ON DOMAIN MONOTONICITY FOR THE PRINCIPAL EIGENVALUE OF THE LAPLACIAN WITH A MIXED DIRICHLET-NEUMANN BOUNDARY CONDITION

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Abstract. Let $\Omega \subset R^d$ be a bounded domain with smooth boundary and let $A \subset\subset \Omega$ be a smooth, compactly embedded subdomain. Consider the operator $-\frac{1}{2}\Delta$ in $\Omega - \bar{A}$ with the Dirichlet boundary condition at ∂A and the Neumann boundary condition at $\partial\Omega$, and let $\lambda_0(\Omega, A) > 0$ denote its principal eigenvalue. We discuss the question of monotonicity of $\lambda_0(\Omega, A)$ in its dependence on the domain Ω.

The main point of this note is to suggest an open problem that is in the spirit of Chavel's question concerning domain monotonicity for the Neumann heat kernal. Let $\Omega \subset R^d$ be a bounded domain with smooth boundary and let $A \subset\subset \Omega$ be a smooth, compactly embedded subdomain. Consider the operator $-\frac{1}{2}\Delta$ in $\Omega - \bar{A}$ with the Dirichlet boundary condition at ∂A and the Neumann boundary condition at $\partial\Omega$, and let $\lambda_0(\Omega, A) > 0$ denote its principal eigenvalue. If instead of the Neumann boundary condition, one imposes the Dirichlet boundary condition at $\partial\Omega$, then it's easy to see that $\lambda_0(\Omega, A)$ is monotone decreasing in Ω and increasing in A. Similarly, in the case at hand, it is clear that $\lambda_0(\Omega, A)$ is monotone increasing in A; however, the question of monotonicity in Ω is not easily resolved. The impetus for studying this question arose in part from a recent paper [5] in which one can find the asymptotic behavior of $\lambda_0(\Omega, A)$ when A is a ball that shrinks to a point,

1991 *Mathematics Subject Classification.* 60C05, 60F05.

Key words and phrases. principal eigenvalue, Neumann boundary condition, Dirichlet boundary condition, mixed boundary condition, domain monotonicity.

This research was supported by the Fund for the Promotion of Research at the Technion and by the V.P.R. Fund.

and in part from the work of Chavel [2], Kendall [3] and Bass and Burdzy [1], where a monotonicity property of the Neumann heat kernel was studied.

Let $B_\epsilon(x)$ denote the ball of radius $\epsilon > 0$ centered at $x \in \Omega$, let $|\Omega|$ denote the volume of Ω and let ω_d denote the volume of the unit ball in R^d. In [5] it was shown that

$$\text{(1.1-a)} \qquad \lim_{\epsilon \to 0} \epsilon^{2-d} \lambda_0(\Omega, B_\epsilon(x)) = \frac{d(d-2)\omega_d}{2|\Omega|}, \text{ if } d \geq 3,$$

and

$$\text{(1.1-b)} \qquad \lim_{\epsilon \to 0} (-\log \epsilon) \lambda_0(\Omega, B_\epsilon(x)) = \frac{\pi}{|\Omega|}, \text{ if } d = 2.$$

In particular then, we obtain from (1.1) the following proposition.

Proposition 1. *If $\Omega_1 \subsetneq \Omega_2$, then for each $x \in \Omega_1$ there exists an $\epsilon_0 = \epsilon_0(x) > 0$ such that*

$$\lambda_0(\Omega_1, B_\epsilon(x)) > \lambda_0(\Omega_2, B_\epsilon(x)), \ \textit{for } \epsilon \in (0, \epsilon_0).$$

Remark. A careful look at the proof of (1.1) shows that $\epsilon_0(x)$ in Proposition 1 may be chosen uniformly for x away from $\partial\Omega_1$.

The question we pose here is this:

Question 1: *Under what "generic" conditions on A, Ω_1 and Ω_2, satisfying $A \subset\subset \Omega_1 \subset \Omega_2$, is it true that $\lambda_0(\Omega_1, A) \geq \Omega_0(\Omega_2, A)$?*

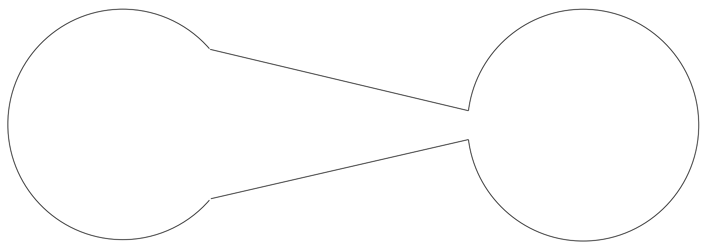
The following well-known probabilistic representation of $\lambda_0(\Omega, A)$ gives some useful intuition for the problem:

$$\text{(1.2)} \qquad \lim_{t \to \infty} \frac{1}{t} \log P_x(\tau_A > t) = -\lambda_0(\Omega, A),$$

where $x \in \Omega - \bar{A}$, and $P_x(\tau_A > t)$ denotes the probability that a Brownian motion starting from $x \in \Omega$ and normally reflected at the barrier $\partial\Omega$ will not reach the set A by time t.

The following example shows that monotoncity in Ω does not hold in complete generality, and that a certain convexity requirement is reasonable. Let $\Omega_1 \subset R^2$

be the skewed, barbell-shaped region pictured below and defined as follows: $\Omega_1 = B_1((-2,0)) \cup B_1((2,0)) \cup T_\delta$, where $\delta \in (0, \frac{1}{4})$ and T_δ is the cone-like region bounded by the line connecting $(-\frac{3}{2}, \sqrt{\frac{3}{4}})$ to $(1+\delta, \sqrt{1-(1-\delta)^2})$ and the line connecting $(-\frac{3}{2}, -\sqrt{\frac{3}{4}})$ to $(1+\delta, -\sqrt{1-(1-\delta)^2})$. Let $\Omega_2 = B_4((0,0))$ and let $A = B_\rho((0,0))$, where $\rho > 0$ is chosen sufficiently small so that $A \subset\subset \Omega_1$ for every $\delta \in (0, \frac{1}{4})$. Then $\lambda_0(\Omega_2, A) > 0$ and doesn't depend on δ, but as is easy to understand from (1.2) and as is not hard to show rigorously, $\lambda_0(\Omega_1, A)$ approaches 0 as $\delta \to 0$.



We prove the following result.

Theorem 1. *Let $\Omega_1 \subset R^d$ be convex and let $B_{r_0}(x_0) \subset\subset \Omega_1$. Let Ω_2 satisfy the following condition: there exists an R such that $\Omega_1 \subset B_R(x_0) \subset \Omega_2$. Then*

$$\lambda_0(\Omega_1, B_{r_0}(x_0)) \geq \lambda_0(\Omega_2, B_{r_0}(x_0)).$$

Remark. In words, the theorem indicates that monotonicity holds if the inner domain A is a ball, and if it is possible to impose a ball which is concentric to A between the two boundaries, $\partial\Omega_1$ and $\partial\Omega_2$.

The method of proof is similar to that used in [2] to prove a certain monotonicity property of the Neumann heat kernel. Before giving the proof, we describe the conjecture raised with regard to the Neumann heat kernel, the results from [2] and [3], and the example constructed in [1].

Consider the operator $-\frac{1}{2}\Delta$ in Ω with the Neumann boundary condition at $\partial\Omega$. Let $p_\Omega(t,x,y)$ denote the corresponding heat kernel. As a function of y, $p_\Omega(t,x,\cdot)$ is the density of the probability distribution corresponding to the position at time t of a Brownian motion starting from x and normally reflected at the barrier $\partial\Omega$; that is,

$$p_\Omega(t,x,y) = P_x(X(t) \in dy). \tag{1.3}$$

It is well-known that $\lim_{t\to\infty} p_\Omega(t,x,y) = \frac{1}{|\Omega|}$. Thus, if $\Omega_1 \subsetneq \Omega_2$, it follows that for each $x, y \in \Omega_1$, there exists a $t_0 = t_0(x,y)$ such that $p_{\Omega_1}(t,x,y) > p_{\Omega_2}(t,x,y)$, for $t > t_0$. We paraphrase the question posed in [2] as follows:

Question 2: *Under what generic conditions on Ω_1 and Ω_2 satisfying $\Omega_1 \subset \Omega_2$ is it true that $p_{\Omega_1}(t,x,y) \geq p_{\Omega_2}(t,x,y)$ for all t and all $x, y \in \Omega_1$?*

It is easy to see that some convexity is needed. Indeed, consider the case that Ω_2 is a square and Ω_1 is a very thin L-shaped subset of Ω_2 running along the lower and left boundaries of Ω_2. Then the intuition gleaned from (1.3) suggests that for fixed t and thin enough Ω_1, the above inequality will be violated if x is chosen from the lower right hand corner of the square and y is chosen from the upper left hand corner.

Using a straightforward integration by parts, Chavel showed in [2] that $p_{\Omega_1}(t,x,y) \geq p_{\Omega_2}(t,x,y)$ when Ω_1 is a convex domain containing x and y, and Ω_2 is a ball centered at either x or y. Building on Chavel's result, Kendall [3] gave a nice argument using couplings of reflected Brownian motions to show that $p_{\Omega_1}(t,x,y) \geq p_{\Omega_2}(t,x,y)$ when Ω_1 is a convex domain containing x and y, and Ω_2 is such that one can fit a ball B, centered at either x or y, between the two domains; that it, $\Omega_1 \subset B \subset \Omega_2$. Note that in their results, the conditions on the domains depend on the points x, y.

What happens if one dispenses with Kendall's assumption concerning the fitting of a ball between Ω_1 and Ω_2? In [1], Bass and Burdzy gave an example showing that the above inequality does not always hold if one only assumes that Ω_1 is convex. They obtained the reverse inequality for a certain pair of domains Ω_1, Ω_2 and for a certain set of points t_0, x_0, y_0. Bass and Burdzy used probabilistic methods starting from (1.3).

Returning to the question in the present paper, we propose the following problem:

Open Problem. *Give an example of a triple A, Ω_1, Ω_2 such that Ω_1 is convex, $A \subset\subset \Omega_1 \subset \Omega_2$ and $\lambda_0(\Omega_2, A) > \lambda_0(\Omega_1, A)$, or alternatively, show that the reverse inequality always holds.*

We suspect that the inequality $\lambda_0(\Omega_2, A) \leq \lambda_0(\Omega_1, A)$ does not always hold when Ω_1 is convex, but we also suspect that it is more difficult to find an example here than it was for the Neumann heat kernel problem. Bass and Burdzy gave an example for a specific (short) time t_0. In the present situation, of course there is no time parameter, and the eigenvalue depends on the entire infinite time interval $[0, \infty)$.

Proof of Theorem 1. We will prove that

$$\lambda_0(\Omega_1, B_{r_0}(x_0)) \geq \lambda_0(B_R(x_0), B_{r_0}(x_0)) \tag{1.4}$$

and that

$$\lambda_0(B_R(x_0), B_{r_0}(x_0)) \geq \lambda_0(\Omega_2, B_{r_0}(x_0)). \tag{1.5}$$

Without loss of generality, we will assume that $x_0 = 0$.

We first consider (1.4). Let $\phi_{0,1}, \phi_{0,2} > 0$ denote the principal eigenfunctions corresponding respectively to $\lambda_0(\Omega_1, B_{r_0}(0))$ and $\lambda_0(B_R(0), B_{r_0}(0))$. Using the fact that $\frac{1}{2}\Delta\phi_{0,1} = -\lambda_0(\Omega_1, B_{r_0}(0))\phi_{0,1}$ and $\frac{1}{2}\Delta\phi_{0,2} = -\lambda_0(B_R(0), B_{r_0}(0))\phi_{0,2}$ for the first equality below, and using integration by parts, the fact that $\phi_{0,1}, \phi_{0,2}$ vanish on $\partial B_{r_0}(0)$, and the fact that $\nabla\phi_{0,1} \cdot n = 0$ on $\partial\Omega_1$ for the second one, we have

$$\begin{aligned} &(\lambda_0(\Omega_1, B_{r_0}(0)) - \lambda_0(B_R(0), B_{r_0}(0)) \int_{\Omega_1 - B_{r_0}(0)} \phi_{0,1}\phi_{0,2} dx \\ &= \frac{1}{2}\int_{\Omega_1 - B_{r_0}(0)} (\phi_{0,1}\Delta\phi_{0,2} - \phi_{0,2}\Delta\phi_{0,1}) dx = \frac{1}{2}\int_{\partial\Omega_1} \phi_{0,1}\nabla\phi_{0,2} \cdot n, \end{aligned} \tag{1.6}$$

where n is the outward unit normal to Ω_1 at $\partial\Omega_1$. In light of (1.6), to complete the proof of (1.4) it suffices to show that $\nabla\phi_{0,2} \cdot n \geq 0$ on $\partial\Omega_1$.

By symmetry, $\phi_{0,2}$ depends only on $|x|$, so we will write $\phi_{0,2}(r)$. By the convexity of Ω_1, in order to show that $\nabla\phi_{0,2} \cdot n \geq 0$ on $\partial\Omega_1$, it suffices to show that $\phi_{0,2}$ is non-decreasing in r. If the contrary were true, then there would exist an $r_1 \in (r_0, R)$ such that $\phi'_{0,2}(r_1) = 0$. But then $\phi_{0,2}$ would constitute a positive eigenfunction corresponding to a positive eigenvalue for the Neumann Laplacian in the annulus $B_R(0) - \bar{B}_{r_0}(0)$. However, this contradicts the fact that the only positive eigenfunctions for the Neumann Laplacian in the annulus are the constant functions, corresponding to the eigenvalue 0. Thus, we conclude that $\phi_{0,2}$ is nondecreasing in r.

We now turn to (1.5). We could use Kendall's construction, but there is no need; (1.5) follows immediately from the variational formuala for $\lambda_0(\cdot,\cdot)$. Indeed, we have

$$\lambda_0(\Omega, A) = \inf \frac{\frac{1}{2}\int_{\Omega - A} |\nabla u|^2 dx}{\int_{\Omega - A} u^2 dx},$$

where the infimum is over those $0 \neq u \in C^1(\bar{\Omega} - A)$ satisfying $u = 0$ on ∂A. The infimum is attained at $u = u_0 \geq 0$, where u_0 is the eigenfunction corresponding to $\lambda_0(\Omega, A)$. In particular, when $\Omega = B_R(0)$ and $A = B_{r_0}(0)$, this eigenfunction is radially symmetric and positive on the interior of its domain of definition, hence equal to a positive constant c_0 on $\partial B_R(0)$. Let

$$u_1(x) = \begin{cases} u_0(x), \text{ if } x \in \bar{B}_R(0) - B_{r_0}(0)) \\ c_0, \text{ if } x \in \Omega_2 - B_R(0). \end{cases}$$

Then by variational formula,

$$\lambda_0(\Omega_2, B_{r_0}(0)) = \inf \frac{\frac{1}{2}\int_{\Omega_2 - B_{r_0}(0))} |\nabla u|^2 dx}{\int_{\Omega_2 - B_{r_0}(0))} u^2 dx} \leq \frac{\frac{1}{2}\int_{\Omega_2 - B_{r_0}(0))} |\nabla u_1|^2 dx}{\int_{\Omega_2 - B_{r_0}(0))} u_1^2 dx}$$
$$\leq \frac{\frac{1}{2}\int_{B_R(0) - B_{r_0}(0))} |\nabla u_0|^2 dx}{\int_{B_R(0) - B_{r_0}(0))} u_0^2 dx} = \lambda_0(B_R(0), B_{r_0}(0)).$$

$\square$

We conclude with an explicit sufficient condition for monotonicity in the case of smooth, bounded two-dimensional , star-shaped domains. A bounded C^1-domain $\Omega \subset R^2$ is star-shaped with respect to the origin if and only if it can be represented in polar coordinates in the form $\Omega = \{(r, \theta) : r < R(\theta)\}$, where $R > 0$ is a C^1 function on the circle.

Theorem 2. *i. Let Ω_1, Ω_2 and A be two-dimensional, star-shaped C^1-domains with respect to the origin, satisfying $A \subset\subset \Omega_1 \subset \Omega_2$. Let $\Omega_i = \{(r, \theta) : r < R_i(\theta)\}$, $i = 1, 2$. Let*

$$f = \frac{R_2}{R_1}.$$

Assume that

$$\frac{(f')^2}{f} \leq (f-1)^2. \tag{1.7}$$

Then

$$\lambda_0(\Omega_1, A) \geq \lambda_0(\Omega_2, A).$$

Furthermore, if the strict inequality $R_2 > R_1$ holds, then $\lambda_0(\Omega_1, A) > \lambda_0(\Omega_2, A)$.

ii. Let $A \subset R^2$ be a star-shaped C^1-domain with respect to the origin and let $\{\Omega_t,\ t \in [0,1]\}$ be a one-parameter family of star-shaped domains with respect to the origin, satisfying $A \subset\subset \Omega_s \subset \Omega_t$, for $s < t$, and defined by $\Omega_t = \{(r,\theta) : r < R(\theta,t)\}$, where $R(\theta,t)$ is a C^2-function. If

$$|(\frac{R_t}{R})_\theta| \leq \frac{R_t}{R},$$

then $\lambda_0(\Omega_t, A)$ is nonincreasing in t.

Remark. Note that in contrast to Theorem 1, no condition is imposed on A in Theorem 2 beyond its being star-shaped and smooth.

Example. Let $R, \bar{R} > 0$ be C^1 functions. Define $\Omega_0 = \{(r,\theta) : r < R(\theta)\}$ and $\Omega_t = \{(r,\theta) : r < R(\theta) + t\bar{R}(\theta)\},\ t > 0$. Let $A \subset\subset \Omega_0$ be star-shaped with respect to the origin and C^1. From Theorem 2-ii it follows that $\lambda_0(\Omega_t, A)$ is nonincreasing in t if $|\frac{\bar{R}'}{\bar{R}} - \frac{R'}{R}| \leq 1$. A particular case of this is when $\bar{R} = R$, so that the Ω_t's all have the same shape, differing only in magnification.

Proof. i. Assume for now that $R_2 > R_1$. Let $A = \{(r,\theta) : r < a(\theta)\}$. Let $\phi \geq 0$ denote the eigenfunction corresponding to $\lambda_0(\Omega_1, A)$. Of course, $\phi = 0$ on ∂A. Extend ϕ to all of Ω_1 by defining $\phi(x) = 0$ for $x \in A$. Let

$$u(r,\theta) = \begin{cases} \phi(\frac{R_1}{R_2}(\theta)r, \theta), & \text{if } (\frac{R_1}{R_2}(\theta)r, \theta) \in \bar{\Omega}_1 - A \\ 0, & \text{otherwise.} \end{cases}$$

Since $\phi(r,\theta) = 0$ for $r \leq a(\theta)$, it follows that $u(r,\theta) = 0$ for $r \leq \frac{R_2(\theta)}{R_1(\theta)}a(\theta)$. Thus, since $R_2 > R_1$, we have $u = 0$ on $\bar{A}$. Note that u cannot be the eigenfunction corresponding to $\lambda_0(\Omega_2, A)$ because it vanishes at all points in $\Omega_2 - A$ which are sufficiently close to A, whereas the eigenfunction is strictly positive in $\Omega_2 - \bar{A}$. Since the right hand side of (1.6) is attained only when u is equal to the eigenfunction, it then follows from the variational formula (1.6) that

$$\lambda_0(\Omega_2, A) < \frac{\frac{1}{2}\int_{\Omega_2 - A} |\nabla u|^2(r,\theta)drd\theta}{\int_{\Omega_2 - A} u^2(r,\theta)drd\theta}. \tag{1.8}$$

We have

$$|\nabla u|^2(r,\theta) = u_r^2(r,\theta) + \frac{1}{r^2}u_\theta^2(r,\theta) =$$
$$\phi_r^2(\frac{R_1}{R_2}(\theta)r,\theta)(\frac{R_1}{R_2}(\theta))^2 + \frac{1}{r^2}\left(\phi_r(\frac{R_1}{R_2}(\theta)r,\theta)r(\frac{R_1}{R_2}(\theta))' + \phi_\theta(\frac{R_1}{R_2}(\theta)r,\theta)\right)^2.$$

Thus, making the change of variables $s = \frac{R_1}{R_2}(\theta)r$, we obtain

$$\begin{gathered}\int_{\Omega_2 - A} |\nabla u|^2(r,\theta)drd\theta = \int_0^{2\pi} d\theta \int_{\frac{R_2(\theta)}{R_1(\theta)}a(\theta)}^{R_2(\theta)} dr\ (u_r^2 + \frac{1}{r^2}u_\theta^2)(r,\theta) = \\ \int_0^{2\pi} d\theta \int_{a(\theta)}^{R_1(\theta)} dr\ [\phi_r^2(s,\theta)\frac{R_1}{R_2}(\theta) + \phi_r^2(s,\theta)((\frac{R_1}{R_2}(\theta))')^2\frac{R_2}{R_1}(\theta) + \\ \frac{1}{s^2}\phi_\theta^2(s,\theta)\frac{R_1}{R_2}(\theta) + \frac{2}{s}\phi_r(s,\theta)\phi_\theta(s,\theta)(\frac{R_1}{R_2}(\theta))'].\end{gathered} \tag{1.9}$$

Let $\delta(\theta) > 0$ be an as yet unspecified function. Using the inequality $2ab \le a^2 + b^2$, we have

$$\frac{2}{s}\phi_r(s,\theta)\phi_\theta(s,\theta)(\frac{R_1}{R_2}(\theta))' \le \frac{1}{\delta(\theta)}\phi_r^2(s,\theta)((\frac{R_1}{R_2}(\theta))')^2\frac{R_2}{R_1}(\theta) + \frac{1}{s^2}\delta(\theta)\phi_\theta^2(s,\theta)\frac{R_1}{R_2}(\theta). \tag{1.10}$$

Substituting (1.10) in (1.9), we have

$$\int_{\Omega_2 - A} |\nabla u|^2(r,\theta)drd\theta \le \int_0^{2\pi} d\theta \int_{a(\theta)}^{R_1(\theta)} dr\ [\phi_r^2(s,\theta)C(\theta) + \frac{1}{s^2}\phi_\theta^2(s,\theta)D(\theta)], \tag{1.11}$$

where $C(\theta) = \frac{R_1}{R_2}(\theta) + ((\frac{R_1}{R_2}(\theta))')^2\frac{R_2}{R_1}(\theta)(1 + \frac{1}{\delta(\theta)})$ and $D(\theta) = \frac{R_1}{R_2}(\theta)(1 + \delta(\theta))$.

From (1.11), we conclude that

$$\int_{\Omega_2 - A} |\nabla u|^2(r,\theta)drd\theta \le \int_{\Omega_1 - A} |\nabla \phi|^2(r,\theta)drd\theta, \text{ if } C(\theta), D(\theta) \le 1. \tag{1.12}$$

The same change of variables also shows that

$$\begin{gathered}\int_{\Omega_2 - A} u^2(r,\theta)drd\theta = \int_0^{2\pi} d\theta \int_{a(\theta)}^{R_1(\theta)} dr\ \phi^2(s,\theta)\frac{R_2}{R_1}(\theta) \ge \\ \int_0^{2\pi} d\theta \int_{a(\theta)}^{R_1(\theta)} dr\ \phi^2(s,\theta) = \int_{\Omega_1 - A} \phi^2 drd\theta.\end{gathered} \tag{1.13}$$

Since $\lambda_0(\Omega_1,A)=\frac{\frac{1}{2}\int_{\Omega_1-A}|\nabla\phi|^2(r,\theta)drd\theta}{\int_{\Omega_1-A}\phi^2(r,\theta)drd\theta}$, it follows from (1.8), (1.12) and (1.13) that $\lambda_0(\Omega_1,A)>\lambda_0(\Omega_2,A)$, if $C(\theta),D(\theta)\le 1$. To obtain $D(\theta)\le 1$, we need

$$\delta(\theta)\le\frac{R_2}{R_1}(\theta)-1. \tag{1.14}$$

Doing a little algebra, we conclude that to obtain $C(\theta)\le 1$, we need

$$\delta(\theta)\ge\frac{((\frac{R_1}{R_2}(\theta))')^2}{\frac{R_1}{R_2}(\theta)-(\frac{R_1}{R_2}(\theta))^2-((\frac{R_1}{R_2}(\theta))')^2}; \tag{1.15-a}$$

$$\frac{R_1}{R_2}(\theta)-(\frac{R_1}{R_2}(\theta))^2-((\frac{R_1}{R_2}(\theta))')^2>0. \tag{1.15-b}$$

Recall that by assumption, R_1 is strictly smaller than R_2. Thus, in order that there exist a function $\delta>0$ satisfying (1.14) and (1.15-a), it is necessary and sufficient that

$$\frac{((\frac{R_1}{R_2}(\theta))')^2}{\frac{R_1}{R_2}(\theta)-(\frac{R_1}{R_2}(\theta))^2-((\frac{R_1}{R_2}(\theta))')^2}\le\frac{R_2}{R_1}(\theta)-1. \tag{1.16}$$

After a little algebra, one finds that (1.15-b) and (1.16) together are equivalent to (1.15-b) and the inequality

$$((\frac{R_1}{R_2})')^2+2(\frac{R_1}{R_2})^2\le\frac{R_1}{R_2}+(\frac{R_1}{R_2})^3. \tag{1.17}$$

It is easy to see that if (1.15-b) does not hold, then (1.17) does not hold. Thus, (1.17) is necessary and sufficient to guarantee the existence of a $\delta(\theta)>0$ such that $C(\theta),D(\theta)\le 1$. Note that (1.17) is equivalent to (1.7) with f replaced by $g\equiv\frac{1}{f}$. It is easy to see that (1.7) holds for f if and only if it holds for $g=\frac{1}{f}$. This completes the proof in the case that $R_1<R_2$.

Now assume that $R_1\le R_2$. Let $R_{2,\delta}=R_2+\delta$, for $\delta>0$, and let $\Omega_{2,\delta}$ denote the corresponding domain. Using the variational formula (1.6), it is easy to see that $\lambda_0(\Omega_{2,\delta},A)$ is lower semicontinuous in $\delta>0$. Since $R_{2,\delta}>R_1$, we have $\lambda_0(\Omega_1,A)>\lambda_0(\Omega_{2,\delta},A)$, and thus $\lambda_0(\Omega_1,A)\ge\lambda_0(\Omega_2,A)$.

ii. It suffices to show that for each $t\in[0,1)$, there exists an $\epsilon_0=\epsilon_0(t)>0$ such that

$$\lambda_0(\Omega_s,A)\le\lambda_0(\Omega_t,A),\ \text{for } s\in(t,t+\epsilon_0].$$

For $\epsilon > 0$, we write

$$R(\theta, t+\epsilon) = R(\theta, t) + R_t(\theta, t)\epsilon + \frac{1}{2}R_{tt}(\theta, t)\epsilon^2 + o(\epsilon^2), \text{ as } \epsilon \to 0. \tag{1.18}$$

The assumption that R is C^2 guarantees that the term $o(\epsilon^2)$ in (1.18) is uniform in t and θ. We now apply the result in part (i) with R_1 replaced by $R(\cdot, t)$ and R_2 replaced by $R(\cdot, t+\epsilon)$, for $\epsilon > 0$. The function denoted by f in the statement of part (i) is given in the present context by $f_\epsilon(\theta) = \frac{R(\theta, t+\epsilon)}{R(\theta, t)}$. Thus, from (1.18), we have

$$f_\epsilon(\theta) = 1 + \frac{R_t(\theta, t)}{R(\theta, t)}\epsilon + \frac{1}{2}\frac{R_{tt}(\theta, t)}{R(\cdot, t)}\epsilon^2 + o(\epsilon^2), \text{ as } \epsilon \to 0. \tag{1.19}$$

Substituting f_ϵ for f in (1.7) and equating powers of ϵ, one finds that the leading order non-vanishing term is ϵ^2 and that (1.7) will hold for sufficiently small $\epsilon > 0$ if $((\frac{R_t}{R})_\theta)^2 < (\frac{R_t}{R})^2$, or equivalently, if $|(\frac{R_t}{R})_\theta)| < \frac{R_t}{R}$. The limiting case, $|(\frac{R_t}{R})_\theta)| \le \frac{R_t}{R}$, is treated as in part (i). $\square$

Acknowledgement. I thank Yehuda Pinchover for suggesting the simple proof that $\phi_{0,2}$ is nondecreasing in r, in place of my more complicated one in the original draft.

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