UNIVERSAL BOUND INDEPENDENT OF GEOMETRY FOR SOLUTION TO SYMMETRIC STEADY-STATE DIFFUSION EQUATION IN EXTERIOR DOMAIN WITH BOUNDARY FLUX

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ABSTRACT. Fix $R > 0$ and let B_R denote the ball of radius R centered at the origin in R^d, $d \geq 2$. Let $D \subset B_R$ be an open set with smooth boundary and such that $R^d - \bar{D}$ is connected, and let

$$L = \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}$$

be a second order elliptic operator. Consider the following linear heat equation in the exterior domain $R^d - \bar{D}$ with boundary flux:

$$Lu = 0 \text{ in } R^d - \bar{D};$$

$$a\nabla u \cdot \bar{n} = -h \text{ on } \partial D;$$

$$u > 0 \text{ is minimal,}$$

where $h \gneq 0$ is continuous, and where $\bar{n}$ is the unit inward normal to the domain $R^d - \bar{D}$. The operator L must possess a Green's function in order that a solution u exist. An important feature of the equation is that there is no a priori bound on the supremum $\sup_{x \in R^d - \bar{D}} u(x)$ of the solution exclusively in terms of the boundary flux h, the hyper-surface measure of ∂D and the coefficients of L; rather the geometry of $D \subset B_R$ plays an essential role. However, we prove that in the case that L is a *symmetric operator* with respect to some reference measure, then *outside of* $\bar{B}_R$, the solution to (1.2) is uniformly bounded, independent of the particular choice of $D \subset B_R$. The proof uses a combination of analytic and probabilistic techniques.

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1. Introduction and Statement of Results

Fix $R>0$ once and for all and let B_R denote the ball of radius R centered at the origin in R^d, $d\ge 2$. Let $D\subset B_R$ be an open set with smooth boundary and such that $R^d-\bar D$ is connected. Let

$$L=\sum_{i,j=1}^d a_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}-\sum_{i=1}^d b_i\frac{\partial}{\partial x_i} \tag{1.1}$$

be a strictly elliptic operator in R^d-D with smooth coefficients $a=\{a_{i,j}\}_{i,j=1}^d$ and $b=\{b_i\}_{i=1}^d$. Consider the following linear steady-state heat equation in the exterior domain $R^d-\bar D$ with boundary flux:

$$\begin{aligned}&Lu=0 \text{ in } R^d-\bar D;\\ &a\nabla u\cdot\bar n=-h \text{ on } \partial D;\\ &u>0 \text{ is minimal,}\end{aligned} \tag{1.2}$$

where $h\gneq 0$ is continuous, and where $\bar n$ is the unit inward normal to the domain $R^d-\bar D$. By minimal, we mean that the solution u satisfies $u=\lim_{n\to\infty}u_n$, where for $n>R$, u_n solves

$$\begin{aligned}&Lu=0 \text{ in } B_n-\bar D;\\ &a\nabla u\cdot\bar n=-h \text{ on } \partial D;\\ &u=0 \text{ on } \partial B_n.\end{aligned} \tag{1.3}$$

We note that a certain restriction must be placed on the operator L in order that a solution u exist to (1.2). This will be discussed below. An important feature of (1.2) is that there is no a priori bound on the supremum $\sup_{x\in R^d-\bar D}u(x)$ of the solution exclusively in terms of the boundary flux h, the hyper-surface measure of ∂D and the coefficients of L; rather the geometry of $D\subset B_R$ plays an essential role. Here is a simple example.

Spherical Shell Example. Consider a spherical shell centered at the origin in R^3 with the radii of the inner and outer boundary spheres given by R_1 and R_2 respectively, where $R_1<R_2<R$. Now fix some direction $\theta_0\in S^2$

and puncture the shell with a spherical bullet of radius $\frac{1}{n}$ centered in the θ_0 direction. Thus, a cylindrical-like region of radius $\frac{1}{n}$ and length $R_2 - R_1$ has been removed from the spherical shell. Denote by D_n the open set obtained by taking this punctured spherical shell and deleting its boundary. (If one insists that D_n have a smooth boundary, one can smooth out the edges where the bullet enters and exits.) Let $\Gamma_n = \partial B_{R_1} \cap (\bar{D}_n)^c$ denote the punctured part of the inner sphere. By the maximum principle, the solution u_n to (1.2) with $L = \Delta$ and $h = 1$ satisfies $u_n(x) \geq v_n(x)$, for $|x| < R_1$, where v_n is the solution of

$$\begin{aligned}
&\Delta v_n = 0 \text{ in } B_{R_1};\\
&\nabla v_n \cdot \bar{n} = -1 \text{ on } \partial B_{R_1} - \bar{\Gamma}_n;\\
&v_n = 0 \text{ on } \Gamma_n,
\end{aligned}$$

with $\bar{n}$ being the inward unit normal to B_{R_1} at ∂B_{R_1}. By the maximum principle, $v_n(x)$ is increasing in n; let $v(x) \equiv \lim_{n\to\infty} v_n(x)$. By Harnack's inequality, v is either finite everywhere or infinite everywhere in B_{R_1}. If v is finite, then v must satisfy $\Delta v = 0$ in B_{R_1} and satisfy the Neumann boundary condition $\nabla v \cdot \bar{n} = -1$ on all of ∂B_{R_1}. But this is impossible because solvability of the above equation with Neumann boundary data requires the compatibility condition $\int_{\partial B_{R_1}} (\nabla v \cdot \bar{n}) d\sigma(x) = -\int_{B_{R_1}} \Delta v \, dx = 0$, where $d\sigma$ denotes Lebesgue hyper-surface measure on ∂B_{R_1}. Thus, $v \equiv \infty$ and consequently, $\lim_{n\to\infty} u_n(x) = \infty$, for $x \in B_{R_1}$. Yet, trivially, the sequence $\{|\partial D_n|\}_{n=1}^{\infty}$ is bounded in n.

However, we will prove that in the case that L is a symmetric operator with respect to some reference measure, then for any $\delta > 0$, *outside of* $B_{R+\delta}$, the solution to (1.2) is uniformly bounded, independent of the particular choice of $D \subset B_R$. More precisely, we will show that for any $x \in R^d - \bar{B}_R$, the solution $u(x)$ is *bounded uniformly over all* $D \subset B_R$, in terms of

i. the boundary flux h;

ii. the hyper-surface measure of ∂D with respect to the measure whose density with respect to Lebesgue measure is the trace of the above mentioned reference measure;

iii. the behavior of the coefficients of L outside of $\bar{B}_R$.

In particular, letting $\text{dist}_{\text{geod};D_n}(x_n, \bar{B}_{R+\delta})$ denote the length of a shortest path in $R^d - D_n$ from x_n to $\bar{B}_{R+\delta}$, we will give an example of a sequence of open sets $\{D_n\}_{n=1}^\infty \subset B_R$ and points $x_n \in B_R - D_n$ with $\lim_{\delta\to 0} \lim_{n\to\infty} \text{dist}_{\text{geod};D_n}(x_n, \bar{B}_{R+\delta})$ equal to an arbitrarily small positive number and such that the solution u_{D_n} to (1.2) with $D = D_n$ satisfies $\lim_{n\to\infty} u_{D_n}(x_n) = \infty$, yet for all $x \in R^d - \bar{B}_R$, one has $\sup_n u_{D_n}(x) < \infty$.

The proof of our result will use a combination of analysis and probability. The analysis will consist of representing the solution in terms of an appropriate Green's function and using symmetry, while the probability will consist of representing this Green's function stochastically in terms of the occupation time of a diffusion process.

Before stating our result, we discuss the existence and uniqueness of a solution to (1.2). It is standard that the solution u_n to the linear equation (1.3) with co-normal boundary data at ∂D and homogeneous Dirichlet data at ∂B_n exists and is unique. By the maximum principle, u_n is positive off of ∂B_n and attains its maximum on ∂D. The maximum principle also shows that u_n is nondecreasing in n. Thus, if $\lim_{n\to\infty} \sup_{x\in\partial D} u_n(x) < \infty$, then we obtain existence and uniqueness for the solution u to (1.2). The maximum principle shows that u attains its maximum on the boundary ∂D.

With regard to existence, we begin with a physical description. When $d = 3$, u can be thought of as the equilibrium quantity of a reactant after having undergone a long period of L-diffusion and convection in an exterior domain which is being supplied with the reactant via a boundary flux h, and where complete and instantaneous absorption occurs far away. Note that in (1.1), the drift term b has been written with a minus sign; with this convention, the reactant is being convected in the direction b. If this convection vector field is pointing away from D with sufficient strength, then the convection of the reactant from the boundary ∂D into the domain $R^3 - \bar{D}$ will not allow for an equilibrium state.

In order to obtain existence, in fact it is necessary and sufficient to assume that the operator L possesses an appropriate Green's function.

Assumption G. The operator L with the co-normal boundary condition at ∂D possesses a Green's function.

Recall that the Green's function $G^D_{\text{Neu}}(x,y)$ for an operator L with the co-normal boundary condition at ∂D is the minimal positive function $g(x,y)$ satisfying the following conditions: for each $y \in R^d - \bar{D}$ the function $g(\cdot,y)$ satisfies $Lg(\cdot,y) = -\delta_y$ in $R^d - \bar{D}$ and satisfies the homogeneous co-normal boundary condition at ∂D: $a\nabla g(\cdot,y)\cdot\bar{n} = 0$ on ∂D. The Green's function for an operator L on all of R^d is the minimal positive function $g(x,y)$ satisfying $Lg(\cdot,y) = -\delta_y$ in R^d, for each $y \in R^d$. We make several remarks concerning the Green's function.

Remark 1. Assumption G is equivalent to each of the following assumptions:

i. If L is extended to be a smooth strictly elliptic operator on all of R^d, then this extension possesses a Green's function;

ii. For $n > R$, let V_n denote the solution to the equation

$$\begin{aligned} &LV_n = 0 \text{ in } B_n - \bar{B}_R; \\ &V_n = 1 \text{ on } \partial B_R,\ V_n = 0 \text{ on } \partial B_n. \end{aligned} \tag{1.4}$$

Then

$$V \equiv \lim_{n\to\infty} V_n \tag{1.5}$$

is not the constant function 1.

iii. The diffusion process $X(t)$ in $R^d - D$ corresponding to the operator L and with co-normal reflection at ∂D is transient, that is, $P_x(\lim_{t\to\infty}|X(t)| = \infty) = 1$, or equivalently, $P_x(\tau_R < \infty) < 1$, for $x \in R^d - \bar{B}_R$, where P_x denotes probabilities for the diffusion starting from x and $\tau_R = \inf\{t \geq 0 : X(t) \in$

$\bar{B}_R\}$ is the first hitting time of the ball $\bar{B}_R$. Indeed, the function V defined in (1.4)-(1.5) satisfies

$$V(x) = P_x(\tau_R < \infty). \tag{1.6}$$

For details, see [3].

Remark 2. If Assumption G is not in force, then the solution u_n to (1.3) will satisfy $\lim_{n\to\infty} u_n = \infty$.

Remark 3. For generic choices of b such that L satisfies Assumption G, the solution u to (1.2) will satisfy $\lim_{x\to\infty} u(x) = 0$. Necessarily, one has $\liminf_{x\to\infty} u(x) = 0$. One can obtain $\limsup_{x\to\infty} u(x) > 0$ by choosing the convection term b pointing very strongly away from the origin when x is in certain sectors. (The restriction to certain sectors is necessary for otherwise the Green's function would not exist.)

Our result requires that L be symmetric with respect to a weight e^Q; that is, L must be of the form

$$L = e^{-Q}\nabla \cdot e^Q a\nabla = \nabla \cdot a\nabla + a\nabla Q\nabla. \tag{1.7}$$

(Note that such an L is of the form (1.1) with $b_i = -\sum_{j=1}^d \frac{\partial a_{i,j}}{\partial x_j} - \sum_{j=1}^d a_{i,j}\frac{\partial Q}{\partial x_j}$.) Note that the boundary condition in (1.1) involves the co-normal derivative; as is well-known, the operator L in (1.7) with the homogeneous co-normal boundary condition is symmetric with resect to the weight e^Q: $\int_{R^d-D} gLfe^Q dx = \int_{R^d-D} fLge^Q dx$, for all smooth, compactly supported f, g that satisfy $a\nabla f \cdot \bar{n} = a\nabla g \cdot \bar{n} = 0$ on ∂D.

We first present a theorem for the case $L = \Delta$ with $d \geq 3$. (We must restrict to $d \geq 3$ because there is no Green's function when $d = 2$.) We can obtain tighter results in this case than in the case of generic L. We note that the Green's function for Δ in R^d is $G(x, y) = \frac{|x-y|^{2-d}}{(d-2)\omega_d}$ where ω_d denotes the Lebesgue hyper-surface measure of the unit sphere in R^d. Lebesgue hyper-surface measure on ∂D is denoted by $d\sigma$.

Theorem 1. *Let $L=\Delta$ in R^d, $d\ge 3$. Let $R>0$ and assume that $D\subset B_R$. Let ω_d denote the surface measure of the unit sphere in R^d. Then for every $\gamma>1$, the solution u to* (1.2) *satisfies*

$$\left(\int_{\partial D} h(z)d\sigma(z)\right)\frac{c^-_{\gamma,d}\,|x|^{2-d}}{(d-2)\omega_d}\le u(x)\le \left(\int_{\partial D} h(z)d\sigma(z)\right)\frac{c^+_{\gamma,d}\,|x|^{2-d}}{(d-2)\omega_d},$$

for $|x|\ge\gamma R$, where $c^\pm_{\gamma,d}$ are independent of D and R and satisfy $\lim_{\gamma\to\infty}c^\pm_{\gamma,d}=1$.

For the case of a generic operator L, we need one more definition before we can state the result. Let G^R_{Dir} denote the Green's function for L in $R^d-\bar B_R$ with the Dirichlet boundary condition at ∂B_R. That is, $G^R_{\text{Dir}}(x,y)$ is the minimal positive function $g(x,y)$ satisfying $Lg(\cdot,y)=-\delta_y$ in $R^d-\bar B_R$ and satisfying the zero Dirichlet boundary condition at ∂B_R, for each $y\in R^d-\bar B_R$. As an aside, we note that the Green's function G^R_{Dir} always exists, even if Assumption G is not satisfied.

Theorem 2. *Assume that the operator L satisfies Assumption G and is symmetric with respect to the weight function e^Q as in* (1.7). *Let $R>0$ and assume that $D\subset B_R$. Let $R'>R$. Then the solution u to* (1.2) *satisfies*

$$\begin{aligned}&\left(\int_{\partial D} h(y)e^{Q(y)}d\sigma(y)\right)\frac{\min_{|z|=R'}G^R_{Dir}(z,x)}{1-\min_{|z|=R'}V(z)}e^{-Q(x)}\le u(x)\le\\ &\left(\int_{\partial D} h(y)e^{Q(y)}d\sigma(y)\right)\frac{\max_{|z|=R'}G^R_{Dir}(z,x)}{1-\max_{|z|=R'}V(z)}e^{-Q(x)},\ \text{for } |x|>R',\end{aligned}\tag{1.8}$$

where G^R_{Dir} is the Green's function for L in R^d-B_R with the Dirichlet boundary condition at ∂B_R, and V is as in (1.4)-(1.6).

Remark 1. Recall that $V\not\equiv 1$ is equivalent to Assumption G. By the maximum principal, $V<1$ in $R^d-\bar B_R$. In the generic case one has $\lim_{x\to\infty}V(x)=0$. Necessarily one has $\liminf_{x\to\infty}V(x)=0$. One can obtain $\limsup_{x\to\infty}V(x)>0$ by choosing Q in the manner noted in Remark 3 following Assumption G. If $\lim_{x\to\infty}V(x)=0$, then of course $\lim_{R'\to\infty}\frac{1-\min_{|z|=R'}V(z)}{1-\max_{|z|=R'}V(z)}=1$. For certain classes of operators one has $\lim_{x\to\infty}\frac{\min_{|z|=R'}G^R_{\text{Dir}}(z,x)}{\max_{|z|=R'}G^R_{\text{Dir}}(z,x)}=1$. If the above two limits hold, then by choosing R' large, the ratio of the left hand side to the right hand side of (1.8) can be made arbitrarily close to 1 for large $|x|$.

Remark 2. One can of course choose a sequence $\{D_n\}_{n=1}^\infty$ of domains satisfying $D_n \subset B_R$, for all n, and $\lim_{n\to\infty} |\partial D_n| = \infty$. Letting u_n denote the solution to (1.2) with, say, $h = 1$ on ∂D_n, it follows that $\lim_{n\to\infty} u_n(x) = \infty$, for $|x| > R$.

Theorems 1 and 2 show that given an $M > 0$, then for any $\delta > 0$, the solution to (1.2) is bounded in $R^d - B_{R+\delta}$ uniformly over all $D \subset B_R$ satisfying $|\partial D| \le M$ and over all fluxes h satisfying $|h| \le M$, even though the solution can be arbitrarily large at points inside B_R and very close to ∂B_R. For example, consider the case of the punctured spherical shell, defined above, except let the inner and outer radii, R_1 and R_2, also depend on n, setting $R_1 = R - \frac{2}{n}$ and $R_2 = R - \frac{1}{n}$. Then for any $\delta > 0$, the solutions $\{u_{D_n}\}_{n=1}^\infty$ will be uniformly bounded over $R^d - B_{R+\delta}$, even though for x_n satisfying $|x_n| = 1 - \frac{3}{n}$ and $\arg(\frac{x_n}{|x_n|}) = \theta \ne \theta_0$, one has that $\lim_{n\to\infty} u_{D_n}(x_n) = \infty$. Recalling the definition of $\text{dist}_{\text{geod};D_n}(\cdot,\cdot)$, note that $\lim_{\delta\to 0} \lim_{n\to\infty} \text{dist}_{\text{geod};D_n}(x_n, \bar{B}_{R+\delta})$ can be made as small as one likes by choosing θ sufficiently close to θ_0.

The proofs of Theorems 1 and 2 depend in an essential way on the fact that the operator L is symmetric. Of course, the statement of the result also depends on symmetry, as the weight function e^Q appears in the boundary integrals.

Open Question. In the case that the operator L is not symmetric, is the size of the solution to (1.2) at $x \in R^d - \bar{B}_R$ governed independently of the geometry of D? Or alternatively, can one, say, give an example of a class of domains $\{D_n\}_{n=1}^\infty \subset B_R$ with $\sup_n |\partial D_n| < \infty$ and such that the corresponding solutions $\{u_{D_n}\}_{n=1}^\infty$ to (1.2) with, say, $h \equiv 1$ satisfy $\sup_n \sup_{|x|>R+\delta} u_{D_n}(x) = \infty$, for some $\delta > 0$?

The proof of Theorem 1 is given in section 2. In section 3 we sketch how to amend the proof of Theorem 1 to obtain the proof of Theorem 2.

2. Proof of Theorem 1

Proof. For $n > R$, let u_n denote the solution to

$$\begin{aligned} &\Delta u_n = 0 \text{ in } B_n - \bar{D};\\ &\nabla u_n \cdot n = -h \text{ on } \partial D;\\ &u_n = 0 \text{ on } \partial B_n. \end{aligned} \tag{2.1}$$

Let $G_n^D(x,y)$ denote the Green's function for Δ in $B_n - \bar{D}$ with the Dirichlet boundary condition at ∂B_n and the Neumann boundary condition at ∂D. One has

$$\begin{aligned} &\Delta_x G_n^D(x,y) = -\delta_y \text{ for } x, y \in B_n - \bar{D};\\ &G_n^D(x,y) = 0, \text{ for } x \in \partial B_n,\ y \in B_n - \bar{D};\\ &\frac{\partial G_n^D}{\partial \bar{n}_x}(x,y) = 0, \text{ for } x \in \partial D,\ y \in B_n - \bar{D}. \end{aligned} \tag{2.2}$$

By the maximum principle, G_n^D is increasing in n; let $G_{\text{Neu}}^D(x,y) = \lim_{n\to\infty} G_n^D(x,y)$. This limiting function is the Green's function for Δ in $R^d - \bar{D}$ with the Neumann boundary condition at ∂D.

Recall the fundamental property of the Green's function: if v is a smooth function in $B_n - D$ and continuous up to ∂B_n, then

$$v(x) = -\int_{B_n - D} (\Delta v)(y) G_n^D(x,y)dy + \int_{\partial B_n} v(y)(\nabla_y G_n^D \cdot \bar{n})(x,y)d\sigma(y) - \int_{\partial D} (\nabla v \cdot \bar{n})(y) G_n^D(x,y)d\sigma(y),$$

where $\bar{n}$ denotes the exterior unit normal vector to D at ∂D and the interior unit normal vector to B_n at ∂B_n. Since u_n is harmonic and since u_n vanishes on ∂B_n and $\nabla u_n \cdot \bar{n} = -h$ on ∂D, one has the representation

$$u_n(x) = \int_{\partial D} G_n^D(x,y)h(y)d\sigma(y), \text{ for } x \in B_n - D.$$

Letting $n \to \infty$ gives

$$u(x) = \int_{\partial D} G^D_{\mathrm{Neu}}(x,y)h(y)d\sigma(y), \text{ for } x \in R^d - \bar{D}. \tag{2.3}$$

To complete the proof of the theorem, we appeal to spectral theory and then to the probabilistic representation of the Green's function $G^D_{\mathrm{Neu}}(x,y)$.

Since the operator Δ in $B_n - \bar{D}$ with the Dirichlet boundary condition at ∂B_n and the Neumann boundary condition at ∂D is symmetric, it follows that the Green's function G^D_n, which is the integral kernel of the inverse operator, is symmetric; that is, $G^D_n(x,y) = G^D_n(y,x)$. Thus, also

$$G^D_{\mathrm{Neu}}(x,y) = G^D_{\mathrm{Neu}}(y,x). \tag{2.4}$$

We now turn to the probabilistic representation of the Green's function. Let $B(t)$ be a d-dimensional Brownian motion in $R^d - D$, normally reflected at ∂D, and corresponding to the operator Δ. (A standard Brownian motion $\beta(t)$ corresponds to the operator $\frac{1}{2}\Delta$; our $B(t)$ can be obtained as $\sqrt{2}\beta(t)$, or alternatively, as $\beta(2t)$.) Let P_x denote probabilities and let E_x denote the corresponding expectations for the Brownian motion starting from $x \in R^d - D$. For $x \in R^d - D$, define the expected occupation measure by $\mu^D_x(A) = E_x \int_0^\infty 1_A(B(t))dt$, for Borel sets $A \subset R^d - D$. Since $d \geq 3$, the Brownian motion is transient (that is, $P_x(\lim_{t\to\infty} |B(t)| = \infty) = 1$) and from this one can show that $\mu^D_x(A) < \infty$ for all bounded A. The measure $\mu^D_x(dy)$ possesses a density and the density is given by $G^D(x,y)$ [3]. From now on we will write $G^D(x,A) \equiv \mu^D_x(A)$. Using this probabilistic representation, we will show that for $\gamma > 1$,

$$\begin{aligned} &G^D_{\mathrm{Neu}}(x,y) \leq \frac{1}{(d-2)\omega_d} c^+_{\gamma,d}|y|^{2-d}, \text{ for } x \in \partial D, |y| \geq \gamma R; \\ &G^D_{\mathrm{Neu}}(x,y) \geq \frac{1}{(d-2)\omega_d} c^-_{\gamma,d}|y|^{2-d}, \text{ for } \ x \in \partial D, |y| \geq \gamma R, \end{aligned} \tag{2.5}$$

where $\lim_{\gamma\to\infty} c^+_{\gamma,d} = \lim_{\gamma\to\infty} c^-_{\gamma,d} = 1$, $c^-_{\gamma,d} > 0$ and $c^\pm_{\gamma,d}$ are independent of D and R. The theorem then follows from (2.3), (2.4) and (2.5).

To prove (2.5), we define a sequence of hitting times for the Brownian motion. Let $\gamma > \rho > 1$. Define $\tau_1 = \inf\{t \geq 0 : |B(t)| = \rho R\}$, and then by induction define $\tau_{2n} = \inf\{t > \tau_{2n-1} : |B(t)| = R\}$ and $\tau_{2n+1} = \inf\{t > \tau_{2n} : |B(t)| = \rho R\}$. In words, τ_1 is the first time the Brownian motion hits $\partial B_{\rho R}$, τ_2 is the first time after τ_1 that the Brownian motion hits ∂B_R, τ_3 is the first time after τ_2 that the Brownian motion hits $\partial B_{\rho R}$, etc. Since the Brownian motion is transient, almost surely only a finite number of the τ_n will be finite. For $x \in \partial D$ and $A \subset R^d - B_{\gamma R}$, we have

$$G^D_{\mathrm{Neu}}(x, A) = E_x \int_0^\infty 1_A(B(t))dt = \sum_{n=1}^\infty E_x \int_{\tau_{2n-1}}^{\tau_{2n}} 1_A(B(t))dt. \tag{2.6}$$

This equality holds because for times $s \in [\tau_{2n}, \tau_{2n+1}]$ one has $|B(s)| \leq \rho R$ and thus $B(s) \notin A$.

Note that for $z \in \partial B_{\rho R}$, one has $P_z(\tau_1 = 0) = 1$ and thus under P_z one has that τ_2 is the first hitting time of B_R. Let $\phi(r) = (\frac{R}{r})^{d-2}$, $r = |x|$, be the radially symmetric harmonic function in $R^d - \bar{B}_R$ which equals 1 on the boundary and decays to 0 at ∞. It is well-known that starting from z with $|z| > r$, the probability that the first hitting time of ∂B_R is finite is $\phi(|z|)$; thus, $P_z(\tau_2 < \infty) = \rho^{2-d}$, for $z \in \partial B_{\rho R}$. By the strong Markov property, conditioned on $\tau_{2n-1} < \infty$, the probability that $\tau_{2n} < \infty$ is again ρ^{2-d}. Thus, one has

$$P_x(\tau_{2n-1} < \infty) = \rho^{(2-d)(n-1)}, \text{ for } x \in \partial D. \tag{2.7}$$

Also from the strong Markov property, the conditional expectation $E_x(\int_{\tau_{2n-1}}^{\tau_{2n}} 1_A(B(t))dt | \tau_{2n-1} < \infty)$ is some averaging of the values of $E_z \int_0^{\tau_2} 1_A(B(t))dt$, as z varies over $\partial B_{\rho R}$. That is, there exists a probability measure $\nu_{n,x}$ on $\partial B_{\rho R}$ such that

$$E_x(\int_{\tau_{2n-1}}^{\tau_{2n}} 1_A(B(t))dt | \tau_{2n-1} < \infty) = \int_{\partial B_{\rho R}} \left(E_z \int_0^{\tau_2} 1_A(B(t))dt\right) \nu_{n,x}(dz). \tag{2.8}$$

Now let $W(t)$ be a Brownian motion in all of R^d corresponding to the operator Δ and let $\mathcal{E}_x$ denote the expectation for this Brownian motion starting from x. Since the Brownian motion reflected at ∂D and the Brownian motion on all of R^d behave the same when they are in $R^d - B_R$, one has

$$E_z\int_0^{\tau_2}1_A(B(t))dt=\mathcal{E}_z\int_0^{\tau_2}1_A(W(t))dt,\ \ \text{for } z\in\partial B_{\rho R}. \tag{2.9}$$

We now prove the upper bound in (2.5). From (2.9) we have

$$E_z\int_0^{\tau_2}1_A(B(t))dt\le\mathcal{E}_z\int_0^{\infty}1_A(W(t))dt,\ \ \text{for } z\in\partial B_{\rho R}. \tag{2.10}$$

Recall that the Green's function for Δ on all of R^d is given by $G(x,y)\equiv\frac{|x-y|^{2-d}}{(d-2)\omega_d}$, where ω_d denotes the surface measure of the unit sphere in R^d. The probabilistic representation of the Green's function described above also holds for the Brownian motion in all of R^d; that is, $G(x,y)$ is the density of the measure $\mu_x(A)=\mathcal{E}_x\int_0^\infty 1_A(W(t))dt$. Thus, one has

$$\mathcal{E}_z\int_0^{\infty}1_A(W(t))dt=\int_A G(z,y)dy=\int_A\frac{|z-y|^{2-d}}{(d-2)\omega_d}dy. \tag{2.11}$$

Since $E_x(\int_{\tau_{2n-1}}^{\tau_{2n}}1_A(B(t))dt|\tau_{2n-1}=\infty)=0$, we have from (2.6)-(2.11) that

(2.12)

$$\begin{aligned}
&G^D_{\text{Neu}}(x,A)=E_x\int_0^\infty 1_A(B(t))dt=\sum_{n=1}^\infty E_x\int_{\tau_{2n-1}}^{\tau_{2n}}1_A(B(t))dt=\\
&\sum_{n=1}^\infty E_x(\int_{\tau_{2n-1}}^{\tau_{2n}}1_A(B(t))dt|\tau_{2n-1}<\infty)P_x(\tau_{2n-1}<\infty)\le\\
&\left(\sup_{z\in\partial B_{\rho R}}\mathcal{E}_z\int_0^\infty 1_A(W(t))dt\right)\sum_{n=1}^\infty P_x(\tau_{2n-1}<\infty)=\\
&\left(\sup_{z\in\partial B_{\rho R}}\int_A\frac{|z-y|^{2-d}}{(d-2)\omega_d}dy\right)\sum_{n=1}^\infty\rho^{(2-d)(n-1)}\le\\
&\int_A\frac{1}{(d-2)\omega_d}\frac{1}{1-\rho^{2-d}}\left(\frac{\gamma}{\gamma-\rho}\right)^{d-2}|y|^{2-d}dy,\ \text{for } x\in\partial D,\ A\subset R^d-B_{\gamma R},
\end{aligned}$$

where in the last inequality we have used the fact that $|y|\le\frac{\gamma}{\gamma-\rho}|y-z|$, for $|z|=\rho R$ and $|y|\ge\gamma R$. Now $\rho\in(1,\gamma)$ is a free parameter. One can check that

$$\sup_{\rho\in(1,\gamma)}(1-\rho^{2-d})(\gamma-\rho)^{d-2}=\frac{(\gamma^{\frac{d-2}{d-1}}-1)(\gamma-\gamma^{\frac{1}{d-1}})^{d-2}}{\gamma^{\frac{d-2}{d-1}}}.$$

The maximum is attained at $\rho = \gamma^{\frac{1}{d-1}}$. Using this value of ρ in the right hand side of (2.12) gives

$$G^D_{\text{Neu}}(x,A) \le \int_A \frac{c^+_{\gamma,d}}{(d-2)\omega_d}|y|^{2-d}dy, \tag{2.13}$$

for $x \in \partial D$ and $A \subset R^d - B_{\gamma R}$, where

$$c^+_{\gamma,d} = \frac{\gamma^{\frac{d-2}{d-1}}\gamma^{d-2}}{(\gamma^{\frac{d-2}{d-1}} - 1)(\gamma - \gamma^{\frac{1}{d-1}})^{d-2}}.$$

Note that $\lim_{\gamma\to\infty} c^+_{\gamma,d} = 1$. Now the upper bound in (2.5) follows from (2.13).

We now prove the lower bound in (2.5). Let $G^R_{\text{Dir}}(x,y)$ denote the Green's function for Δ in $R^d - \bar{B}_R$ with the Dirichlet boundary condition at ∂B_R. The probabilistic representation of the Green's function gives

$$G^R_{\text{Dir}}(z,A) = \mathcal{E}_z \int_0^{\tau_2} 1_A(W(t))dt, \text{ for } |z| > R. \tag{2.14}$$

Using reflection with respect to ∂B_R allows one to calculate the Green's function explicitly [2]:

(2.15)

$$G^R_{\text{Dir}}(x,y) = \frac{1}{(d-2)\omega_d}|x-y|^{2-d} - \frac{1}{(d-2)\omega_d}(\frac{|y|}{R})^{2-d}|x - \frac{R^2}{|y|^2}y|^{2-d} = \frac{1}{(d-2)\omega_d}|x-y|^{2-d} - \frac{1}{(d-2)\omega_d}\left|\frac{|y|}{R}x - \frac{R}{|y|}y\right|^{2-d}.$$

(In the proof of the upper bound, we could have used this Green's function and (2.9) instead of the Green's function $G(x,y)$ for all of R^d and (2.10), and this would have yielded a slightly smaller value of $c^+_{\gamma,d}$. However, it was simpler to work with $G(x,y)$. For the lower bound we have no choice but to work with $G^R_{\text{Dir}}(x,y)$.)

For $|x| = \rho R$ and $|y| = \gamma' R$, with $\gamma' \ge \gamma$, one has

$$\left|\frac{|y|}{R}x - \frac{R}{|y|}y\right| \ge \frac{|y||x|}{R} - R = R(\gamma'\rho - 1)$$

and

$$|x-y| \le |x| + |y| = R(\gamma' + \rho).$$

Using this with (2.15), one has

(2.16)
$$\frac{(d-2)\omega_d G^R_{\mathrm{Dir}}(x,y)}{|y|^{2-d}}\ge (\frac{\gamma'}{\gamma'+\rho})^{d-2}-(\frac{\gamma'}{\gamma'\rho-1})^{d-2}, \text{for } |x|=\rho R,\ |y|=\gamma' R.$$

One can check that the right hand side of (2.16) is increasing in γ'; thus

(2.17)
$$\frac{(d-2)\omega_d G^R_{\mathrm{Dir}}(x,y)}{|y|^{2-d}}\ge (\frac{\gamma}{\gamma+\rho})^{d-2}-(\frac{\gamma}{\gamma\rho-1})^{d-2},\ \text{for } |x|=\rho R,\ |y|\ge\gamma R.$$

Then similar to (2.12), but using (2.9) and (2.14) instead of (2.10) and (2.11), we have

(2.18)
$$G^D_{\mathrm{Neu}}(x,A)=E_x\int_0^\infty 1_A(B(t))dt=\sum_{n=1}^\infty E_x\int_{\tau_{2n-1}}^{\tau_{2n}}1_A(B(t))dt=$$
$$\sum_{n=1}^\infty E_x(\int_{\tau_{2n-1}}^{\tau_{2n}}1_A(B(t))dt|\tau_{2n-1}<\infty)P_x(\tau_{2n-1}<\infty)\ge$$
$$\left(\inf_{z\in\partial B_{\rho R}}\mathcal{E}_z\int_0^{\tau_2}1_A(W(t))dt\right)\sum_{n=1}^\infty P_x(\tau_{2n-1}<\infty)=$$
$$\left(\inf_{z\in\partial B_{\rho R}}\int_A G^R_{\mathrm{Dir}}(z,y)dy\right)\sum_{n=1}^\infty\rho^{(2-d)(n-1)}=\left(\inf_{z\in\partial B_{\rho R}}\int_A G^R_{\mathrm{Dir}}(z,y)dy\right)\frac{1}{1-\rho^{2-d}}\ge$$
$$\int_A\frac{1}{(d-2)\omega_d}\frac{1}{1-\rho^{2-d}}\left((\frac{\gamma}{\gamma+\rho})^{d-2}-(\frac{\gamma}{\gamma\rho-1})^{d-2}\right)|y|^{2-d}dy,$$
for $x\in\partial D,\ A\subset R^d-B_{\gamma R}$,

where the inequality follows from (2.17).

As a function of $\rho\in(1,\gamma)$, the expression $\frac{1}{1-\rho^{2-d}}\left((\frac{\gamma}{\gamma+\rho})^{d-2}-(\frac{\gamma}{\gamma\rho-1})^{d-2}\right)$ is non-positive if $\gamma\le 1+\sqrt{2}$. For $\gamma>1+\sqrt{2}$, this expression has a positive maximum. The formula for the maximum is complicated and doesn't add much so we will just define $c^-_{\gamma,d}$ as follows. For γ sufficiently large so that the above expression is positive for $\rho=\gamma^{\frac{1}{d-1}}$, let this value be $c^-_{\gamma,d}$. Then, in particular, for some $\gamma_0>1$, one has

$$c^-_{\gamma,d}=\frac{1}{1-\gamma^{\frac{2-d}{d-1}}}\left(\left(\frac{\gamma}{\gamma+\gamma^{\frac{1}{d-1}}}\right)^{d-2}-\left(\frac{\gamma}{\gamma^{\frac{d}{d-1}}-1}\right)^{d-2}\right),\ \text{for } \gamma\ge\gamma_0.$$

Note that $\lim_{\gamma\to\infty} c^-_{\gamma,d} = 1$. From (2.18) and the above remarks, we have now shown that

$$G^D_{\text{Neu}}(x,y) \ge \frac{1}{(d-2)\omega_d} c^-_{\gamma,d}|y|^{2-d}, \text{ for } x \in \partial D, |y| \ge \gamma R, \gamma \ge \gamma_0. \tag{2.19}$$

This is the lower bound in (2.5) except that one has here $\gamma \ge \gamma_0$ instead of $\gamma > 1$. To extend (2.19) to $\gamma > 1$, one notes that since $G^R_{\text{Dir}}(z,y)$ is positive for $z, y \in R^d - \bar{B}_R$, one has trivially $\inf_{\gamma R \le |y| \le \gamma_0 R, |z|=\rho R} G^R_{\text{Dir}}(z,y) > 0$, for any choice of ρ, γ satisfying $1 < \rho < \gamma < \gamma_0$, and in fact by scaling, the left hand side is independent of R. Using this along with the fact that $G^D_{\text{Neu}}(x,A) \ge \left(\inf_{z\in\partial B_{\rho R}} \int_A G^R_{\text{Dir}}(z,y)dy\right) \frac{1}{1-\rho^{2-d}}$, which follows from (2.18), we can define $c^-_{\gamma,d} > 0$ for all $\gamma > 1$ so that (2.19) holds.

□

3. Proof of Theorem 2

In the present case, (2.3) still holds, where G^D_{Neu} is now the Green's function for the operator L on $R^d - \bar{D}$ with the co-normal boundary condition at ∂D. Since the operator L is symmetric with respect to the weight e^Q, similar to (2.4) we have

$$e^{Q(x)} G^D_{\text{Neu}}(x,y) = e^{Q(y)} G^D_{\text{Neu}}(y,x). \tag{3.1}$$

To see this, note that since Green's function $G^D_{\text{Neu}}(x,y)$ satisfies $LG^D_{\text{Neu}}(\cdot,y) = -\delta_y$ with the homogeneous co-normal boundary condition at ∂D, it follows that for any compactly supported sufficiently smooth function f defined on $R^d - D$, the function $v(x) = \int_{R^d-D} G^D_{\text{Neu}}(x,y)f(y)dy$ is the minimal positive solution of $Lv = -f$ in $R^d - D$, with the homogeneous co-normal boundary condition at ∂D. Now on the one hand, since L is in the form (1.7), it follows that v solves $\nabla \cdot e^Q a \nabla v = -e^Q f$ with the homogeneous co-normal boundary condition, but on the other hand, by the same reasoning as above, we have $v(x) = \int_{R^d-D} G_{\text{sym}}(x,y) e^Q(y) f(y)dy$, where G_{sym} is the Green's function for the operator $\nabla \cdot e^Q a \nabla$ on $R^d - D$ with the co-normal boundary condition at ∂D. Since this holds for all nice f, we conclude that

$G^D_{\rm Neu}(x,y)=G_{\rm sym}(x,y)e^{Q(y)}$. Since $G_{\rm sym}$ is the Green's function of an operator that is symmetric with respect to Lebesgue measure, it follows as in (2.4) that $G_{\rm sym}(x,y)=G_{\rm sym}(y,x)$. Now (3.1) follows from this.

In light of (2.3) and (3.1), to prove the theorem it suffices to show that for $R'>R$, one has

(3.2)
$$\frac{\min_{|z|=R'}G^R_{\rm Dir}(z,x)}{1-\min_{|z|=R'}V(z)}\le G^D_{\rm Neu}(y,x)\le\frac{\max_{|z|=R'}G^R_{\rm Dir}(z,x)}{1-\max_{|z|=R'}V(z)},\ \text{for } |x|>R' \text{ and } y\in\partial D.$$

As in the previous section, we consider the probabilistic representation of the Green's function. Let $X(t)$ be the diffusion process in R^d-D, conormally reflected at ∂D, and corresponding to the operator L [4]. Let P_y denote probabilities and let E_y denote the corresponding expectations for the diffusion starting from $y\in R^d-D$. For $y\in R^d-D$, define the expected occupation measure by $\mu^D_y(A)=E_y\int_0^\infty 1_A(X(t))dt$, for Borel sets $A\subset R^d-D$. By assumption, the process $X(t)$ is transient, (that is, $P_y(\lim_{t\to\infty}|X(t)|=\infty)=1$), and from this one can show that $\mu^D_y(A)<\infty$ for all bounded A. The measure $\mu^D_y(dx)$ possesses a density and the density is given by $G^D_{\rm Neu}(y,x)$. From now on we will write $G^D_{\rm Neu}(y,A)\equiv\mu^D_y(A)$.

Define $\tau_1=\inf\{t\ge 0:|X(t)|=R_1\}$, and then by induction define $\tau_{2n}=\inf\{t>\tau_{2n-1}:|X(t)|=R\}$ and $\tau_{2n+1}=\inf\{t>\tau_{2n}:|X(t)|=R_1\}$. Similar to the proof of Theorem 1 (see (2.6), (2.8) and the line between (2.11) and (2.12)), we have for $y\in\partial D$ and $A\subset R^d-\bar B_{R_1}$,

$$\begin{aligned}&G^D_{\rm Neu}(y,A)=E_y\int_0^\infty 1_A(X(t))dt=\sum_{n=1}^\infty E_y\int_{\tau_{2n-1}}^{\tau_{2n}}1_A(X(t))dt=\\ &\sum_{n=1}^\infty E_y(\int_{\tau_{2n-1}}^{\tau_{2n}}1_A(X(t))dt|\tau_{2n-1}<\infty)P_y(\tau_{2n-1}<\infty)=\\ &\sum_{n=1}^\infty\int_{\partial B_{R_1}}\left(E_z\int_0^{\tau_2}1_A(X(t))dt\right)\nu_{n,y}(dz)P_y(\tau_{2n-1}<\infty),\end{aligned}\tag{3.3}$$

where $\nu_{n,y}$ is some probability measure on ∂B_{R_1}. By the strong Markov property,

$$(3.4)\qquad (\min_{|z|=R'} V(z))^{n-1} \le P_y(\tau_{2n-1}<\infty) \le (\max_{|z|=R'} V(z))^{n-1},$$

for $y \in \partial D$, where V is as in (1.6). Now (3.2) follows from (3.3), (3.4) and the fact that $E_z \int_0^{\tau_2} 1_A(X(t))dt = G^R_{\text{Dir}}(z, A)$.

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