SOME CONNECTIONS BETWEEN PERMUTATION CYCLES AND TOUCHARD POLYNOMIALS AND BETWEEN PERMUTATIONS THAT FIX A SET AND COVERS OF MULTISETS

ROSS G. PINSKY

Dedicated to the memory of my cousin, Mark A. Pinsky (1940-2016)

Abstract. We present a new proof of a fundamental result concerning cycles of random permutations which gives some intuition for the connection between Touchard polynomials and the Poisson distribution. We also introduce a rather novel permutation statistic and study its distribution. This quantity, indexed by m, is the number of sets of size m fixed by the permutation. This leads to a new and simpler derivation of the exponential generating function for the number of covers of certain multisets.

1. Introduction and Statement of Results

In this paper, we present a new and perhaps simpler proof of a fundamental result concerning cycles of random permutations which gives some intuition for the connection between Touchard polynomials and the Poisson distribution. We also introduce a rather novel permutation statistic and study its distribution. This quantity, indexed by m, is the number of sets of size m fixed by the permutation. This leads to a new and simpler derivation of the exponential generating function for the number of covers of certain multisets.

We begin by recalling some basic facts concerning Bell numbers and Touchard polynomials, and their connection to Poisson distributions. The

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facts noted below without proof can be found in many books on combinatorics; for example, in [12], [14]. The Bell number B_n denotes the number of partitions of a set of n distinct elements. Elementary combinatorial reasoning yields the recursive formula

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k, \ n \geq 0, \tag{1.1}$$

where $B_0 = 1$. Let

$$E_B(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n, \tag{1.2}$$

denote the exponential moment generating function of $\{B_n\}_{n=0}^{\infty}$. Using (1.1) it is easy to show that $E'_B(x) = e^x E_B(x)$, from which it follows that

$$E_B(x) = e^{e^x - 1}. \tag{1.3}$$

A random variable X has the Poisson distribution Pois(λ), $\lambda > 0$, if $P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, \ldots$. Let $M_\lambda(t) = Ee^{tX}$ denote the moment generating function of X, and let $\mu_{n;\lambda} = EX^n$ denote the nth moment of X. Since $M_\lambda^{(n)}(0) = \mu_{n;\lambda}$, we have

$$M_\lambda(t) = \sum_{n=0}^{\infty} \frac{\mu_{n;\lambda}}{n!} t^n. \tag{1.4}$$

However a direct calculation gives

$$M(t) = \sum_{k=0}^{\infty} e^{tk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{\lambda(e^t - 1)}. \tag{1.5}$$

From (1.2)-(1.5), it follows that the nth moment $\mu_{n;1}$ of a Pois(1)-distributed random variable satisfies

$$\mu_{n;1} = B_n. \tag{1.6}$$

Since

$$\mu_{n;1} = EX^n = \sum_{k=0}^{\infty} (e^{-1} \frac{1}{k!}) k^n,$$

we conclude that

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}, \tag{1.7}$$

which is known as *Dobínski's formula.*

The Stirling number of the second kind $\left\{ {n \atop k} \right\}$ denotes the number of partitions of a set of n distinct elements into k nonempty sets. We will need the formula

$$x^n = \sum_{j=0}^{n} \left\{ {n \atop j} \right\} (x)_j, \tag{1.8}$$

where $(x)_j = x(x-1)\cdots(x-j+1)$ is the falling factorial, and one defines $(x)_0 = \left\{ {0 \atop 0} \right\} = 1$. It is enough to prove this for positive integers x, in which case x^n is the number of functions $f : [n] \to X$, where $|X| = x$. We now count such functions in another way. For each $j \in [x]$, consider all those functions whose range contains exactly j elements. The inverse images of these elements give a partition of $[n]$ into j nonempty sets, $\{B_i\}_{i=1}^{j}$. We can choose the particular j elements in $\binom{x}{j}$ ways, and we can order the sets $\{B_i,\}_{i=1}^{j}$ in $j!$ ways. Thus there are $\left\{ {n \atop j} \right\}\binom{x}{j} j! = \left\{ {n \atop j} \right\}(x)_j$ such functions.

Now using (1.8) and the fact that $(k)_j = 0$ for $j > k$, we can write the nth moment $\mu_{n;\lambda}$ of a Pois(λ)-distributed random variable as

$$\begin{aligned} &\mu_{n;\lambda} = \sum_{k=0}^{\infty} (e^{-\lambda} \frac{\lambda^k}{k!}) k^n = \sum_{k=0}^{\infty} (e^{-\lambda} \frac{\lambda^k}{k!}) \Big(\sum_{j=0}^{n} \left\{ {n \atop j} \right\} (k)_j \Big) = \\ &\sum_{k=0}^{\infty} (e^{-\lambda} \frac{\lambda^k}{k!}) \Big(\sum_{j=0}^{k \wedge n} \left\{ {n \atop j} \right\} (k)_j \Big) = e^{-\lambda} \sum_{j=0}^{n} \left\{ {n \atop j} \right\} \sum_{k=j}^{\infty} \lambda^k \frac{(k)_j}{k!} = \\ &\sum_{j=0}^{n} \left\{ {n \atop j} \right\} \lambda^j. \end{aligned} \tag{1.9}$$

The Touchard polynomials $T_n(x)$, $n \geq 0$, are defined by

$$T_n(x) = \sum_{j=0}^{n} \left\{ {n \atop j} \right\} x^j.$$

Thus, (1.9) gives the formula

$$\mu_{n;\lambda} = T_n(\lambda). \tag{1.10}$$

Since

$$\mu_{n;\lambda} = \sum_{k=0}^{\infty} (e^{-\lambda} \frac{\lambda^k}{k!}) k^n,$$

we conclude from (1.10) that

$$T_n(x) = e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k. \tag{1.11}$$

Since $T_n(1) = \sum_{j=0}^{n} \left\{ {n \atop j} \right\} = B_n$, the Dobínski formula (1.7) is contained in (1.11).

Let S_n denote the set of permutations of $[n] =: \{1,\dots,n\}$. For $\sigma \in S_n$, let $C_m^{(n)}(\sigma)$ denote the number of cycles of length m in σ. Let P_n denote the uniform probability measure on S_n. We can now think of $\sigma \in S_n$ as random, and of $C_m^{(n)}$ as a random variable. Using generating function techniques and/or inclusion-exclusion formulas, one can show that under P_n, the distribution of the random variable $C_m^{(n)}$ converges weakly to $Z_{\frac{1}{m}}$, where $Z_{\frac{1}{m}}$ has the Pois($\frac{1}{m}$)-distribution; equivalently;

$$\lim_{n\to\infty} P_n(C_m^{(n)} = j) = e^{-\frac{1}{m}}\frac{1}{m^j j!},\ j = 0,1\dots. \tag{1.12}$$

More generally, we consider the Ewens sampling distributions, $P_{n;\theta}$, $\theta > 0$, on S_n as follows. Let $N^{(n)}(\sigma)$ denote the number of cycles in the permutation $\sigma \in S_n$, and let $s(n,k) = |\{\sigma \in S^n : N^{(n)}(\sigma) = k\}|$ denote the number of permutations in S_n with k cycles. It is known that the polynomial $\sum_{k=1}^{n} s(n,k)\theta^k$ is equal to the rising factorial $\theta^{(n)}$, defined by $\theta^{(n)} = \theta(\theta+1)\cdots(\theta+n-1)$. For $\theta > 0$, define the probability measure $P_{n;\theta}$ on S_n by

$$P_{n;\theta}(\{\sigma\}) = \frac{\theta^{N^{(n)}(\sigma)}}{\theta^{(n)}}.$$

Of course, $P_{n;1}$ reduces to the uniform measure P_n. The following theorem is well-known.

Theorem C. *Under $P_{n,\theta}$, the random vector $(C_1^{(n)}, C_2^{(n)}, \cdots, C_m^{(n)}, \dots)$ converges weakly to $(Z_\theta, Z_{\frac{\theta}{2}}, \cdots, Z_{\frac{\theta}{m}}, \cdots)$, where the random variables $\{Z_{\frac{\theta}{m}}\}_{m=1}^{\infty}$ are independent, and $Z_{\frac{\theta}{m}}$ has the Pois($\frac{\theta}{m}$)-distribution:*

$$(C_1^{(n)}, C_2^{(n)}, \cdots C_m^{(n)}, \dots) \stackrel{w}{\Rightarrow} (Z_\theta, Z_{\frac{\theta}{2}}, \cdots, Z_{\frac{\theta}{m}} \cdots); \tag{1.13}$$

equivalently,

$$\lim_{n\to\infty} P_{n;\theta}(C_1^{(n)} = j_1, C_2^{(n)} = j_2, \ldots, C_m^{(n)} = j_m) = \prod_{k=1}^{m} e^{-\frac{\theta}{k}} \frac{(\frac{\theta}{k})^{j_k}}{j_k!}, \tag{1.14}$$

for all $m \geq 1$ *and* $j_1, \ldots j_m \in \mathbb{Z}_+$.

Theorem C can be proved using moment generating functions; see for example, [1], [11]. We will use the method of moments to give a new and perhaps simpler proof of Theorem C. More significantly, it which will give intuition for (1.10), or equivalently, for (1.11); that is for the connection between the moments of Poisson random variables and Touchard polynomials. The key to this connection comes by representing the random variables $C_m^{(n)}$ as sums of indicator random variables. We note that, at least in the case of uniformly distributed permutations, one can ostensibly find a different proof by the moment method in the literature [2], but that proof seems more complicated than ours, and more importantly, doesn't provide the above-noted intuition. The paper [2] gives a formula from [13] on factorial moments, and then states that with this formula the proof follows by the moment method. However, the derivation of that formula in [13] is not explicit and seems more complicated than our proof.

We now consider a permutation statistic that hasn't been studied much. (Indeed, it was only after completing the first version of this paper that we were directed to any papers on this subject.) For $\sigma \in S_n$ and $A \subset [n]$, define $\sigma(A) = \{\sigma_j : j \in A\}$. If $\sigma(A) = A$, we will say that σ fixes A. Let $\mathcal{F}_m^{(n)}(\sigma)$ denote the number of sets of cardinality m that are fixed by σ. (Note that $\mathcal{F}_1^{(n)}(\sigma) = C_1^{(n)}(\sigma)$, the number of fixed points of σ.) A little thought reveals that

$$\mathcal{F}_m^{(n)}(\omega) = \sum_{\substack{(l_1,\ldots,l_m): \sum_{j=1}^m j l_j = m \\ l_j \leq C_j^{(n)}, j \in [m]}} \prod_{j=1}^{m} \binom{C_j^{(n)}(\omega)}{l_j}. \tag{1.15}$$

For example, if $\sigma \in S_9$ is written in cycle notation as $\sigma = (379)(24)(16)(5)(8)$, then $\mathcal{F}_4^{(9)}(\omega) = 5$, with the sets $A \subset [9]$ for which $|A| = 4$ and $\sigma(A) = A$ being $\{3,5,7,9\}, \{3,7,8,9\}, \{1,2,4,6\}, \{2,4,5,8\}, \{1,5,6,8\}$.

We consider the uniform measure $P_n = P_{n;1}$ on S_n. From Theorem C and (1.15) it follows that the random variable $\mathcal{F}_m^{(n)}$ under P_n converges weakly as $n \to \infty$ to the random variable

$$\mathcal{F}_m =: \sum_{\substack{(l_1,\ldots,l_m):\sum_{j=1}^m jl_j=m\\ l_j\le Z_{\frac1j},j\in[m]}} \prod_{j=1}^m \binom{Z_{\frac1j}}{l_j}, \tag{1.16}$$

where $\{Z_{\frac1j}\}_{j=1}^m$ are independent and $Z_{\frac1j}$ has the Pois($\frac1j$)-distribution.

Remark. Note that $\mathcal{F}_1 = Z_1$, $\mathcal{F}_2 = \binom{Z_1}{2} + Z_{\frac12}$, $\mathcal{F}_3 = Z_{\frac13} + Z_{\frac12}Z_1 + \binom{Z_1}{3}$.

For $k, m \in \mathbb{N}$, consider the multiset consisting of m copies of the set $[k]$. A collection $\{\Gamma_l\}_{l=1}^r$ such that each Γ_l is a nonempty subset of $[k]$, and such that each $j \in [k]$ appears in exactly m from among the r sets $\{\Gamma_l\}_{l=1}^r$, is called an *m-cover of $[k]$ of order r*. Denote the total number of m covers of $[k]$, regardless of order, by $v_{k;m}$. Note that when $m = 1$, we have $v_{k;1} = B_k$, the kth Bell number, denoting the number of partitions of a set of k elements. Also, it's very easy to see that $v_{1;m} = 1$ and $v_{2,m} = m + 1$.

By calculating directly the moments of $\mathcal{F}_m^{(n)}$, we will prove the following theorem.

Theorem 1. *For $m, k \in \mathbb{N}$,*

$$E\mathcal{F}_m^k = v_{k;m}.$$

In particular, $E\mathcal{F}_m = 1$ and $E\mathcal{F}_m^2 = m + 1$; thus, $Var(\mathcal{F}_m) = m$.

Remark. It is natural to suspect that $\mathcal{F}_m$ converges weakly to 0 as $m \to \infty$; that is, $\lim_{m\to\infty} P(\mathcal{F}_m \ge 1) = 0$. This is in fact a hard problem. In [9] it was shown that $P_n(\mathcal{F}_m^{(n)}) \le Am^{-\frac{1}{100}}$, for $1 \le m \le \frac{n}{2}$ and $n \ge 2$. Thus indeed, $\mathcal{F}_m$ converges weakly to 0 as $m \to \infty$. A lower bound on $P(\mathcal{F}_m \ge 1)$ of the form $A\frac{\log m}{m}$ was obtained in [6]. These results were dramatically improved in [10] where it was shown that $P(\mathcal{F}_m \ge 1) = m^{-\delta+o(1)}$ as $m \to \infty$, where $\delta = 1 - \frac{1+\log\log 2}{\log 2} \approx 0.08607$. And very recently, in [8], this latter bound has been refined to $A_1 m^{-\delta}(1 + \log m)^{-\frac32} \le P(\mathcal{F}_m \ge 1) \le A_2 m^{-\delta}(1 + \log m)^{-\frac32}$.

Let

$$V_m(x) = \sum_{k=1}^{\infty} \frac{v_{k;m}}{k!} x^k$$

denote the exponential generating function of the sequence $\{v_{k;m}\}_{k=1}^{\infty}$. Of course, by Theorem 1 V_m is also the moment generating function of the random variable $\mathcal{F}_m$: $V_m(x) = Ee^{x\mathcal{F}_m}$. Using (1.16) and Theorem 1, we will give an almost immediate proof of the following representation theorem for $V_m(x)$. We use the notation $[z^m]P(z) = a_m$, where $P(z) = \sum_{m=0}^{\infty} a_m z^m$.

Theorem 2.

$$V_m(x) = Ee^{x\mathcal{F}_m} = e^{-\sum_{j=1}^{m} \frac{1}{j}} \sum_{u_1,\ldots,u_m \geq 0} \Big(\prod_{j=1}^{m} \frac{j^{-u_j}}{u_j!}\Big)\ e^{x\gamma_m(u)}, \tag{1.17}$$

where

$$\gamma_m(u) = [z^m] \prod_{j=1}^{m} (1 + z^j)^{u_j}.$$

Remark. When $m = 2, 3$, the above formula reduces to

$$V_2(x) = e^{-\frac{3}{2}} e^{\frac{1}{2}e^x} \sum_{r=0}^{\infty} \frac{e^{\binom{r}{2}x}}{r!};$$

$$V_3(x) = e^{-\frac{11}{6}} e^{\frac{e^x}{3}} \sum_{r=0}^{\infty} \frac{e^{\binom{r}{3}x + \frac{1}{2}e^{rx}}}{r!}.$$

The formula for $m = 2$ was proved by Comtets [4] and the formula for $m = 3$ was proved by Bender [3]. The case of general m was proved by Devitt and Jackson [5]. They also prove that there exists a number c such that the extraction of the coefficient $v_{k;m}$ from the exponential generating function $V_m(x)$ can be done in no more than $ck^m \log k$ arithmetic operations.

In section 2 we will give our new proof of Theorem C via the method of moments. In section 3 we prove Theorems 1 and 2.

2. A proof of Theorem C via the method of moments

If a sequence of nonnegative random variables $\{X_n\}_{n=1}^{\infty}$ satisfies $\sup_{n\geq 1} EX_n < \infty$, then the sequence is tight, that is, pre-compact with respect to weak convergence. Let X be distributed as one of the accumulation points. If for some $k\in\mathbb{N}$, $\lim_{n\to\infty} EX_n^k$ exists and equals μ_k, and $\sup_{n\ge1} EX_n^{k+1}<\infty$, then the $\{X_n^k\}_{n=1}^\infty$ are uniformly integrable, and thus $EX^k=\mu_k$. Thus, if

$$\mu_k =: \lim_{n\to\infty} EX_n^k \text{ exists for all } k\in\mathbb{N}, \tag{2.1}$$

then $EX^k=\mu_k$, for all k. The Stieltjes moment theorem states that if

$$\sup_{k\ge1}\frac{\mu_k^{\frac1k}}{k}<\infty, \tag{2.2}$$

then the sequence $\{\mu_k\}_{k=1}^\infty$ uniquely characterizes the distribution [7]. We conclude then that if a sequence of nonnegative random variables $\{X_n\}_{n=1}^\infty$ satisfies (2.1) and (2.2), then the sequence is weakly convergent to a random variable X satisfying $EX^k=\mu_k$.

An extremely crude argument shows that the Bell numbers satisfy $B_k\le k^k$; thus

$$\sup_{k\ge1}\frac{B_k^{\frac1k}}{k}<\infty. \tag{2.3}$$

By (1.10), the kth moment $\mu_{k;\frac{\theta}{m}}$ of the Pois($\frac{\theta}{m}$)-distributed random variable $Z_{\frac{\theta}{m}}$ is equal to $T_k(\frac{\theta}{m})$. Now $T_k(\frac{\theta}{m})$ is bounded from above by $T_k(\theta)$, for all $m\ge1$, and $T_k(\theta)\le B_k\max(1,\theta^k)$. Thus, in light of (2.3) and the previous paragraph, if we prove that

$$\lim_{n\to\infty} E_{n;\theta}(C_m^{(n)})^k = T_k(\frac{\theta}{m}),\ k,m\in\mathbb{N}, \tag{2.4}$$

where $E_{n;\theta}$ denotes the expectation with respect to $P_{n;\theta}$, then we will have proved that $C_m^{(n)}$ under $P_{n;\theta}$ converges weakly to $Z_{\frac{\theta}{m}}$, for all $m\in\mathbb{N}$. And if we then prove that

$$\lim_{n\to\infty} E_{n;\theta}\prod_{j=1}^m (C_j^{(n)})^{k_j} = \prod_{j=1}^m T_{k_j}(\frac{\theta}{j}),\ m\ge2, k_j\in\mathbb{N}, j=1,\dots,m, \tag{2.5}$$

then we will have completed the proof of Theorem C.

We first prove (2.4). In fact, we will first prove (2.4) in the case of the uniform measure, $P_n=P_{n;1}$. Once we have this, the case of general θ will follow after a short explanation. Assume that $n\ge mk$. For $D\subset[n]$ with

$|D| = m$, let $1_D(\sigma)$ be equal to 1 or 0 according to whether or not $\sigma \in S_n$ possesses an m-cycle consisting of the elements of D. Then we have

$$C_m^{(n)}(\sigma) = \sum_{\substack{D \subset [n] \\ |D|=m}} 1_D(\sigma), \tag{2.6}$$

and

$$E_n (C_m^{(n)})^k = \sum_{\substack{D_j \subset [n], |D_j|=m \\ j \in [k]}} E_n \prod_{j=1}^k 1_{D_j}. \tag{2.7}$$

Now $E_n \prod_{j=1}^k 1_{D_j} \neq 0$ if and only if for some $l \in [k]$, there exist disjoint sets $\{A_i\}_{i=1}^l$ such that $\{D_j\}_{j=1}^k = \{A_i\}_{i=1}^l$. If this is the case, then

$$E_n \prod_{j=1}^k 1_{D_j} = \frac{(n-lm)!((m-1)!)^l}{n!}. \tag{2.8}$$

(Here we have used the assumption that $n \geq mk$, since otherwise $n - ml$ will be negative for certain $l \in [k]$.) The number of ways to construct a collection of l disjoint sets $\{A_i\}_{i=1}^l$, each of which consists of m elements from $[n]$, is $\frac{n!}{(m!)^l (n-lm)!\, l!}$. Given the $\{A_i\}_{i=1}^l$, the number of ways to choose the sets $\{D_j\}_{j=1}^k$ so that $\{D_j\}_{j=1}^k = \{A_i\}_{i=1}^l$ is equal to the Stirling number $\left\{ {k \atop l} \right\}$, the number of ways to partition a set of size k into l nonempty parts, multiplied by $l!$, since the labeling must be taken into account. From these facts along with (2.7) and (2.8), we conclude that for $n \geq mk$,

$$\begin{aligned} E_n (C_m^{(n)})^k &= \sum_{l=1}^k \Big(\frac{(n-lm)!((m-1)!)^l}{n!}\Big)\Big(\frac{n!}{(m!)^l (n-lm)!}\Big) \left\{ {k \atop l} \right\} = \\ &\sum_{l=1}^k \frac{1}{m^l} \left\{ {k \atop l} \right\} = T_k\Big(\frac{1}{m}\Big), \end{aligned} \tag{2.9}$$

proving (2.4) in the case $\theta = 1$.

For the case of general θ, we note that the only change that must be made in the above proof is in (2.8). Recalling that $s(n, k)$ denotes the number of permutations in S_n with k cycles, we have

(2.10)

$$E_{n;\theta}\prod_{j=1}^{k}1_{D_j}=\frac{((m-1)!)^l\sum_{k=1}^{n-ml}s(n-lm,k)\theta^{k+l}}{\theta^{(n)}}=\frac{\theta^{(n-ml)}((m-1)!)^l}{\theta^{(n)}}\theta^l\sim$$

$$\frac{(n-ml)!((m-1)!)^l}{n!}\theta^l=\theta^lE_n\prod_{j=1}^{k}1_{D_j},\ \text{as } n\to\infty.$$

Thus, instead of (2.9), we have

$$E_{n;\theta}(C_m^{(n)})^k\sim\sum_{l=1}^{k}(\frac{\theta}{m})^l\left\{{k \atop l}\right\}=T_k(\frac{\theta}{m}),\ \text{as } n\to\infty.$$

We now turn to (2.5). The method of proof is simply the natural extension of the one used to prove (2.4); thus, since the notation is cumbersome we will suffice with illustrating the method by proving that

$$\lim_{n\to\infty}E_n(C_{m_1}^{(n)})^{k_1}(C_{m_2}^{(n)})^{k_2}=T_{k_1}(\frac{1}{m_1})T_{k_2}(\frac{1}{m_2}).\tag{2.11}$$

Let $n\ge m_1k_1+m_2k_2$. By (2.6), we have

$$E_n(C_{m_1}^{(n)})^{k_1}(C_{m_2}^{(n)})^{k_2}=\sum_{\substack{D_j^1\subset[n],|D_j^1|=m_1\\ j\in[k_1]}}\ \sum_{\substack{D_j^2\subset[n],|D_j^2|=m_2\\ j\in[k_2]}}E_n\prod_{j=1}^{k_1}1_{D_j^1}\prod_{j=1}^{k_2}1_{D_j^2}.\tag{2.12}$$

Now $E_n\prod_{j=1}^{k_1}1_{D_j^1}\prod_{j=1}^{k_2}1_{D_j^2}\ne0$ if and only if for some $l_1\in[k_1]$ and some $l_2\in[k_2]$, there exist disjoint sets $\{A_i^1\}_{i=1}^{l_1}$, $\{A_i^2\}_{i=1}^{l_2}$ such that $\{D_j^r\}_{j=1}^{k_r}=\{A_i^r\}_{i=1}^{l_r}$, $r=1,2$. If this is the case, then

$$E_n\prod_{j=1}^{k_1}1_{D_j^1}\prod_{j=1}^{k_2}1_{D_j^2}=\frac{(n-l_1m_1-l_2m_2)!((m_1-1)!)^{l_1}(m_2-1)!)^{l_2}}{n!}.\tag{2.13}$$

The number of ways to construct disjoint sets $\{A_i^1\}_{i=1}^{l_1},\{A_i^2\}_{i=1}^{l_2}$, with the A_i^1 each consisting of m_1 elements from $[n]$ and the A_i^2 each consisting of m_2 elements from $[n]$, is $\frac{n!}{(m_1!)^{l_1}(m_2!)^{l_2}(n-l_1m_1-l_2m_2)!\,l_1!l_2!}$. Given $\{A_i^1\}_{i=1}^{l_1},\{A_i^2\}_{i=1}^{l_2}$, the number of ways to choose the sets $\{D_j^1\}_{j=1}^{k_1},\{D_j^2\}_{j=1}^{k_2}$ so that $\{D_j^1\}_{j=1}^{k_1}=\{A_i^1\}_{i=1}^{l_1}$ and $\{D_j^2\}_{j=1}^{k_2}=\{A_i^2\}_{i=1}^{l_2}$ is equal to $\big(\left\{{k_1 \atop l_1}\right\}l_1!\big)\big(\left\{{k_2 \atop l_2}\right\}l_2!\big)$. We have

$$\frac{(n-l_1m_1-l_2m_2)!((m_1-1)!)^{l_1}(m_2-1)!)^{l_2}}{n!}\times$$

$$\frac{n!}{(m_1!)^{l_1}(m_2!)^{l_2}(n-l_1m_1-l_2m_2)!l_1!l_2!}=\frac{1}{m_1^{l_1}m_2^{l_2}l_1!l_2!}.$$

From these fact along with (2.12) and (2.13), we conclude that for $n \geq m_1k_1 + m_2k_2$,

$$E_n C_{m_1}^{(n)})^{k_1}(C_{m_2}^{(n)})^{k_2} = \sum_{l_2=1}^{k_2}\sum_{l_1=1}^{k_1} \frac{1}{m_1^{l_1} m_2^{l_2}} \left\{ {k_1 \atop l_1} \right\} \left\{ {k_2 \atop l_2} \right\} = T_{k_1}(\frac{1}{m_1})T_{k_2}(\frac{1}{m_2}),$$

proving (2.11). □

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. Since $\mathcal{F}_m^{(n)}$ converges weakly to $\mathcal{F}_m$, it follows from the discussion in the first paragraph of section 2 that it suffices to show that

$$\lim_{n\to\infty} E_n(\mathcal{F}_m^{(n)})^k = v_{k;m}. \tag{3.1}$$

Let $n \geq km$. For $D \subset [n]$, let $1_D(\sigma)$ equal 1 or 0 according to whether or not $\sigma \in S_n$ induces an embedded permutation on D. Then we have

$$\mathcal{F}_m^{(n)}(\omega) = \sum_{\substack{D\subset[n] \\ |D|=m}} 1_D(\omega), \tag{3.2}$$

and

$$E_n(\mathcal{F}_m^{(n)})^k = \sum_{\substack{D_j\subset[n],|D_j|=m \\ j\in[k]}} E_n \prod_{j=1}^{k} 1_{D_j}. \tag{3.3}$$

There is a one-to-one correspondence between collections $\{D_j\}_{j=1}^k$, satisfying $D_j \subset [n]$ and $|D_j| = m$, and collections $\{A_I\}_{I\subset[k]}$ of disjoint sets satisfying $A_I \subset [n]$ and satisfying

$$\sum_{I:i\in I} l_I = m, \text{ for all } i \in [k], \tag{3.4}$$

where

$$l_I = |A_I|, \ I \subset [k]. \tag{3.5}$$

The correspondence is through the formula

$$A_I = \big(\cap_{i\in I} D_i\big) \cap \big(\cap_{i\in[k]-I}\left([n] - D_i\right)\big), \ I \subset [k]. \tag{3.6}$$

Now $\prod_{j=1}^k 1_{D_j}(\omega)=1$ if and only if for all $I\subset[k]$, σ induces an embedded permutation on A_I. Thus, we have

$$E_n\prod_{j=1}^k 1_{D_j}=\frac{(n-\sum_{I\subset[k]}l_I)!\prod_{I\subset[k]}l_I!}{n!}. \tag{3.7}$$

Given the values $l_I=|A_I|$, $I\subset[k]$, the number of ways to construct the disjoint sets $\{A_I\}_{I\subset[k]}$ is $\frac{n!}{(n-\sum_{I\subset[k]}l_I)!\prod_{I\subset[k]}l_I!}$. Using this with (3.3)-(3.7), it follows that $E_n(\mathcal{F}_m^{(n)})^k$ equals the number of solutions $\{l_I\}_{I\subset[k]}$ to (3.4).

To complete the proof, we will show that the number of solutions to (3.4) is $v_{k;m}$. Consider the set $\cup_{j=1}^k D_j\subset[n]$. Label the elements of this set by $\{x_i\}_{i=1}^r$. Of course, $m\le r\le km$. Now construct the sets $\{\Gamma_i\}_{i=1}^r$ by $\Gamma_i=\{j:x_i\in D_j\}$. By construction, the sets $\{\Gamma_i\}_{i=1}^r$ form an m-cover of $[k]$ (of order r). There is a one-to-one correspondence between solutions to (3.4) and m covers of $[k]$; indeed, $l_I=|\{i\in[k]:\Gamma_i=I\}|$. □

Proof of Theorem 2. We use (1.16) to calculate $V_m(x)=Ee^{x\mathcal{F}_m}$. We have

$$\begin{aligned}&V_m(x)=E\exp(x\mathcal{F}_m)=E\exp\big(x\sum_{\substack{(l_1,\dots,l_m):\sum_{j=1}^m jl_j=m\\ l_j\le Z_{\frac1j},j\in[m]}}\prod_{j=1}^m\binom{Z_{\frac1j}}{l_j}\big)=\\ &\sum_{u_1\ge0,\cdots,u_m\ge0}\prod_{j=1}^m(\frac1j)^{u_j}\frac{1}{u_j!}e^{-\frac1j}\exp\big(x\sum_{\substack{(l_1,\dots,l_m):\sum_{j=1}^m jl_j=m\\ l_j\le u_j,j\in[m]}}\prod_{j=1}^m\binom{u_j}{l_j}\big).\end{aligned} \tag{3.8}$$

Thus, to complete the proof, we only need to show that

$$\sum_{\substack{(l_1,\dots,l_m):\sum_{j=1}^m jl_j=m\\ l_j\le u_j,j\in[m]}}\prod_{j=1}^m\binom{u_j}{l_j}=\gamma_m(u), \tag{3.9}$$

where

$$\gamma_m(u)=[z^m]\prod_{j=1}^m(1+z^j)^{u_j}. \tag{3.10}$$

Expanding with the binomial formula, we have

$$\prod_{j=1}^m(1+z^j)^{u_j}=\prod_{j=1}^m\big(\sum_{l_j=0}^{u_j}\binom{u_j}{l_j}z^{jl_j}\big). \tag{3.11}$$

From (3.11) and (3.10), it follows that (3.9) holds. □

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Department of Mathematics, Technion—Israel Institute of Technology, Haifa, 32000, Israel

E-mail address: pinsky@math.technion.ac.il

URL: http://www.math.technion.ac.il/~pinsky/