UNIVERSAL BOUND INDEPENDENT OF GEOMETRY 
FOR SOLUTION TO SYMMETRIC STEADY-STATE 
DIFFUSION EQUATION IN EXTERIOR DOMAIN WITH 
BOUNDARY FLUX 

ROSS G. PINsky 

Abstract. Fix $R > 0$ and let $B_R$ denote the ball of radius $R$ centered 
at the origin in $\mathbb{R}^d$, $d \geq 2$. Let $D \subset B_R$ be an open set with smooth 
boundary and such that $\mathbb{R}^d - \bar{D}$ is connected, and let 

$$L = \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}$$ 

be a second order elliptic operator. Consider the following linear heat 
equation in the exterior domain $\mathbb{R}^d - \bar{D}$ with boundary flux: 

$$Lu = 0 \text{ in } \mathbb{R}^d - \bar{D};$$ 

$$a \nabla u \cdot \bar{n} = -h \text{ on } \partial D;$$ 

$$u > 0 \text{ is minimal},$$ 

where $h \geq 0$ is continuous, and where $\bar{n}$ is the unit inward normal to 
the domain $\mathbb{R}^d - \bar{D}$. The operator $L$ must possess a Green's function in 
order that a solution $u$ exist. An important feature of the equation is 
that there is no a priori bound on the supremum $\sup_{x \in \mathbb{R}^d - \bar{D}} u(x)$ of the 
solution exclusively in terms of the boundary flux $h$, the hyper-surface 
measure of $\partial D$ and the coefficients of $L$; rather the geometry of $D \subset B_R$ 
plays an essential role. However, we prove that in the case that $L$ is a 
symmetric operator with respect to some reference measure, then outside 
of $B_R$, the solution to (1.2) is uniformly bounded, independent of the 
particular choice of $D \subset B_R$. The proof uses a combination of analytic 
and probabilistic techniques.

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co-normal boundary condition, boundary flux.
1. Introduction and Statement of Results

Fix $R > 0$ once and for all and let $B_R$ denote the ball of radius $R$ centered at the origin in $\mathbb{R}^d$, $d \geq 2$. Let $D \subset B_R$ be an open set with smooth boundary and such that $\mathbb{R}^d - \bar{D}$ is connected. Let

\begin{equation}
L = \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}
\end{equation}

be a strictly elliptic operator in $\mathbb{R}^d - D$ with smooth coefficients $a = \{a_{i,j}\}_{i,j=1}^{d}$ and $b = \{b_i\}_{i=1}^{d}$. Consider the following linear steady-state heat equation in the exterior domain $\mathbb{R}^d - \bar{D}$ with boundary flux:

\begin{equation}
Lu = 0 \text{ in } \mathbb{R}^d - \bar{D};
\end{equation}

\begin{equation}
a \nabla u \cdot \bar{n} = -h \text{ on } \partial D;
\end{equation}

\begin{equation}
u > 0 \text{ is minimal},
\end{equation}

where $h \geq 0$ is continuous, and where $\bar{n}$ is the unit inward normal to the domain $\mathbb{R}^d - \bar{D}$. By minimal, we mean that the solution $u$ satisfies $u = \lim_{n \to \infty} u_n$, where for $n > R$, $u_n$ solves

\begin{equation}
Lu = 0 \text{ in } B_n - \bar{D};
\end{equation}

\begin{equation}
a \nabla u \cdot \bar{n} = -h \text{ on } \partial D;
\end{equation}

\begin{equation}
u = 0 \text{ on } \partial B_n.
\end{equation}

We note that a certain restriction must be placed on the operator $L$ in order that a solution $u$ exist to (1.2). This will be discussed below. An important feature of (1.2) is that there is no a priori bound on the supremum $\sup_{x \in \mathbb{R}^d - \bar{D}} u(x)$ of the solution exclusively in terms of the boundary flux $h$, the hyper-surface measure of $\partial D$ and the coefficients of $L$; rather the geometry of $D \subset B_R$ plays an essential role. Here is a simple example.

**Spherical Shell Example.** Consider a spherical shell centered at the origin in $\mathbb{R}^3$ with the radii of the inner and outer boundary spheres given by $R_1$ and $R_2$ respectively, where $R_1 < R_2 < R$. Now fix some direction $\theta_0 \in S^2$
and puncture the shell with a spherical bullet of radius $\frac{1}{n}$ centered in the $\theta_0$ direction. Thus, a cylindrical-like region of radius $\frac{1}{n}$ and length $R_2 - R_1$ has been removed from the spherical shell. Denote by $D_n$ the open set obtained by taking this punctured spherical shell and deleting its boundary. (If one insists that $D_n$ have a smooth boundary, one can smooth out the edges where the bullet enters and exits.) Let $\Gamma_n = \partial B_{R_1} \cap (\bar{D}_n)^c$ denote the punctured part of the inner sphere. By the maximum principle, the solution $u_n$ to (1.2) with $L = \Delta$ and $h = 1$ satisfies $u_n(x) \geq v_n(x)$, for $|x| < R_1$, where $v_n$ is the solution of

$$
\begin{align*}
\Delta v_n &= 0 \text{ in } B_{R_1}; \\
\nabla v_n \cdot \bar{n} &= -1 \text{ on } \partial B_{R_1} - \bar{\Gamma}_n; \\
v_n &= 0 \text{ on } \Gamma_n,
\end{align*}
$$

with $\bar{n}$ being the inward unit normal to $B_{R_1}$ at $\partial B_{R_1}$. By the maximum principle, $v_n(x)$ is increasing in $n$; let $v(x) \equiv \lim_{n \to \infty} v_n(x)$. By Harnack’s inequality, $v$ is either finite everywhere or infinite everywhere in $B_{R_1}$. If $v$ is finite, then $v$ must satisfy $\Delta v = 0$ in $B_{R_1}$ and satisfy the Neumann boundary condition $\nabla v \cdot \bar{n} = -1$ on all of $\partial B_{R_1}$. But this is impossible because solvability of the above equation with Neumann boundary data requires the compatibility condition $\int_{\partial B_{R_1}} (\nabla v \cdot \bar{n}) d\sigma(x) = -\int_{\partial B_{R_1}} \Delta v dx = 0$, where $d\sigma$ denotes Lebesgue hyper-surface measure on $\partial B_{R_1}$. Thus, $v \equiv \infty$ and consequently, $\lim_{n \to \infty} u_n(x) = \infty$, for $x \in B_{R_1}$. Yet, trivially, the sequence $\{|\partial D_n|\}_{n=1}^\infty$ is bounded in $n$.

However, we will prove that in the case that $L$ is a symmetric operator with respect to some reference measure, then for any $\delta > 0$, outside of $B_{R+\delta}$, the solution to (1.2) is uniformly bounded, independent of the particular choice of $D \subset B_R$. More precisely, we will show that for any $x \in R^d - B_{R}$, the solution $u(x)$ is bounded uniformly over all $D \subset B_R$, in terms of

1. the boundary flux $h$;
ii. the hyper-surface measure of $\partial D$ with respect to the measure whose density with respect to Lebesgue measure is the trace of the above mentioned reference measure;

iii. the behavior of the coefficients of $L$ outside of $\bar{B}_R$.

In particular, letting $\text{dist}_{\text{geod};D_n}(x_n, \bar{B}_{R+\delta})$ denote the length of a shortest path in $R^d - D_n$ from $x_n$ to $\bar{B}_{R+\delta}$, we will give an example of a sequence of open sets $\{D_n\}_{n=1}^{\infty} \subset B_R$ and points $x_n \in B_R - D_n$ with

$$\lim_{\delta \to 0} \lim_{n \to \infty} \text{dist}_{\text{geod};D_n}(x_n, \bar{B}_{R+\delta}) = \text{arbitrarily small positive number}$$

such that the solution $u_{D_n}$ to (1.2) with $D = D_n$ satisfies

$$\lim_{n \to \infty} u_{D_n}(x_n) = \infty, \text{ yet for all } x \in R^d - \bar{B}_R, \text{ one has } \sup_{x} u_{D_n}(x) < \infty.$$
convection vector field is pointing away from $D$ with sufficient strength, then the convection of the reactant from the boundary $\partial D$ into the domain $R^d - \bar{D}$ will not allow for an equilibrium state.

In order to obtain existence, in fact it is necessary and sufficient to assume that the operator $L$ possesses an appropriate Green’s function.

**Assumption G.** The operator $L$ with the co-normal boundary condition at $\partial D$ possesses a Green’s function.

Recall that the Green’s function $G^D_{\text{Neu}}(x, y)$ for an operator $L$ with the co-normal boundary condition at $\partial D$ is the minimal positive function $g(x, y)$ satisfying the following conditions: for each $y \in R^d - \bar{D}$ the function $g(\cdot, y)$ satisfies $Lg(\cdot, y) = -\delta_y$ in $R^d - \bar{D}$ and satisfies the homogeneous co-normal boundary condition at $\partial D$: $a \nabla g(\cdot, y) \cdot \bar{n} = 0$ on $\partial D$. The Green’s function for an operator $L$ on all of $R^d$ is the minimal positive function $g(x, y)$ satisfying $Lg(\cdot, y) = -\delta_y$ in $R^d$, for each $y \in R^d$. We make several remarks concerning the Green’s function.

**Remark 1.** Assumption G is equivalent to each of the following assumptions:

i. If $L$ is extended to be a smooth strictly elliptic operator on all of $R^d$, then this extension possesses a Green’s function;

ii. For $n > R$, let $V_n$ denote the solution to the equation

\begin{equation}
LV_n = 0 \text{ in } B_n - \bar{B}_R;
\end{equation}

\begin{equation} 
V_n = 1 \text{ on } \partial B_R, \quad V_n = 0 \text{ on } \partial B_n.
\end{equation}

Then

\begin{equation}
V \equiv \lim_{n \to \infty} V_n
\end{equation}

is not the constant function 1.

iii. The diffusion process $X(t)$ in $R^d - D$ corresponding to the operator $L$ and with co-normal reflection at $\partial D$ is transient, that is, $P_x(\lim_{t \to \infty} |X(t)| = \infty) = 1$, or equivalently, $P_x(\tau_R < \infty) < 1$, for $x \in R^d - \bar{B}_R$, where $P_x$ denotes probabilities for the diffusion starting from $x$ and $\tau_R = \inf\{t \geq 0 : X(t) \in$
\( B_R \) is the first hitting time of the ball \( B_R \). Indeed, the function \( V \) defined in (1.4)-(1.5) satisfies

\[
(1.6) \quad V(x) = P_x(\tau_R < \infty).
\]

For details, see [3].

**Remark 2.** If Assumption G is not in force, then the solution \( u_n \) to (1.3) will satisfy \( \lim_{n \to \infty} u_n = \infty \).

**Remark 3.** For generic choices of \( b \) such that \( L \) satisfies Assumption G, the solution \( u \) to (1.2) will satisfy \( \lim_{x \to \infty} u(x) = 0 \). Necessarily, one has \( \liminf_{x \to \infty} u(x) = 0 \). One can obtain \( \limsup_{x \to \infty} u(x) > 0 \) by choosing the convection term \( b \) pointing very strongly away from the origin when \( x \) is in certain sectors. (The restriction to certain sectors is necessary for otherwise the Green’s function would not exist.)

Our result requires that \( L \) be symmetric with respect to a weight \( e^Q \); that is, \( L \) must be of the form

\[
(1.7) \quad L = e^{-Q} \nabla \cdot e^Q a \nabla = \nabla \cdot a \nabla + a \nabla Q \nabla.
\]

(Note that such an \( L \) is of the form (1.1) with \( b_i = -\sum_{j=1}^d \frac{\partial a_{ij}}{\partial x_j} - \sum_{j=1}^d a_{ij} \frac{\partial Q}{\partial x_j} \).)

Note that the boundary condition in (1.1) involves the co-normal derivative; as is well-known, the operator \( L \) in (1.7) with the homogeneous co-normal boundary condition is symmetric with respect to the weight \( e^Q \):

\[
\int_{\partial \Omega} g L e^Q dx = \int_{\partial \Omega} f L e^Q dx, \quad \text{for all smooth, compactly supported} \quad f, g \quad \text{that satisfy} \quad a \nabla f \cdot \bar{n} = a \nabla g \cdot \bar{n} = 0 \quad \text{on} \partial \Omega.
\]

We first present a theorem for the case \( L = \Delta \) with \( d \geq 3 \). (We must restrict to \( d \geq 3 \) because there is no Green’s function when \( d = 2 \).) We can obtain tighter results in this case than in the case of generic \( L \). We note that the Green’s function for \( \Delta \) in \( \mathbb{R}^d \) is \( G(x, y) = \frac{|x-y|^{2-d}}{(d-2)\omega_d} \) where \( \omega_d \) denotes the Lebesgue hyper-surface measure of the unit sphere in \( \mathbb{R}^d \). Lebesgue hyper-surface measure on \( \partial \Omega \) is denoted by \( d\sigma \).
Theorem 1. Let $L = \Delta$ in $\mathbb{R}^d$, $d \geq 3$. Let $R > 0$ and assume that $D \subset B_R$. Let $\omega_d$ denote the surface measure of the unit sphere in $\mathbb{R}^d$. Then for every $\gamma > 1$, the solution $u$ to (1.2) satisfies

$$\left( \int_{\partial D} h(z) d\sigma(z) \right) \frac{c_{\gamma,d}^- |x|^{2-d}}{(d-2)\omega_d} \leq u(x) \leq \left( \int_{\partial D} h(z) d\sigma(z) \right) \frac{c_{\gamma,d}^+ |x|^{2-d}}{(d-2)\omega_d},$$

for $|x| \geq \gamma R$, where $c_{\gamma,d}^\pm$ are independent of $D$ and $R$ and satisfy $\lim_{\gamma \to \infty} c_{\gamma,d}^\pm = 1$.

For the case of a generic operator $L$, we need one more definition before we can state the result. Let $G_{\text{Dir}}^R$ denote the Green’s function for $L$ in $\mathbb{R}^d - B_R$ with the Dirichlet boundary condition at $\partial B_R$. That is, $G_{\text{Dir}}^R(x,y)$ is the minimal positive function $g(x,y)$ satisfying $Lg(\cdot,y) = -\delta_y$ in $\mathbb{R}^d - B_R$ and satisfying the zero Dirichlet boundary condition at $\partial B_R$, for each $y \in \mathbb{R}^d - B_R$. As an aside, we note that the Green’s function $G_{\text{Dir}}^R$ always exists, even if Assumption G is not satisfied.

Theorem 2. Assume that the operator $L$ satisfies Assumption G and is symmetric with respect to the weight function $e^Q$ as in (1.7). Let $R > 0$ and assume that $D \subset B_R$. Let $R' > R$. Then the solution $u$ to (1.2) satisfies

$$\left( \int_{\partial D} h(y) e^{Q(y)} d\sigma(y) \right) \frac{\min_{|z|=|R'|} G_{\text{Dir}}^R(z,x)}{1 - \min_{|z|=|R'|} V(z)} e^{-Q(x)} \leq u(x) \leq \left( \int_{\partial D} h(y) e^{Q(y)} d\sigma(y) \right) \frac{\max_{|z|=|R'|} G_{\text{Dir}}^R(z,x)}{1 - \max_{|z|=|R'|} V(z)} e^{-Q(x)}, \text{ for } |x| > R',$$

where $G_{\text{Dir}}^R$ is the Green’s function for $L$ in $\mathbb{R}^d - B_R$ with the Dirichlet boundary condition at $\partial B_R$, and $V$ is as in (1.4)-(1.6).

Remark 1. Recall that $V \not= 1$ is equivalent to Assumption G. By the maximum principal, $V < 1$ in $\mathbb{R}^d - B_R$. In the generic case one has $\lim_{x \to \infty} V(x) = 0$. Necessarily one has $\lim \inf_{x \to \infty} V(x) = 0$. One can obtain $\lim \sup_{x \to \infty} V(x) > 0$ by choosing $Q$ in the manner noted in Remark 3 following Assumption G. If $\lim_{x \to \infty} V(x) = 0$, then of course $\lim_{R' \to \infty} \frac{1 - \min_{|z|=R'} V(z)}{1 - \max_{|z|=R'} V(z)} = 1$. For certain classes of operators one has $\lim_{x \to \infty} \frac{\min_{|z|=R'} G_{\text{Dir}}^R(z,x)}{\max_{|z|=R'} G_{\text{Dir}}^R(z,x)} = 1$. If
the above two limits hold, then by choosing $R'$ large, the ratio of the left hand side to the right hand side of (1.8) can be made arbitrarily close to 1 for large $|x|$.

**Remark 2.** One can of course choose a sequence $\{D_n\}_{n=1}^{\infty}$ of domains satisfying $D_n \subset B_R$, for all $n$, and $\lim_{n \to \infty} |\partial D_n| = \infty$. Letting $u_n$ denote the solution to (1.2) with, say, $h = 1$ on $\partial D_n$, it follows that $\lim_{n \to \infty} u_n(x) = \infty$, for $|x| > R$.

Theorems 1 and 2 show that given an $M > 0$, then for any $\delta > 0$, the solution to (1.2) is bounded in $R^d - B_R + \delta$ uniformly over all $D \subset B_R$ satisfying $|\partial D| \leq M$ and over all fluxes $h$ satisfying $|h| \leq M$, even though the solution can be arbitrarily large at points inside $B_R$ and very close to $\partial B_R$. For example, consider the case of the punctured spherical shell, defined above, except let the inner and outer radii, $R_1$ and $R_2$, also depend on $n$, setting $R_1 = R - \frac{2}{n}$ and $R_2 = R - \frac{1}{n}$. Then for any $\delta > 0$, the solutions $\{u_{D_n}\}_{n=1}^{\infty}$ will be uniformly bounded over $R^d - B_{R+\delta}$, even though for $x_n$ satisfying $|x_n| = 1 - \frac{3}{n}$ and $\arg(\frac{x_n}{|x_n|}) = \theta \neq \theta_0$, one has that $\lim_{n \to \infty} u_{D_n}(x_n) = \infty$. Recalling the definition of $\text{dist}_{\text{good;}D_n}(\cdot, \cdot)$, note that $\lim_{\delta \to 0} \lim_{n \to \infty} \text{dist}_{\text{good;}D_n}(x_n, B_R + \delta)$ can be made as small as one likes by choosing $\theta$ sufficiently close to $\theta_0$.

The proofs of Theorems 1 and 2 depend in an essential way on the fact that the operator $L$ is symmetric. Of course, the statement of the result also depends on symmetry, as the weight function $e^Q$ appears in the boundary integrals.

**Open Question.** In the case that the operator $L$ is not symmetric, is the size of the solution to (1.2) at $x \in R^d - B_R$ governed independently of the geometry of $D$? Or alternatively, can one, say, give an example of a class of domains $\{D_n\}_{n=1}^{\infty} \subset B_R$ with $\sup_n |\partial D_n| < \infty$ and such that the corresponding solutions $\{u_{D_n}\}_{n=1}^{\infty}$ to (1.2) with, say, $h \equiv 1$ satisfy $\sup_n \sup_{|x| > R + \delta} u_{D_n}(x) = \infty$, for some $\delta > 0$?
2. PROOF OF THEOREM 1

Proof. For $n > R$, let $u_n$ denote the solution to
\begin{align}
\Delta u_n &= 0 \text{ in } B_n - \bar{D}; \\
\nabla u_n \cdot n &= -h \text{ on } \partial D; \\
u_n &= 0 \text{ on } \partial B_n.
\end{align}

Let $G_n^D(x, y)$ denote the Green’s function for $\Delta$ in $B_n - \bar{D}$ with the Dirichlet boundary condition at $\partial B_n$ and the Neumann boundary condition at $\partial D$. One has
\begin{align}
\Delta_x G_n^D(x, y) &= -\delta_y \text{ for } x, y \in B_n - \bar{D}; \\
G_n^D(x, y) &= 0, \text{ for } x \in \partial B_n, y \in B_n - \bar{D}; \\
\frac{\partial G_n^D(x, y)}{\partial \bar{n}_x}(x, y) &= 0, \text{ for } x \in \partial D, y \in B_n - \bar{D}.
\end{align}

By the maximum principle, $G_n^D$ is increasing in $n$; let $G_n^{\text{Neu}}(x, y) = \lim_{n \to \infty} G_n^D(x, y)$. This limiting function is the Green’s function for $\Delta$ in $\mathbb{R}^d - \bar{D}$ with the Neumann boundary condition at $\partial D$.

Recall the fundamental property of the Green’s function: if $v$ is a smooth function in $B_n - D$ and continuous up to $\partial B_n$, then
\begin{align}
v(x) &= -\int_{B_n - D} (\Delta v)(y) G_n^D(x, y) dy + \int_{\partial B_n} v(y) (\nabla_y G_n^D \cdot \bar{n})(x, y) d\sigma(y) - \\
&\quad \int_{\partial D} (\nabla v \cdot \bar{n})(y) G_n^D(x, y) d\sigma(y),
\end{align}
where $\bar{n}$ denotes the exterior unit normal vector to $D$ at $\partial D$ and the interior unit normal vector to $B_n$ at $\partial B_n$. Since $u_n$ is harmonic and since $u_n$ vanishes on $\partial B_n$ and $\nabla u_n \cdot \bar{n} = -h$ on $\partial D$, one has the representation
\begin{align}u_n(x) &= \int_{\partial D} G_n^D(x, y) h(y) d\sigma(y), \text{ for } x \in B_n - D.\end{align}
Letting $n \to \infty$ gives

\begin{equation}
  u(x) = \int_{\partial D} G_{\text{Neu}}^D(x, y) h(y) d\sigma(y), \quad \text{for } x \in R^d - \bar{D}.
\end{equation}

To complete the proof of the theorem, we appeal to spectral theory and then to the probabilistic representation of the Green’s function $G_{\text{Neu}}^D(x, y)$.

Since the operator $\Delta$ in $B_n - \bar{D}$ with the Dirichlet boundary condition at $\partial B_n$ and the Neumann boundary condition at $\partial D$ is symmetric, it follows that the Green’s function $G_{\text{Neu}}^D$, which is the integral kernel of the inverse operator, is symmetric; that is, $G_{\text{Neu}}^D(x, y) = G_{\text{Neu}}^D(y, x)$. Thus, also

\begin{equation}
  G_{\text{Neu}}^D(x, y) = G_{\text{Neu}}^D(y, x).
\end{equation}

We now turn to the probabilistic representation of the Green’s function. Let $B(t)$ be a $d$-dimensional Brownian motion in $R^d - D$, normally reflected at $\partial D$, and corresponding to the operator $\Delta$. (A standard Brownian motion $\beta(t)$ corresponds to the operator $\frac{1}{2} \Delta$; our $B(t)$ can be obtained as $\sqrt{2}\beta(t)$, or alternatively, as $\beta(2t)$.) Let $P_x$ denote probabilities and let $E_x$ denote the corresponding expectations for the Brownian motion starting from $x \in R^d - D$. For $x \in R^d - D$, define the expected occupation measure by

\[ \mu_{\text{Neu}}^D(x) (A) = \mathbb{E}_x \int_0^\infty 1_{A}(B(t)) dt, \] for Borel sets $A \subset R^d - D$. Since $d \geq 3$, the Brownian motion is transient (that is, $P_x(\lim_{t \to \infty} |B(t)| = \infty) = 1$) and from this one can show that $\mu_{\text{Neu}}^D(A) < \infty$ for all bounded $A$. The measure $\mu_{\text{Neu}}^D(dy)$ possesses a density and the density is given by $G_{\text{Neu}}^D(x, y)$ [3]. From now on we will write $G_{\text{Neu}}^D(x, A) \equiv \mu_{\text{Neu}}^D(A)$. Using this probabilistic representation, we will show that for $\gamma > 1$,

\begin{equation}
  G_{\text{Neu}}^D(x, y) \leq \frac{1}{(d - 2)\omega_d} c_{\gamma, d}^+ |y|^{2-d}, \quad \text{for } x \in \partial D, |y| \geq \gamma R;
\end{equation}

\begin{equation}
  G_{\text{Neu}}^D(x, y) \geq \frac{1}{(d - 2)\omega_d} c_{\gamma, d}^- |y|^{2-d}, \quad \text{for } x \in \partial D, |y| \geq \gamma R,
\end{equation}

where $\lim_{\gamma \to \infty} c_{\gamma, d}^+ = \lim_{\gamma \to \infty} c_{\gamma, d}^- = 1$, $c_{\gamma, d}^- > 0$ and $c_{\gamma, d}^\pm$ are independent of $D$ and $R$. The theorem then follows from (2.3), (2.4) and (2.5).

To prove (2.5), we define a sequence of hitting times for the Brownian motion. Let $\gamma > \rho > 1$. Define $\tau_1 = \inf\{t \geq 0 : |B(t)| = \rho R\}$, and then
by induction define $\tau_{2n} = \inf\{t > \tau_{2n-1} : |B(t)| = R\}$ and $\tau_{2n+1} = \inf\{t > \tau_{2n} : |B(t)| = \rho R\}$. In words, $\tau_1$ is the first time the Brownian motion hits $\partial B\rho R$, $\tau_2$ is the first time after $\tau_1$ that the Brownian motion hits $\partial B\rho R$, $\tau_3$ is the first time after $\tau_2$ that the Brownian motion hits $\partial B\rho R$, etc. Since the Brownian motion is transient, almost surely only a finite number of the $\tau_n$ will be finite. For $x \in \partial D$ and $A \subset R^d - B\gamma R$, we have

$$G_{\text{Neu}}^D(x, A) = E_x \int_0^{\infty} 1_A(B(t)) dt = \sum_{n=1}^{\infty} E_x \int_{\tau_{2n-1}}^{\tau_{2n}} 1_A(B(t)) dt.$$  

This equality holds because for times $s \in [\tau_{2n}, \tau_{2n+1}]$ one has $|B(s)| \leq \rho R$ and thus $B(s) \notin A$.

Note that for $z \in \partial B\rho R$, one has $P_z(\tau_1 = 0) = 1$ and thus under $P_z$ one has that $\tau_2$ is the first hitting time of $B\rho R$. Let $\phi(r) = (\frac{2}{r})^{d-2}$, $r = |x|$, be the radially symmetric harmonic function in $R^d - \bar{B}R$ which equals 1 on the boundary and decays to 0 at $\infty$. It is well-known that starting from $z$ with $|z| > r$, the probability that the first hitting time of $\partial B\rho R$ is finite is $\phi(|z|)$; thus, $P_z(\tau_2 < \infty) = \rho^{2-d}$, for $z \in \partial B\rho R$. By the strong Markov property, conditioned on $\tau_{2n-1} < \infty$, the probability that $\tau_{2n} < \infty$ is again $\rho^{2-d}$. Thus, one has

$$P_x(\tau_{2n-1} < \infty) = \rho^{(2-d)(n-1)}, \text{ for } x \in \partial D.$$  

Also from the strong Markov property, the conditional expectation $E_x(\int_{\tau_{2n-1}}^{\tau_{2n}} 1_A(B(t)) dt | \tau_{2n-1} < \infty)$ is some averaging of the values of $E_z \int_0^{\tau_2} 1_A(B(t)) dt$, as $z$ varies over $\partial B\rho R$. That is, there exists a probability measure $\nu_{n,x}$ on $\partial B\rho R$ such that

$$E_x(\int_{\tau_{2n-1}}^{\tau_{2n}} 1_A(B(t)) dt | \tau_{2n-1} < \infty) = \int_{\partial B\rho R} \left( E_z \int_0^{\tau_2} 1_A(B(t)) dt \right) \nu_{n,x}(dz).$$  

Now let $W(t)$ be a Brownian motion in all of $R^d$ corresponding to the operator $\Delta$ and let $E_x$ denote the expectation for this Brownian motion starting from $x$. Since the Brownian motion reflected at $\partial D$ and the Brownian motion on all of $R^d$ behave the same when they are in $R^d - B\gamma R$, one
has

\[(2.9) \quad E_z \int_0^{\tau_2} 1_A(B(t))dt = \mathcal{E}_z \int_0^{\tau_2} 1_A(W(t))dt, \quad \text{for } z \in \partial B_{\rho R}.
\]

We now prove the upper bound in (2.5). From (2.9) we have

\[(2.10) \quad E_z \int_0^{\tau_2} 1_A(B(t))dt \leq \mathcal{E}_z \int_0^{\tau_2} 1_A(W(t))dt, \quad \text{for } z \in \partial B_{\rho R}.
\]

Recall that the Green’s function for \(\Delta\) on all of \(R^d\) is given by

\[
G(x, y) \equiv \frac{|x - y|^{2-d}}{(d-2)\omega_d},
\]

where \(\omega_d\) denotes the surface measure of the unit sphere in \(R^d\).

The probabilistic representation of the Green’s function described above also holds for the Brownian motion in all of \(R^d\); that is, \(G(x, y)\) is the density of

\[
\mu_x(A) = E_x \int_0^{\tau_2} 1_A(W(t))dt.
\]

Thus, one has

\[(2.11) \quad E_z \int_0^{\tau_2} 1_A(W(t))dt = \int_A G(z, y) dy = \int_A \frac{|z - y|^{2-d}}{(d-2)\omega_d} dy.
\]

Since \(E_x(\int_0^{\tau_{2n}} 1_A(B(t))dt) = 0\), we have from (2.6)-(2.11) that

\[(2.12) \quad G_{\text{Neu}}(x, A) = E_x \int_0^{\tau_{2n}} 1_A(B(t))dt = \sum_{n=1}^{\infty} E_x \int_{\tau_{2n-1}}^{\tau_{2n}} 1_A(B(t))dt = \sum_{n=1}^{\infty} \mathcal{E}_x \int_0^{\tau_{2n}} 1_A(W(t))dt |\tau_{2n-1} < \infty) P_x(\tau_{2n-1} < \infty) \leq \left( \sup_{z \in \partial B_{\rho R}} \mathcal{E}_z \int_0^{\tau_{2n}} 1_A(W(t))dt \right) \sum_{n=1}^{\infty} P_x(\tau_{2n-1} < \infty) = \left( \sup_{z \in \partial B_{\rho R}} \int_A \frac{|z - y|^{2-d}}{(d-2)\omega_d} dy \right) \sum_{n=1}^{\infty} \rho^{(2-d)(n-1)} \leq \int_A \frac{1}{(d-2)\omega_d} \frac{1}{1 - \rho^{2-d}} \left( \frac{\gamma}{\gamma - \rho} \right)^{d-2} |y|^{2-d} dy, \quad \text{for } x \in \partial D, A \subset R^d - B_{\gamma R},
\]

where in the last inequality we have used the fact that |y| \leq \frac{\gamma - \rho}{\gamma - \rho} |y - z|, for |z| = \rho R and |y| \geq \gamma R. Now \(\rho \in (1, \gamma)\) is a free parameter. One can check that

\[
\sup_{\rho \in (1, \gamma)} (1 - \rho^{2-d})(\gamma - \rho)^{d-2} = \frac{(\frac{\gamma d}{\gamma - 1}) - 1)(\gamma - \gamma \frac{d}{\gamma - 1})^{d-2}}{\gamma^{d-2}}.
\]
The maximum is attained at \( \rho = \gamma \frac{1}{2-d} \). Using this value of \( \rho \) in the right hand side of (2.12) gives

\[
(2.13) \quad G_{\text{Neu}}^D(x, A) \leq \int_A \frac{c^+_{\gamma,d}}{(d-2)\omega_d} |y|^{2-d} dy,
\]

for \( x \in \partial D \) and \( A \subset R^d - B_{\rho R} \), where

\[
c^+_{\gamma,d} = \frac{\gamma^{\frac{d-2}{2}}}{(\gamma^{\frac{d-2}{2}} - 1)(\gamma - \frac{1}{2-d})}.
\]

Note that \( \lim_{\gamma \to \infty} c^+_{\gamma,d} = 1 \). Now the upper bound in (2.5) follows from (2.13).

We now prove the lower bound in (2.5). Let \( G_{\text{Dir}}^R(x, y) \) denote the Green’s function for \( \Delta \) in \( R^d - \bar{B}_R \) with the Dirichlet boundary condition at \( \partial B_R \). The probabilistic representation of the Green’s function gives

\[
(2.14) \quad G_{\text{Dir}}^R(z, A) = \mathcal{E}_z \int_0^{\tau_z} 1_A(W(t)) dt, \text{ for } |z| > R.
\]

Using reflection with respect to \( \partial B_R \) allows one to calculate the Green’s function explicitly \[2\]:

\[
(2.15) \quad G_{\text{Dir}}^R(x, y) = \frac{1}{(d-2)\omega_d} |x - y|^{2-d} - \frac{1}{(d-2)\omega_d} \left( \frac{|y|}{R} \right)^{2-d} |x - \frac{R^2}{|y|^2} y|^{2-d} =
\]

\[
= \frac{1}{(d-2)\omega_d} |x - y|^{2-d} - \frac{1}{(d-2)\omega_d} \left( \frac{|y|}{R} \right)^{2-d} \left| \frac{x}{|y|} - \frac{R}{|y|} y \right|^2.
\]

(In the proof of the upper bound, we could have used this Green’s function and (2.9) instead of the Green’s function \( G(x, y) \) for all of \( R^d \) and (2.10), and this would have yielded a slightly smaller value of \( c^+_{\gamma,d} \). However, it was simpler to work with \( G(x, y) \). For the lower bound we have no choice but to work with \( G_{\text{Dir}}^R(x, y) \).)

For \( |x| = \rho R \) and \( |y| = \gamma' R \), with \( \gamma' \geq \gamma \), one has

\[
\left| \frac{|y|}{R} x - \frac{R}{|y|} y \right| \geq |y| |x| - R = R(\gamma' \rho - 1)
\]

and

\[
|x - y| \leq |x| + |y| = R(\gamma' + \rho).
\]
Using this with (2.15), one has
\[(d - 2)\omega_d G_{\text{Dir}}^R(x, y) \geq (\frac{\gamma'}{\gamma + \rho})^{d-2} \cdot (\frac{\gamma'}{\gamma/\rho - 1})^{d-2}, \text{ for } |x| = \rho R, \ |y| = \gamma' R.\]

One can check that the right hand side of (2.16) is increasing in \(\gamma'\); thus
\[(d - 2)\omega_d G_{\text{Dir}}^R(x, y) \geq (\frac{\gamma}{\gamma + \rho})^{d-2} \cdot (\frac{\gamma}{\gamma/\rho - 1})^{d-2}, \text{ for } |x| = \rho R, \ |y| \geq \gamma R.\]

Then similar to (2.12), but using (2.9) and (2.14) instead of (2.10) and (2.11), we have
\[(2.18) \quad G_{\text{Neu}}^D(x, A) = E_x \int_0^\infty 1_A(B(t))dt = \sum_{n=1}^\infty E_x \int_{\tau_{2n-1}}^{\tau_{2n}} 1_A(B(t))dt = \]
\[\sum_{n=1}^\infty E_x (\int_{\tau_{2n-1}}^{\tau_{2n}} 1_A(B(t))dt|\tau_{2n-1} < \infty)P_x(\tau_{2n-1} < \infty) \geq \]
\[\left( \inf_{z \in \partial B_{\rho R}} E_z \int_0^{\tau_2} 1_A(W(t))dt \right) \sum_{n=1}^\infty P_x(\tau_{2n-1} < \infty) = \]
\[\left( \inf_{z \in \partial B_{\rho R}} \int_A G_{\text{Dir}}^R(z, y)dy \right) \sum_{n=1}^\infty \rho^{(2-d)(n-1)} = \left( \inf_{z \in \partial B_{\rho R}} \int_A G_{\text{Dir}}^R(z, y)dy \right) \frac{1}{1 - \rho^{2-d}} \geq \]
\[\int_A \frac{1}{(d - 2)\omega_d 1 - \rho^{2-d}} \left( \frac{\gamma}{\gamma + \rho} \right)^{d-2} \cdot \left( \frac{\gamma}{\gamma/\rho - 1} \right)^{d-2} |y|^{2-d} dy, \]
for \(x \in \partial D, \ A \subset R^d - B_{\gamma R},\)

where the inequality follows from (2.17).

As a function of \(\rho \in (1, \gamma),\) the expression \(\frac{1}{1 - \rho^{2-d}} \left( \frac{\gamma}{\gamma + \rho} \right)^{d-2} \cdot \left( \frac{\gamma}{\gamma/\rho - 1} \right)^{d-2} \)
is non-positive if \(\gamma \leq 1 + \sqrt{2}.\) For \(\gamma > 1 + \sqrt{2},\) this expression has a positive maximum. The formula for the maximum is complicated and doesn’t add much so we will just define \(c_{\gamma, d}^-\) as follows. For \(\gamma\) sufficiently large so that the above expression is positive for \(\rho = \gamma^{\frac{1}{d-1}},\) let this value be \(c_{\gamma, d}^-\). Then, in particular, for some \(\gamma_0 > 1,\) one has
\[c_{\gamma, d}^- = \frac{1}{1 - \gamma^{\frac{2}{d-1}}} \left( \frac{\gamma}{\gamma + \gamma^{\frac{2}{d-1}}} \right)^{d-2} \cdot \left( \frac{\gamma}{\gamma^{\frac{2}{d-1}} - 1} \right)^{d-2}, \text{ for } \gamma \geq \gamma_0.\]
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Note that \( \lim_{\gamma \to \infty} c_{\gamma,d}^- = 1 \). From (2.18) and the above remarks, we have now shown that

\[
(2.19) \quad G_{\text{Neu}}^D(x, y) \geq \frac{1}{(d-2)\omega_d} c_{\gamma,d}^- |y|^{2-d}, \text{ for } x \in \partial D, |y| \geq \gamma R, \gamma \geq \gamma_0.
\]

This is the lower bound in (2.5) except that one has here \( \gamma \geq \gamma_0 \) instead of \( \gamma > 1 \). To extend (2.19) to \( \gamma > 1 \), one notes that since \( G_{\text{Dir}}^R(z, y) \) is positive for \( z, y \in \mathbb{R}^d - \bar{B}_R \), one has trivially \( \inf_{\gamma R \leq |y| \leq \gamma_0 R, |z|=\rho R} G_{\text{Dir}}^R(z, y) > 0 \), for any choice of \( \rho, \gamma \) satisfying \( 1 < \rho < \gamma < \gamma_0 \), and in fact by scaling, the left hand side is independent of \( R \). Using this along with the fact that \( G_{\text{Neu}}^D(x, A) \geq (\inf_{z \in \partial B_{\rho R}} \int_A G_{\text{Dir}}^R(z, y)dy) \frac{1}{1-\rho^2-a} \), which follows from (2.18), we can define \( c_{\gamma,d}^- > 0 \) for all \( \gamma > 1 \) so that (2.19) holds.

\( \square \)

3. PROOF OF THEOREM 2

In the present case, (2.3) still holds, where \( G_{\text{Neu}}^D \) is now the Green’s function for the operator \( L \) on \( \mathbb{R}^d - \bar{D} \) with the co-normal boundary condition at \( \partial D \). Since the operator \( L \) is symmetric with respect to the weight \( e^Q \), similar to (2.4) we have

\[
e^Q(x) G_{\text{Neu}}^D(x, y) = e^Q(y) G_{\text{Neu}}^D(y, x).
\]

To see this, note that since Green’s function \( G_{\text{Neu}}^D(x, y) \) satisfies \( LG_{\text{Neu}}^D(\cdot, y) = -\delta_y \) with the homogeneous co-normal boundary condition at \( \partial D \), it follows that for any compactly supported sufficiently smooth function \( f \) defined on \( \mathbb{R}^d - D \), the function \( u(x) = \int_{\mathbb{R}^d - D} G_{\text{Neu}}^D(x, y) f(y)dy \) is the minimal positive solution of \( Lu = -f \) in \( \mathbb{R}^d - D \), with the homogeneous co-normal boundary condition at \( \partial D \). Now on the one hand, since \( L \) is in the form (1.7), it follows that \( u \) solves \( \nabla \cdot e^Q a \nabla u = -e^Q f \) with the homogeneous co-normal boundary condition, but on the other hand, by the same reasoning as above, we have \( v(x) = \int_{\mathbb{R}^d - D} G_{\text{sym}}(x, y) e^Q(y) f(y)dy \), where \( G_{\text{sym}} \) is the Green’s function for the operator \( \nabla \cdot e^Q a \nabla \) on \( \mathbb{R}^d - D \) with the co-normal boundary condition at \( \partial D \). Since this holds for all nice \( f \), we conclude that
G_{Neu}^D(x, y) = G_{sym}(x, y)e^{Q(y)}. Since \( G_{sym} \) is the Green’s function of an operator that is symmetric with respect to Lebesgue measure, it follows as in (2.4) that \( G_{sym}(x, y) = G_{sym}(y, x) \). Now (3.1) follows from this.

In light of (2.3) and (3.1), to prove the theorem it suffices to show that for \( R' > R \), one has

\[
\min_{|z|=R'} G_{Dir}^R(z, x) \leq G_{Neu}^D(y, x) \leq \max_{|z|=R'} G_{Dir}^R(z, x), \quad \text{for } |x| > R' \text{ and } y \in \partial D.
\]

As in the previous section, we consider the probabilistic representation of the Green’s function. Let \( X(t) \) be the diffusion process in \( \mathbb{R}^d - D \), co-normally reflected at \( \partial D \), and corresponding to the operator \( L \) [4]. Let \( P_y \) denote probabilities and let \( E_y \) denote the corresponding expectations for the diffusion starting from \( y \in \mathbb{R}^d - D \). For \( y \in \mathbb{R}^d - D \), define the expected occupation measure by

\[
\mu_{D,y}(A) = E_y \int_0^\infty 1_A(X(t))dt ,
\]

for Borel sets \( A \subset \mathbb{R}^d - D \). By assumption, the process \( X(t) \) is transient, (that is, \( P_y(\lim_{t \to \infty} |X(t)| = \infty) = 1 \)), and from this one can show that \( \mu_{D,y}(A) < \infty \) for all bounded \( A \). The measure \( \mu_{D,y}(dx) \) possesses a density and the density is given by \( G_{Neu}^D(y, x) \). From now on we will write \( G_{Neu}^D(y, A) \equiv \mu_{D,y}(A) \).

Define \( \tau_1 = \inf \{ t \geq 0 : |X(t)| = R_1 \} \), and then by induction define \( \tau_2 = \inf \{ t > \tau_{n-1} : |X(t)| = R \} \) and \( \tau_{n+1} = \inf \{ t > \tau_n : |X(t)| = R \} \). Similar to the proof of Theorem 1 (see (2.6), (2.8) and the line between (2.11) and (2.12)), we have for \( y \in \partial D \) and \( A \subset \mathbb{R}^d - \bar{B}_{R_1} \),

\[
G_{Neu}^D(y, A) = E_y \int_0^\infty 1_A(X(t))dt = \sum_{n=1}^\infty E_y \int_{\tau_{n-1}}^{\tau_n} 1_A(X(t))dt =
\]

\[
\sum_{n=1}^\infty E_y \left( \int_{\tau_{n-1}}^{\tau_n} 1_A(X(t))dt \right) \nu_{n,y}(dz) P_y(\tau_{n-1} < \infty),
\]

As in the previous section, we consider the probabilistic representation of the Green’s function. Let \( X(t) \) be the diffusion process in \( \mathbb{R}^d - D \), co-normally reflected at \( \partial D \), and corresponding to the operator \( L \) [4]. Let \( P_y \) denote probabilities and let \( E_y \) denote the corresponding expectations for the diffusion starting from \( y \in \mathbb{R}^d - D \). For \( y \in \mathbb{R}^d - D \), define the expected occupation measure by \( \mu_{D,y}(A) = E_y \int_0^\infty 1_A(X(t))dt \), for Borel sets \( A \subset \mathbb{R}^d - D \). By assumption, the process \( X(t) \) is transient, (that is, \( P_y(\lim_{t \to \infty} |X(t)| = \infty) = 1 \)), and from this one can show that \( \mu_{D,y}(A) < \infty \) for all bounded \( A \). The measure \( \mu_{D,y}(dx) \) possesses a density and the density is given by \( G_{Neu}^D(y, x) \). From now on we will write \( G_{Neu}^D(y, A) \equiv \mu_{D,y}(A) \).

Define \( \tau_1 = \inf \{ t \geq 0 : |X(t)| = R_1 \} \), and then by induction define \( \tau_2 = \inf \{ t > \tau_{n-1} : |X(t)| = R \} \) and \( \tau_{n+1} = \inf \{ t > \tau_n : |X(t)| = R \} \). Similar to the proof of Theorem 1 (see (2.6), (2.8) and the line between (2.11) and (2.12)), we have for \( y \in \partial D \) and \( A \subset \mathbb{R}^d - \bar{B}_{R_1} \),

\[
G_{Neu}^D(y, A) = E_y \int_0^\infty 1_A(X(t))dt = \sum_{n=1}^\infty E_y \int_{\tau_{n-1}}^{\tau_n} 1_A(X(t))dt =
\]

\[
\sum_{n=1}^\infty E_y \left( \int_{\tau_{n-1}}^{\tau_n} 1_A(X(t))dt \right) \nu_{n,y}(dz) P_y(\tau_{n-1} < \infty) =
\]

\[
\sum_{n=1}^\infty \int_{\partial B_{R_1}} \left( E_z \int_0^{\tau_z} 1_A(X(t))dt \right) \nu_{n,y}(dz) P_y(\tau_{n-1} < \infty),
\]
where $\nu_{n,y}$ is some probability measure on $\partial B_{R_1}$. By the strong Markov property,

\begin{equation}
(\min_{|z|=R'} V(z))^{n-1} \leq P_y(\tau_{2n-1} < \infty) \leq (\max_{|z|=R'} V(z))^{n-1},
\end{equation}

for $y \in \partial D$, where $V$ is as in (1.6). Now (3.2) follows from (3.3), (3.4) and the fact that $E_z \int_0^{\tau_2} 1_A(X(t)) dt = G_{Du}(z,A)$.

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**References**


**Department of Mathematics, Technion—Israel Institute of Technology, Haifa, 32000, Israel**

E-mail address: pinsky@math.technion.ac.il

URL: http://www.math.technion.ac.il/~pinsky/