} \int_0^{\sigma_{r,w}} (\widehat{b}-b)^2/a(X(s))ds \\ &= u(r) P_x^{a,b}(\{\sigma_r < \sigma_w\}) + u(w) P_x^{a,b}(\{\sigma_w < \sigma_r\}) \\ &\quad -\mathbb{E}_x^{a,b} \int_0^{\sigma_{r,w}} (\widehat{b}-b)^2/a(X(s))ds. \end{aligned}$$

Now Letting $w \to \infty$, the monotone convergence theorem yields

$$u(x) = u(r) P_x^{a,b}(\{\sigma_r < \infty\}) + M P_x^{a,b}(\{\sigma_r = \infty\}) - \mathbb{E}_x^{a,b} \int_0^{\sigma_r} (\widehat{b}-b)^2/a(X(s))ds. \tag{4.11}$$

Therefore

$$\begin{aligned} 0 &\le \mathbb{E}_x^{a,b} \mathbf{1}_{\{\sigma_r=\infty\}} \int_0^{\infty} (\widehat{b}-b)^2/a(X(s))ds \\ &\le \mathbb{E}_x^{a,b} \int_0^{\sigma_r} (\widehat{b}-b)^2/a(X(s))ds \\ &\le M \vee u(r) - u(x) < \infty. \end{aligned}$$

Consequently,

$$P_x^{a,b}(\{\int_0^\infty (\widehat{b}-b)^2/a(X(s))ds < \infty\}) \ge P_x^{a,b}(\{\sigma_r = \infty\}) > 1-\epsilon.$$

As ϵ is arbitrary, we conclude from Theorem 1-i that $P_x^{a,b} \ll P_x^{a,\widehat{b}}$.

Now let

$$\widehat{u}(x) = \int_0^x \exp\left(-2\int_0^y \frac{\widehat{b}}{a}(s)ds\right)\left(\int_0^y \frac{(\widehat{b}-b)^2}{a^2}(z)\exp\left(2\int_0^z \frac{\widehat{b}}{a}(s)ds\right)dz\right)dy.$$

Since $P_x^{a,b} \ll P_x^{a,\widehat{b}}$, it follows that $\lim_{x\to\infty} \widehat{u}(x) < \infty$, since otherwise we would conclude from the first part of the proof that $P_x^{a,b} \perp P_x^{a,\widehat{b}}$. Therefore, we can repeat the above argument with u replaced by $\widehat{u}$ and the roles of $P_x^{a,b}$ and $P_x^{a,\widehat{b}}$ switched, to obtain $P_x^{a,\widehat{b}} \ll P_x^{a,b}$.

We now turn to the proof of (1.3). Assume that $P_x^{a,b} \sim P_x^{a,\widehat{b}}$. Rearranging terms in (4.11), we have

$$\mathbb{E}_x^{a,b} \int_0^{\sigma_r} (\widehat{b} - b)^2 / a(X(s)) ds = u(r) P_x^{a,b}(\{\sigma_r < \infty\}) + M P_x^{a,b}(\{\sigma_r = \infty\}) - u(x). \tag{4.12}$$

Therefore,

$$H(P_x^{a,b}; P_x^{a,\widehat{b}}) < \infty \text{ if and only if } \lim_{r \to -\infty} u(r) P_x^{a,b}(\{\sigma_r < \infty\}) < \infty.$$

Note that the existence of the limit on the right-hand side above as an extended real number is a consequence of (4.12). By Theorem 1 and (4.12)

$$H(P_x^{a,b}; P_x^{a,\widehat{b}}) = \frac{1}{2} \left(\lim_{r \to -\infty} u(r) P_x^{a,b}(\{\sigma_r < \infty\}) + M - u(x) \right). \tag{4.13}$$

From (4.2) and (4.4), we have

$$\begin{aligned} P_x^{a,b}(\{\sigma_r < \infty\}) &= 1 - P_x^{a,b}(\{\sigma_r = \infty\}) \\ &= 1 - \lim_{z_2 \to \infty} v_{r,z_2}(x) \\ &= \frac{\int_x^\infty \exp\left(-2 \int_0^y b/a(s) ds\right) dy}{\int_r^\infty \exp\left(-2 \int_0^y b/a(s) ds\right) dy}. \end{aligned}$$

Note that according to (4.1) both integrals above are finite. To simplify the notation we let $A(y) = 2 \int_0^y b/a(u) du$. Let

$$\lim_{r \to -\infty} u(r) P_x^{a,b}(\{\sigma_r < \infty\}) \equiv (*).$$

Then

$$\begin{aligned} (*) &= \lim_{r \to -\infty} \frac{2 \int_0^r \exp(-A(y)) \left(\int_0^y (\widehat{b} - b)^2 / a^2(z) \exp(A(z)) dz \right) dy \int_x^\infty \exp(-A(y)) dy}{\int_r^\infty \exp(-A(y)) dy} \\ &= 2 \int_x^\infty \exp(-A(y)) dy \int_{-\infty}^0 (\widehat{b} - b)^2 / a^2(z) \exp(A(z)) dz, \end{aligned} \tag{4.14}$$

where we have applied l'Hôpital's rule to obtain the second equality. Now

$$M - u(x) = 2 \int_x^\infty \exp(-A(y)) \left(\int_0^y (\widehat{b} - b)^2 / a^2(z) \exp(A(z)) dz \right) dy. \tag{4.15}$$

Plugging (4.14) and (4.15) in (4.13) gives

$$H(P_x^{a,b}; P_x^{a,\widehat{b}}) = \int_x^\infty \exp(-A(y)) \left(\int_{-\infty}^y (\widehat{b} - b)^2 / a^2(z) \exp(A(z)) dz \right) dy.$$

□

5. Proofs of Propositions 1,2 and 3

Proof of Proposition 1. Fix $x > 0$. As will become clear in the course of the proof, we can assume $a(x) = 1$, without loss of generality. We define the random process

$$g(t) = \gamma \int_0^t X(s)^{-l} ds.$$

Since $\{P_x^{a,b}\}_{x>0}$ solves the martingale problem for $\mathrm{L}_{a,b}$ on $(0, \infty)$, there exists an $\{\mathcal{F}_t\}_{t \geq 0}$ measurable Brownian motion $B(t) = B(t, \omega),\ t \geq 0,\ \omega \in \Omega$ and $B(0) = x$, a.s. $[P_x^{a,b}]$, such that

$$X(t) = B(t) + g(t), \text{ a.s. } [P_x^{a,b}]. \tag{5.1}$$

We will show that the drift is the dominant factor; that is, asymptotically, the presence of the random contribution in (5.1) can be neglected. The unique solution of

$$y(t) = \gamma \int_0^t y(s)^{-l} ds,\ t \geq 0$$

is

$$y(t) = (Ct)^{1/(1+l)},\ C = (1 + l)\gamma.$$

Since $y(\cdot)$ is strictly increasing, it has an inverse $t(y) = y^{(1+l)}/C$. We will denote $X(t(y)),\ g(t(y))$ and $B(t(y))$ by $X(y),\ g(y)$ and $B(y)$, respectively. With this change of variables, we have

$$\begin{aligned} g(y) &= \int_0^y X^{-l}(z) z^l dz, \text{ and} \\ X(y) &= B(y) + g(y), \text{ a.s. } [P_x^{a,b}]. \end{aligned} \tag{5.2}$$

These imply

$$g'(y) = (g(y)/y + B(y)/y)^{-l}, \text{ a.s. } [P_x^{a,b}]. \tag{5.3}$$

As $1/(1 + l) > 1/2$,

$$\lim_{y \to \infty} B(y)/y = \lim_{t \to \infty} B(t)/(Ct)^{1/(1+l)} = 0, \text{ a.s. } [P_x^{a,b}].$$

At this point, one can see why we may assume $a(x) = 1$. The limit is not affected when B is multiplied by a bounded process. Restricting the discussion to some set of $P_x^{a,b}$ measure 1, we regard the last equations as true for all paths. Let

$$\alpha = \limsup_{y \to \infty} g(y)/y. \tag{5.4}$$

Then (5.3) implies $\liminf_{y\to\infty} g'(y) \geq \alpha^{-l}$. Integration of this inequality yields

$$\liminf_{y\to\infty} g(y)/y \geq \alpha^{-l}. \tag{5.5}$$

Using this in (5.3) gives $\limsup_{y\to\infty} g'(y) \leq \alpha^{l^2}$. Consequently, by integration,

$$\limsup_{y\to\infty} g(y)/y \leq \alpha^{l^2}. \tag{5.6}$$

Since by assumption $|l| < 1$, (5.4) and (5.5) imply $\alpha \geq 1$. Similarly (5.4) and (5.6) imply $\alpha \leq 1$. Therefore $\alpha = 1$. The same line of reasoning gives $\liminf_{y\to\infty} g(y)/y = 1$. Finally, dividing (5.2) by y while letting $y \to \infty$,

$$\lim_{t\to\infty} X(t)/(Ct)^{1/(1+l)} = \lim_{y\to\infty} g(y)/y = 1.$$

□

Proof of Proposition 2. By the strong Markov property, the sequence $\{A_n^\rho\}$ consists of independent events. By Borel-Cantelli,

$$P_x^{1,\frac{k-1}{2x}}\{A_j^\rho \text{ i.o.}\} = \begin{cases} 0 & \text{if } \sum_{j=1}^{\infty} P_x^{1,\frac{k-1}{2x}}(A_j^\rho) < \infty, \\ 1 & \text{otherwise.} \end{cases} \tag{5.7}$$

We will show that the (5.7) converges if and only if $\rho > \frac{1}{k-2}$. By (4.2) and (4.3), we have

$$\begin{aligned} P_x^{1,\frac{k-1}{2x}}(A_j^\rho) &= \frac{\int_{((j+1)!)^\rho}^{((j+2)!)^\rho} y^{1-k}dy}{\int_{(j!)^\rho}^{((j+2)!)^\rho} y^{1-k}dy} \\ &= \frac{((j+1)!)^{-\rho(k-2)} - ((j+2)!)^{-\rho(k-2)}}{(j!)^{-\rho(k-2)} - ((j+2)!)^{-\rho(k-2)}} \\ &= (j+1)^{-\rho(k-2)} \frac{1-(j+2)^{-\rho(k-2)}}{1-((j+1)(j+2))^{-\rho(k-2)}}. \end{aligned} \tag{5.8}$$

Thus there exist constants $C_1, C_2 > 0$ such that

$$C_1 j^{-\rho(k-2)} \leq P_x^{1,\frac{k-1}{2x}}(A_j^\rho) \leq C_2 j^{-\rho(k-2)}. \tag{5.9}$$

The convergence/divergence dichotomy for (5.7) now follows from (5.9). □

Proof of Proposition 3. Since $\lim_{t\to\infty} X(t) = \infty$, a.s. $[P_x^{1,b}]$ and a.s. $[P_x^{1,\hat{b}}]$, for all $x \in \mathbb{R}$, it is enough to prove (1.7) for $x \geq 3$. Note that for such x, the random variables $\{N_n^{(\rho)}\}_{n=n_0(x)}^\infty$ only depend on the process when it is in $[2,\infty)$, in which case the drifts b and $\widehat{b}$ are as given in the second and third lines of (1.5).

Fix $\rho=\frac{1-2\beta}{k-2}$ as in the statment of the proposition. Let $\sigma_x=\inf\{t\ge 0: X(t)=x\}$. Similar to (4.2) and (4.3), it follows that $P^{1,B}_{((n+1)!)^\rho}(\sigma_{(n!)^\rho}<\sigma_{((n+1)!)^\rho})=p_n^{(\rho)}(B)$, for $B=b,\widehat{b}$, where $p_n^{(\rho)}(B)$ is as in (1.6). It then follows from the strong Markov property that the sequence of random variables $\{N_n^{(\rho)}\}_{n=n_0(x)}^\infty$ are independent and distributed according to geometric distributions. Specifically

$$P_x^{1,B}(N_n^{(\rho)}=j)=(p_n^{(\rho)}(B))^j(1-p_n^{(\rho)}(B))$$

for $B=b,\widehat{b}$ and $j=0,1,\ldots$. Under $P^{1,B}$, the sequence $\{N_n^{(\rho)}\}_{n=n_0(x)}^\infty$ induces a probabilty measure Q_x^B on $\{0,1,\ldots\}^{\mathbb{N}}$, for $B=b,\widehat{b}$. Denoting the restrictions of Q_x^b and $Q_x^{\widehat{b}}$ to the first n coordinates of $\{0,1,\ldots\}^{\mathbb{N}}$ respectively by $Q_{x,n}^b$ and $Q_{x,n}^{\widehat{b}}$, and letting $F_{x,n}=\frac{dQ_{x,n}^{\widehat{b}}}{dQ_{x,n}^b}$, Lemma 1 shows that

$$\lim_{n\to\infty}F_{x,n}=0 \text{ a.s. } [Q_x^b]\Leftrightarrow \limsup_{n\to\infty}F_{x,n}=\infty \text{ a.s. } [Q_x^{\widehat{b}}]\Leftrightarrow Q_x^{\widehat{b}}\perp Q_x^b. \tag{5.10}$$

As the $\{N_n^{(\rho)}\}$ are geometric random variables, it is easy to see that

$$F_{x,n}(z)=\prod_{j=1}^n\left(\left(\frac{p^{(\rho)}_{n_0(x)+j-1}(\widehat{b})}{p^{(\rho)}_{n_0(x)+j-1}(b)}\right)^{z_j}\frac{1-p^{(\rho)}_{n_0(x)+j-1}(\widehat{b})}{1-p^{(\rho)}_{n_0(x)+j-1}(b)}\right),\ z=(z_1,z_2,\ldots)\in\{0,1,\ldots\}^{\mathbb{N}}. \tag{5.11}$$

Let $\mathcal{Y}_n=\sigma(N_1^{(\rho)},\ldots,N_n^{(\rho)})$ and $\mathcal{Y}=\sigma(N_1^{(\rho)},N_2^{(\rho)},\ldots)$. Clearly, $\mathcal{Y}\subseteq\mathcal{F}$. Let $T:\Omega\to\{0,1\ldots\}^{\mathbb{N}}$ be defined by $T\omega=(z_1,z_2\ldots)\in\{0,1\ldots\}^{\mathbb{N}}$, if $N_j^{(\rho)}(\omega)=z_j,\ j\in\mathbb{N}$. If $A\in\{0,1,\ldots\}^n$, then $T^{-1}(A)\in\mathcal{Y}_n$. Therefore, $Q_x^b(A)=P_x^{1,b}(T^{-1}A)$ and $Q_x^{\widehat{b}}(A)=P_x^{a,\widehat{b}}(T^{-1}A)$. Since $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ generate $\mathcal{Y}$, these equalities determine the restrictions of $P_x^{1,b}$ and $P_x^{1,\widehat{b}}$ to $\mathcal{Y}$. Letting $\overline{F}_{x,n}=F_{x,n}\circ T$, we see that $\overline{F}_{x,n}(\omega)=F_{x,n}((N^{(\rho)}_{n_0(x)}(\omega),N^{(\rho)}_{n_0(x)+1}(\omega),\ldots))$ and we obtain from (5.10)

$$\lim_{n\to\infty}\overline{F}_{x,n}=0 \text{ a.s. } [P_x^{1,b}]\Leftrightarrow \limsup_{n\to\infty}\overline{F}_{x,n}=\infty \text{ a.s. } [P_x^{1,\widehat{b}}]\Rightarrow P_x^{1,\widehat{b}}\perp P_x^{1,b}. \tag{5.12}$$

Since

$$\limsup_{n\to\infty}\overline{F}_{x,n}=0 \text{ a.s. } [P_x^{1,b}]\Leftrightarrow \limsup_{n\to\infty}F_{x,n}=0 \text{ a.s. } [Q_x^b],$$

it follows from (5.12) that in order to complete the proof it is enough to show that $\limsup_{n\to\infty}F_{x,n}=0$, a.s. $[Q_x^b]$. Kakutani's dichotomy [3, Theorem 4.3.5] states that

$$Q_x^{\widehat{b}}\ll Q_x^b \text{ or } Q_x^{\widehat{b}}\perp Q_x^b \text{ according as } \lim_{n\to\infty}\mathbb{E}^{Q_x^b}\sqrt{F_{x,n}}>0 \text{ or } =0.$$

Thus in order to show that $\limsup_{n\to\infty} F_{x,n} = 0$, a.s. $[Q^b_x]$, it follows from (5.10) that it is enough to prove $\lim_{n\to\infty} \mathbb{E}^{Q^b_x}\sqrt{F_{x,n}} = 0$. To simplify notation, we let $p_j = p^{(\rho)}_j(b)$ and $q_j = p^{(\rho)}_j(\widehat{b})$. We have

$$\begin{aligned}
(\mathbb{E}^P\sqrt{F_{x,n}})^2 &= \left(\sum_{m_1,\ldots,m_n=0}^{\infty} \sqrt{\prod_{j=n_0(x)}^{n_0(x)+n-1} \left(\frac{q_j}{p_j}\right)^{m_j} \frac{1-q_j}{1-p_j}} \prod_{j=n_0(x)}^{n_0(x)+n-1} p_j^{m_j}(1-p_j)\right)^2 \\
&= \left(\sum_{m_1,\ldots,m_n=0}^{\infty} \prod_{j=n_0(x)}^{n_0(x)+n-1} \sqrt{(p_jq_j)^{m_j}(1-p_j)(1-q_j)}\right)^2 \\
&= \left(\prod_{j=n_0(x)}^{n_0(x)+n-1} \sqrt{(1-p_j)(1-q_j)} \sum_{m=0}^{\infty}(p_jq_j)^{m/2}\right)^2 \\
&= \prod_{j=n_0(x)}^{n_0(x)+n-1} \frac{(1-q_j)(1-p_j)}{(1-\sqrt{p_jq_j})^2} = \prod_{j=n_0(x)}^{n_0(x)+n-1} 1 - \frac{(\sqrt{p_j}-\sqrt{q_j})^2}{(1-\sqrt{p_jq_j})^2}.
\end{aligned}$$

This product converges if and only if the series $\sum^{\infty} \frac{(\sqrt{p_j}-\sqrt{q_j})^2}{(1-\sqrt{p_jq_j})^2}$ converges. However, as we shall see below, $\lim_{j\to\infty} p_jq_j = 0$. Therefore to prove the first assertion of the proposition, we must show that $\sum^{\infty}(\sqrt{p_j}-\sqrt{q_j})^2 = \infty$. We will find a lower (upper) bound for p_j (q_j). We start with p_j.

Let

$$\zeta = 1 - 2\beta = \rho(k-2)$$

and recall that p_j is defined by (1.6) with $B = b$. Since $\mathrm{L}_{1,b}$ of Proposition 2 and $\mathrm{L}_{1,b}$ of Proposition 3 coincide for $x \geq 2$, we can use the computations in the proof of Proposition 2. By comparing (1.6) and (4.2), it is easily seen that the righthand side of (5.8) gives te formula. That is,

$$\begin{aligned}
p_j &= j^{-\zeta}\left(\frac{1-(j+1)^{-\zeta}}{1-(j(j+1))^{-\zeta}}\right) \\
&= j^{-\zeta}\left(1-(j+1)^{-\zeta}\right)\left(1+(j(j+1))^{-\zeta}+O(j^{-4\zeta})\right) \\
&> j^{-\zeta}(1-j^{-\zeta}),
\end{aligned}$$

for j sufficiently large. Since

$$\sqrt{1-x} > 1 - \frac{3}{4}x,\ x \in (0, \frac{1}{2}),$$

we obtain

$$\sqrt{p_j} > j^{-\zeta/2}(1 - \frac{3}{4}j^{-\zeta}),\ \text{for } j \text{ sufficiently large.} \tag{5.13}$$

The evaluation of q_j requires more work. Let $u(x) = \int_2^x \exp(-2\int_2^y \widehat{b}(z)dz)dy$. Then

$$\begin{aligned}
u((j!)^\rho) - u((m!)^\rho) &= \int_{(m!)^\rho}^{(j!)^\rho} \exp\left(-\int_2^y \frac{k-1}{s}ds - \int_2^y \frac{c}{s\log^\beta s}ds\right) dy \\
&= \int_{(m!)^\rho}^{(j!)^\rho} y^{1-k} \exp\left(-\frac{c}{1-\beta}\log^{1-\beta} y\right) dy \\
&= \frac{y^{2-k}}{2-k} \exp\left(-\frac{c}{1-\beta}\log^{1-\beta} y\right) |_{(m!)^\rho}^{(j!)^\rho} \\
&+ \frac{c}{2-k}\int_{(m!)^\rho}^{(j!)^\rho} y^{1-k}\log^{-\beta} y \exp\left(-\frac{c}{1-\beta}\log^{1-\beta} y\right) dy.
\end{aligned}$$

From (1.6) we know that

$$q_j = \frac{u((j!)^\rho) - u(((j+1)!)^\rho)}{u(((j-1)!)^\rho) - u(((j+1)!)^\rho)}.$$

We can write

$$q_j = \frac{H_j - H_{j+1} + A_{j,j+1}}{H_{j-1} - H_{j+1} + A_{j-1,j+1}} < \frac{H_j + A_{j,j+1}}{H_{j-1} - H_{j+1}}, \tag{5.14}$$

where

$$\begin{aligned}
H_j &= \frac{(j!)^{-\zeta}}{2-k}\exp\left(-\frac{c}{1-\beta}\log^{1-\beta}(j!)^\rho\right), \text{ and} \\
A_{j,j+i} &= \frac{c}{2-k}\int_{(j!)^\rho}^{((j+i)!)^\rho} y^{1-k}\log^{-\beta} y \exp\left(-\frac{c}{1-\beta}\log^{1-\beta} y\right) dy.
\end{aligned}$$

Note that $\{H_j\}$ is a strictly decreasing sequence of positive numbers and that the quantities $\{A_{j,j+i}\}$ are non-negative. Now,

$$\begin{aligned}
A_{j,j+1} &= \frac{c}{2-k}\int_{(j!)^\rho}^{((j+i)!)^\rho} y^{1-k}\log^{-\beta} y \exp\left(-\frac{c}{1-\beta}\log^{1-\beta} y\right) dy \\
&= \frac{c}{2-k}\log^{-\beta}(j!)^\rho \int_{(j!)^\rho}^{((j+i)!)^\rho} y^{1-k}\log^{\beta}(j!)^\rho \log^{-\beta} y \exp\left(-\frac{c}{1-\beta}\log^{1-\beta} y\right) dy.
\end{aligned}$$

Since $\lim_{j\to\infty}\log^\beta(j!)^\rho\log^{-\beta}((j+i)!)^\rho = 1$, we have

$$\begin{aligned}
A_{j,j+1} &= O(1)\log^{-\beta}(j!)^\rho \int_{(j!)^\rho}^{((j+i)!)^\rho} y^{1-k}\exp\left(-\frac{c}{1-\beta}\log^{1-\beta} y\right) dy \\
&= \log^{-\beta}(j!)^\rho O(u((j!)^\rho) - u(((j+i)!)^\rho)),
\end{aligned}$$

from which it follows that

$$A_{j,j+1} = \log^{-\beta}(j!)O(H_j - H_{j+i} + A_{j,j+1}).$$

Since $H_{j+i} = o(H_j)$, we have

$$A_{j,j+1} = \log^{-\beta}(j!)O(H_j + A_{j,j+1}).$$

In particular,

$$\limsup_{j\to\infty} \frac{1}{H_j/A_{j,j+1} + 1} = \limsup_{j\to\infty} \frac{A_{j,j+1}}{H_j + A_{j,j+1}} = 0,$$

which shows that $A_{j,j+1} = o(H_j)$. These last observations lead to

$$A_{j,j+1} = \log^{-\beta}(j!)O(H_j). \tag{5.15}$$

In the sequel C denotes a positive constant which may vary from line to line. It will be convenient to write

$$\frac{H_j}{H_{j-1}} = j^{-\zeta}\exp(-\delta_j),$$

with

$$\delta_j = C\left(\log^{1-\beta}(j!) - \log^{1-\beta}((j-1)!)\right).$$

Now

$$\begin{aligned}
\delta_j &= C\left((\log((j-1)!) + \log j)^{1-\beta} - \log^{1-\beta}((j-1)!)\right) \\
&= C\left(\log^{1-\beta}((j-1)!)\left(1 + \frac{\log j}{\log((j-1)!)}\right)^{1-\beta} - \log^{1-\beta}((j-1)!)\right) \\
&= C\log^{1-\beta}((j-1)!)\left(1 + (1-\beta)\frac{\log j}{\log((j-1)!)} + o(\frac{\log j}{\log((j-1)!)}) - 1\right) \\
&= C\log j\log^{-\beta}((j-1)!) + o(\log j\log^{-\beta}(j!)) \equiv (*)
\end{aligned}$$

Recall that

$$\lim_{j\to\infty}\frac{j\log j}{\log(j!)} = 1, \tag{5.16}$$

which implies

$$(*) = O(j^{-\beta}\log^{1-\beta} j).$$

Thus,

$$\delta_j = O(j^{-\beta}\log^{1-\beta} j). \tag{5.17}$$

We are now in a position to estimate q_j. From (5.14) and (5.15), we have

$$\begin{aligned}
q_j &< \frac{H_j}{H_{j-1}}\frac{1 + O(\log^{-\beta}(j!))}{1 - H_{j+1}/H_{j-1}} \\
&= j^{-\zeta}\exp(-\delta_j)\left(1 + O(\log^{-\beta}(j!))\right)\left(1 + (j(j+1))^{-\zeta}\exp(-\delta_j - \delta_{j+1}) + O(j^{-4\zeta})\right) \\
&< j^{-\zeta}\exp(-\delta_j)\left(1 + O(\log^{-\beta}(j!))\right)C(1 + j^{-2\zeta}).
\end{aligned}$$

Therefore

$$\sqrt{q_j} < Cj^{-\zeta/2}\exp(-\delta_j/2)\left(1 + O(\log^{-\beta}(j!))\right)(1 + \frac{1}{2}j^{-2\zeta}). \tag{5.18}$$

Subtracting (5.18) from (5.13), we have

$$\begin{aligned}\sqrt{p_j} - \sqrt{q_j} \quad > \quad & \underbrace{j^{-\zeta/2}(1 - \exp(-\delta_j/2))}_{(I)} - \frac{3}{4}\underbrace{j^{-3\zeta/2}}_{(II)} - \underbrace{j^{-\zeta/2}O(\log^{-\beta}(j!))}_{(III)} - 1/2j^{-5\zeta/2} \\ & -o(j^{-5\zeta/2}).\end{aligned}$$

Since $\zeta = 1 - 2\beta$, we obtain from (5.17) that $(I) = O(j^{-1/2}\log^{1-\beta} j)$. From (5.16) we see that $(III) = O(j^{-1/2}\log^{-\beta} j)$. Finally, we want (II) to be negligible with respect to (I). Clearly, this can be achieved if and only if $3\zeta \geq 1$. This condition is equivalent to $\beta \leq 1/3$. To conclude, $(\sqrt{p_j} - \sqrt{q_j})^2 \geq O(j^{-1}\log^{2-2\beta} j)$, so

$$\sum^{\infty}(\sqrt{p_j} - \sqrt{q_j})^2 = \infty.$$

□

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