

**ON THE STRANGE DOMAIN OF ATTRACTION TO  
GENERALIZED DICKMAN DISTRIBUTIONS FOR SUMS  
OF INDEPENDENT RANDOM VARIABLES**

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ABSTRACT. Let  $\{B_k\}_{k=1}^\infty, \{X_k\}_{k=1}^\infty$  all be independent random variables. Assume that  $\{B_k\}_{k=1}^\infty$  are  $\{0, 1\}$ -valued Bernoulli random variables satisfying  $B_k \stackrel{\text{dist}}{=} \text{Ber}(p_k)$ , with  $\sum_{k=1}^\infty p_k = \infty$ , and assume that  $\{X_k\}_{k=1}^\infty$  satisfy  $X_k > 0$  and  $\mu_k \equiv EX_k < \infty$ . Let  $M_n = \sum_{k=1}^n p_k \mu_k$ , assume that  $M_n \rightarrow \infty$  and define the normalized sum of independent random variables  $W_n = \frac{1}{M_n} \sum_{k=1}^n B_k X_k$ . We give a general condition under which  $W_n \xrightarrow{\text{dist}} c$ , for some  $c \in [0, 1]$ , and a general condition under which  $W_n$  converges weakly to a distribution from a family of distributions that includes the generalized Dickman distributions  $\text{GD}(\theta), \theta > 0$ . In particular, we obtain the following result, which reveals a strange domain of attraction to generalized Dickman distributions. Assume that  $\lim_{k \rightarrow \infty} \frac{X_k}{\mu_k} \stackrel{\text{dist}}{=} 1$ . Let  $J_\mu, J_p$  be nonnegative integers, let  $c_\mu, c_p > 0$  and let

$$\mu_n \sim c_\mu n^{a_0} \prod_{j=1}^{J_\mu} (\log^{(j)} n)^{a_j}, \quad p_n \sim c_p (n^{b_0} \prod_{j=1}^{J_p} (\log^{(j)} n)^{b_j})^{-1}, \quad b_{J_p} \neq 0.$$

If

$$i. \quad J_p \leq J_\mu;$$

$$ii. \quad b_j = 1, \quad 0 \leq j \leq J_p;$$

$$iii. \quad a_j = 0, \quad 0 \leq j \leq J_p - 1, \quad \text{and} \quad a_{J_p} > 0,$$

then  $\lim_{n \rightarrow \infty} W_n \stackrel{\text{dist}}{=} \frac{1}{\theta} \text{GD}(\theta)$ , where  $\theta = \frac{c_p}{a_{J_p}}$ . Otherwise,  $\lim_{n \rightarrow \infty} W_n \stackrel{\text{dist}}{=} c$ , for some  $c \in [0, 1]$ . We also give an application to the statistics of the number of inversions in certain random shuffling schemes.

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## 1. INTRODUCTION AND STATEMENT OF RESULTS

The Dickman function  $\rho_1$  is the unique function, continuous on  $(0, \infty)$ , and satisfying the differential-delay equation

$$\begin{aligned}\rho_1(x) &= 0, \quad x \leq 0; \\ \rho_1(x) &= 1, \quad x \in (0, 1]; \\ x\rho_1'(x) + \rho_1(x-1) &= 0, \quad x > 1.\end{aligned}$$

This function has an interesting role in number theory and probability, which we describe briefly at the end of this section. With a little work, one can show that the Laplace transform of  $\rho_1$  is given by  $\int_0^\infty \rho_1(x)e^{-\lambda x}dx = \exp(\gamma + \int_0^1 \frac{e^{-\lambda x}-1}{x}dx)$ , where  $\gamma$  is Euler's constant. From this it follows that  $\int_0^\infty \rho_1(x)dx = e^\gamma$ , and consequently, that  $e^{-\gamma}\rho_1$  is a probability density on  $[0, \infty)$ . We will call this probability distribution the *Dickman distribution*. We denote its density by  $p_1(x) = e^{-\gamma}\rho_1(x)$ , and we denote by  $D_1$  a random variable distributed according to the Dickman distribution. Differentiating the Laplace transform  $E \exp(-\lambda D_1) = \exp(\int_0^1 \frac{e^{-\lambda x}-1}{x}dx)$  of  $D_1$  at  $\lambda = 0$  shows that  $ED_1 = 1$ . The distribution decays very rapidly; indeed, it is not hard to show that  $p_1(x) \leq \frac{e^{-\gamma}}{\Gamma(x+1)}$ ,  $x \geq 0$  [6].

In fact, for all  $\theta > 0$ ,  $\exp(\theta \int_0^1 \frac{e^{-\lambda x}-1}{x}dx)$  is the Laplace transform of a probability distribution. (We will prove this directly; however, this fact follows from the theory of infinitely divisible distributions, and shows that the distribution in question is infinitely divisible.) This distribution has density  $p_\theta = \frac{e^{-\theta\gamma}}{\Gamma(\theta)}\rho_\theta$ , where  $\rho_\theta$  satisfies the differential-delay equation

$$(1.1) \quad \begin{aligned}\rho_\theta(x) &= 0, \quad x \leq 0; \\ \rho_\theta(x) &= x^{\theta-1}, \quad 0 < x \leq 1; \\ x\rho_\theta'(x) + (1-\theta)\rho_\theta(x) + \theta\rho_\theta(x-1) &= 0, \quad x > 1.\end{aligned}$$

We will call such distributions *generalized Dickman distributions* and denote them by  $\text{GD}(\theta)$ . We denote by  $D_\theta$  a random variable with the  $\text{GD}(\theta)$  distribution. Differentiating its Laplace transform at  $\lambda = 0$  shows that  $ED_\theta = \theta$ . These distribution decays very rapidly; indeed, it is not hard to show that

$p_\theta(x) \leq \frac{C_\theta}{\Gamma(x+1)}$ ,  $x \geq 1$ , for an appropriate constant  $C_\theta$ . A fundamental fact about these distributions is that

$$(1.2) \quad D_\theta \stackrel{\text{dist}}{=} U^{\frac{1}{\theta}}(D_\theta + 1),$$

where  $U$  is distributed according to the uniform distribution on  $[0, 1]$ , and  $U$  and  $D_\theta$  on the right hand side above are independent. From (1.2) it is immediate that

$$D_\theta \stackrel{\text{dist}}{=} U_1^{\frac{1}{\theta}} + (U_1 U_2)^{\frac{1}{\theta}} + (U_1 U_2 U_3)^{\frac{1}{\theta}} + \dots,$$

where  $\{U_n\}_{n=1}^\infty$  are IID random variables distributed according to the uniform distribution on  $[0, 1]$ . It will follow from the proof of Theorem 1 below that  $\exp(\theta \int_0^1 \frac{e^{-\lambda x} - 1}{x} dx)$  is the Laplace transform of a probability distribution. In section 5 we will prove that a random variable with such a distribution satisfies (1.2), and that if a random variable satisfies (1.2), then it has a density of the form  $c_\theta \rho_\theta$ , where  $\rho_\theta$  satisfies (1.1). Thus, this paper is self-contained with regard to all the above noted facts, with the exception of the rate of decay and the value  $\frac{e^{-\theta\gamma}}{\Gamma(\theta)}$  of the normalizing constant  $c_\theta$  in  $p_\theta$ . For more on these distributions, including a derivation of the normalizing constant, see, for example, [1] and [8].

In fact, the scope of this paper leads us to consider a more general family of distributions than the generalized Dickman distributions. Let  $\mathcal{X} \geq 0$  be a random variable satisfying  $E\mathcal{X} \leq 1$ . Then, as we shall see, for  $\theta > 0$ , there exists a distribution whose Laplace transform is  $\exp(\theta \int_0^1 \frac{E \exp(-\lambda x \mathcal{X}) - 1}{x} dx)$ . We will denote this distribution by  $GD^{(\mathcal{X})}(\theta)$  and we denote a random variable with this distribution by  $D_\theta^{(\mathcal{X})}$ . (When  $\mathcal{X} \equiv 1$ , we revert to the previous notation for generalized Dickman distributions.) Differentiating the Laplace transform at  $\lambda = 0$  shows that  $ED_\theta^{(\mathcal{X})} = \theta E\mathcal{X}$ .

Mimicking the proof of (1.2) that we give in section 5 shows that

$$(1.3) \quad D_\theta^{(\mathcal{X})} = U^{\frac{1}{\theta}}(D_\theta^{(\mathcal{X})} + \mathcal{X}),$$

where  $U$  is distributed according to the uniform distribution on  $[0, 1]$ , and  $U$ ,  $D_\theta^{(\mathcal{X})}$  and  $\mathcal{X}$  on the right hand side above are independent. From (1.3)

it is immediate that

$$D_\theta \stackrel{\text{dist}}{=} \mathcal{X}_1 U_1^{\frac{1}{\theta}} + \mathcal{X}_2 (U_1 U_2)^{\frac{1}{\theta}} + \mathcal{X}_3 (U_1 U_2 U_3)^{\frac{1}{\theta}} + \dots,$$

where  $\{U_n\}_{n=1}^\infty$  and  $\{\mathcal{X}_n\}_{n=1}^\infty$  are mutually independent sequences of IID random variables, with  $U_1$  distributed according to the uniform distribution on  $[0, 1]$  and  $\mathcal{X}_1$  distributed according to the distribution of  $\mathcal{X}$ .

It is known that the generalized Dickman distribution  $GD(\theta)$  arises as the limiting distribution of  $\frac{1}{n} \sum_{k=1}^n k Y_k$ , where the  $\{Y_k\}_{k=1}^\infty$  are independent random variables with  $Y_k$  distributed according to the Poisson distribution with parameter  $\frac{\theta}{k}$  [1]. It is also known that the Dickman distribution  $GD(1)$  arises as the limiting distribution of  $\frac{1}{n} \sum_{k=1}^n k Y_k$  as  $n \rightarrow \infty$ , where the  $\{Y_k\}_{k=1}^\infty$  are independent Bernoulli random variables satisfying  $P(Y_k = 1) = 1 - P(Y_k = 0) = \frac{1}{k}$ . Such behavior is in distinct contrast to the law of large numbers behavior of a “well-behaved” sequence of independent random variables  $\{Z_k\}_{k=1}^\infty$  with finite first moments; namely, that  $\frac{1}{M_n} \sum_{k=1}^n Z_k$  converges in distribution to 1 as  $n \rightarrow \infty$ , where  $M_n = \sum_{k=1}^n E Z_k$ .

The purpose of this paper is to understand when the law of large numbers fails and a distribution from the family  $GD^{(\mathcal{X})}(\theta)$  arises in its stead. From the above examples, we see that generalized Dickman distributions sometimes arise as limits of normalized sums from a sequence  $\{V_k\}_{k=1}^\infty$  of independent random variables which are non-negative and satisfy the following three conditions: (i)  $\lim_{k \rightarrow \infty} P(V_k = 0) = 1$ , (ii)  $\lim_{k \rightarrow \infty} \frac{V_k | V_k > 0}{E(V_k | V_k > 0)} \stackrel{\text{dist}}{=} 1$  and (iii)  $\sum_{k=1}^\infty E V_k = \infty$ . (In the above examples,  $k Y_k$  plays the role of  $V_k$ .) It turns out that these three conditions are very far from sufficient for a generalized Dickman distribution to arise. In fact, as we shall see in Theorem 2 below, such distributions arise only in a strange sequence of very narrow windows of opportunity.

In light of the above discussion, we will consider the following setting. Let  $\{B_k\}_{k=1}^\infty, \{X_k\}_{k=1}^\infty$  be mutually independent sequences of independent random variables. Assume that  $\{B_k\}_{k=1}^\infty$  are Bernoulli random variables

satisfying:

$$(1.4) \quad P(B_k = 1) = 1 - P(B_k = 0) = p_k \in [0, 1),$$

and assume that  $\{X_k\}_{k=1}^\infty$  satisfy:

$$(1.5) \quad X_k > 0, \quad \mu_k \equiv EX_k < \infty.$$

Let

$$(1.6) \quad M_n = \sum_{k=1}^n p_k \mu_k,$$

and define

$$(1.7) \quad W_n = \frac{1}{M_n} \sum_{k=1}^n B_k X_k.$$

We will be interested in the limiting behavior of  $W_n$ . In order to avoid trivialities, we will assume that

$$(1.8) \quad \lim_{n \rightarrow \infty} M_n = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} p_k = \infty,$$

since otherwise  $\sum_{k=1}^{\infty} B_k X_k$  is almost surely finite.

Note that for the example brought with the  $\text{Pois}(\frac{\theta}{k})$ -distribution, we have  $p_k = 1 - e^{-\frac{\theta}{k}}$ ,  $X_k$  is distributed according to  $kY_k | \{Y_k > 0\}$ , where  $Y_k$  has the  $\text{Pois}(\frac{\theta}{k})$  distribution,  $\mu_k = \frac{\theta}{1 - e^{-\frac{\theta}{k}}}$  and  $M_n = n\theta$ . And for the example with the  $\text{Ber}(\frac{1}{k})$ -distribution, we have  $p_k = \frac{1}{k}$ ,  $X_k = k$  deterministically,  $\mu_k = k$  and  $M_n = n$ . In the first of these two examples,  $\frac{X_k}{\mu_k} \xrightarrow{\text{dist}} 1$ , and in the second one,  $\frac{X_k}{\mu_k} \stackrel{\text{dist}}{=} 1$  for all  $k$ .

Our first theorem gives a general condition for  $W_n \xrightarrow{\text{dist}} c$  (which is the law of large numbers if  $c = 1$ ), and a general condition for convergence to a limiting distribution from the family of distributions  $\text{GD}^{(\mathcal{X})}(\theta)$ . Using this theorem, we can prove our second theorem, which reveals the strange domain of attraction to generalized Dickman distributions. (Of course, we are using the term “domain of attraction” not in its classical sense, since our sequence of random variables, although independent, are not identically distributed.) Let  $\delta_c$  denote the degenerate distribution at  $c$ .

**Theorem 1.** Let  $W_n$  be as in (1.7), where  $\{B_k\}_{k=1}^\infty$ ,  $\{X_k\}_{k=1}^\infty$  and  $M_n$  are as in (1.4)-(1.6) and (1.8).

i. Assume that  $\{\frac{X_k}{\mu_k}\}_{k=1}^\infty$  is uniformly integrable (which occurs automatically if  $\lim_{k \rightarrow \infty} \frac{X_k}{\mu_k} \stackrel{dist}{=} 1$ ).

a. Assume also that

$$(1.9) \quad \lim_{n \rightarrow \infty} \frac{\max_{1 \leq k \leq n} \mu_k}{M_n} = 0.$$

Then

$$\lim_{n \rightarrow \infty} W_n \stackrel{dist}{=} 1.$$

b. Assume also that there exists a sequence  $\{K_n\}_{n=1}^\infty$  such that

$$(1.10) \quad \lim_{n \rightarrow \infty} \sum_{k=K_n+1}^n p_k = 0,$$

and

$$(1.11) \quad \lim_{n \rightarrow \infty} \frac{\max_{1 \leq k \leq K_n} \mu_k}{M_n} = 0.$$

If

$$(1.12) \quad c \equiv \lim_{n \rightarrow \infty} \frac{M_{K_n}}{M_n} \text{ exists,}$$

then

$$\lim_{n \rightarrow \infty} W_n \stackrel{dist}{=} c.$$

If (1.12) does not hold, then the distributions of  $\{W_n\}_{n=1}^\infty$  form a tight sequence whose set of accumulation points is  $\{\delta_c : c \in A\}$ , where  $A$  denotes the set of accumulation points of the sequence  $\{\frac{M_{K_n}}{M_n}\}_{n=1}^\infty$ .

ii. Assume that there exists a random variable  $\mathcal{X}$  such that

$$(1.13) \quad \lim_{k \rightarrow \infty} \frac{X_k}{\mu_k} \stackrel{dist}{=} \mathcal{X}.$$

Assume also that  $\{\mu_k\}_{k=1}^\infty$  is increasing, that  $\lim_{k \rightarrow \infty} p_k = 0$  and that there exist  $\theta, L \in (0, \infty)$  such that

$$(1.14) \quad \lim_{k \rightarrow \infty} \frac{p_k \mu_k}{\mu_{k+1} - \mu_k} = \theta, \quad \lim_{k \rightarrow \infty} \frac{\mu_k}{M_k} = L.$$

Then

$$\lim_{n \rightarrow \infty} W_n \stackrel{\text{dist}}{=} LD^{(\mathcal{X})}(\theta),$$

where  $D^{(\mathcal{X})}(\theta)$  is a random variable with the  $GD^{(\mathcal{X})}(\theta)$  distribution.

**Remark 1.** In (1.14), necessarily  $L \leq \frac{1}{\theta}$ . Indeed, if  $\{p_k\}_{k=1}^{\infty}$  and  $\{\theta_k\}_{k=1}^{\infty}$  satisfy the conditions of part (ii), and we choose  $X_k = \mu_k$ , then  $W_n \xrightarrow{\text{dist}} LD_{\theta}$ . Since  $EW_n = 1$  and  $ED_{\theta} = \theta$ , it follows from Fatou's lemma that  $L \leq \frac{1}{\theta}$ . In most cases of interest, one has  $L = \frac{1}{\theta}$ .

**Remark 2.** By Fatou's lemma, the random variable  $\mathcal{X}$  in part (ii) must satisfy  $E\mathcal{X} \leq 1$ .

**Remark 3.** The uniform integrability of  $\{\frac{X_k}{\mu_k}\}_{k=1}^{\infty}$  in part (i) occurs automatically if  $\lim_{k \rightarrow \infty} \frac{X_k}{\mu_k} \stackrel{\text{dist}}{=} 1$ , because if a sequence  $\{Y_k\}_{k=1}^{\infty}$  of random variables satisfies  $Y_k \xrightarrow{\text{dist}} Y$ , and  $E|Y_k| < \infty$ , then  $E|Y_k| \rightarrow E|Y|$  is equivalent to uniform integrability.

**Remark 4.** In the case that  $X_k = \mu_k$ , or more generally, if  $EX_k \leq C\mu_k^2$ , for all  $k$  and some  $C > 0$ , then

$$\text{Var}(W_n) \leq \frac{C \sum_{k=1}^n p_k \mu_k^2}{M_n^2} = C \frac{\sum_{k=1}^n p_k \mu_k^2}{(\sum_{k=1}^n p_k \mu_k)^2} \leq C \frac{\sup_{1 \leq k \leq n} \mu_k}{M_n}.$$

Thus, in this case part (i-a) follows directly from the second moment method.

Using Theorem 1, we can prove the following theorem that exhibits the strange domain of attraction to generalized Dickman distributions. Let  $\log^{(j)}$  denote the  $j$ th iterate of the logarithm, and make the convention  $\prod_{j=1}^0 = 1$ .

**Theorem 2.** Let  $W_n$  be as in (1.7), where  $\{B_k\}_{k=1}^{\infty}$ ,  $\{X_k\}_{k=1}^{\infty}$  and  $M_n$  are as in (1.4)-(1.6). Assume also that  $\lim_{k \rightarrow \infty} \frac{X_k}{\mu_k} \stackrel{\text{dist}}{=} 1$ . Let  $J_{\mu}, J_p$  be nonnegative integers, let  $c_{\mu}, c_p > 0$  and define

$$\mu(x) = c_{\mu} x^{a_0} \prod_{j=1}^{J_{\mu}} (\log^{(j)} x)^{a_j},$$

$$p(x) = c_p (x^{b_0} \prod_{j=1}^{J_p} (\log^{(j)} x)^{b_j})^{-1},$$

with  $b_{J_p} \neq 0$ . Assume that

$$\begin{aligned}\mu_k &\sim \mu(k), \quad p_k \sim p(k); \\ \mu_{k+1} - \mu_k &\sim \mu'(k).\end{aligned}$$

Assume that the exponents  $\{a_j\}_{j=0}^{J_\mu}, \{b_j\}_{j=0}^{J_p}$  have been chosen so that (1.8) holds. If

$$(1.15) \quad \begin{aligned}i. & J_p \leq J_\mu; \\ ii. & b_j = 1, \quad 0 \leq j \leq J_p; \\ iii. & a_j = 0, \quad 0 \leq j \leq J_p - 1, \quad \text{and} \quad a_{J_p} > 0,\end{aligned}$$

then

$$\lim_{n \rightarrow \infty} W_n \stackrel{\text{dist}}{=} \frac{1}{\theta} D_\theta, \quad \text{with } \theta = \frac{c_p}{a_{J_p}},$$

where  $D_\theta$  is a random variable with the  $GD(\theta)$  distribution.

Otherwise,  $\lim_{n \rightarrow \infty} W_n \stackrel{\text{dist}}{=} c$ , where  $c \in \{0, 1\}$ . To determine  $c$ , let

$$\kappa_\mu = \min\{0 \leq j \leq J_\mu : a_j \neq 0\} \quad \text{and} \quad \kappa_p = \min\{0 \leq j \leq J_p : b_j \neq 1\}.$$

If  $\{0 \leq j \leq J_\mu : a_j \neq 0\}$  is not empty,  $a_{\kappa_\mu} > 0$  and either  $\{0 \leq j \leq J_p : b_j \neq 1\}$  is empty and  $\kappa_\mu < J_p$ , or  $\{0 \leq j \leq J_p : b_j \neq 1\}$  is not empty and  $\kappa_\mu < \kappa_p$ , then  $c = 0$ ; otherwise,  $c = 1$ .

**Remark 1.** Note that if one chooses  $\mu_k = \mu(k)$  and  $p_k = p(k)$ , then the condition  $\mu_{k+1} - \mu_k \sim \mu'(k)$  is always satisfied.

**Remark 2.** Theorem 2 shows that to obtain a generalized Dickman distribution,  $\{p_k\}_{k=1}^\infty$  in particular must be set in a very restricted fashion. For some intuition regarding this phenomenon, take the situation where  $X_k = \mu_k$ , and consider the sequence  $\{\sigma^2(W_n)\}_{n=1}^\infty$  of variances. This sequence converges to 0 in the cases where  $W_n$  converges to 1, converges to  $\infty$  in the cases where  $W_n$  converges to 0, and converges to a positive number in the cases where  $W_n$  converges to a generalized Dickman distribution.

We now state explicitly what Theorem 2 yields in the cases  $J_p = 0, 1$ .



$\mathbf{J}_p = \mathbf{0}$ . We have

$$p_n \sim \frac{c_p}{n^{b_0}}, \quad b_0 > 0, \quad \mu_n \sim c_\mu n^{a_0} \prod_{j=1}^{J_\mu} (\log^{(j)} n)^{a_j}.$$

In order that (1.8) hold, we require  $b_0 \leq 1$ . We also require either:  $a_0 - b_0 > -1$ ; or  $a_0 - b_0 = -1$  and  $a_1 > -1$ ; or  $a_0 - b_0 = a_1 = -1$  and  $a_2 > -1$ ; etc.

If  $b_0 = 1$  and  $a_0 > 0$ , then

$$\lim_{n \rightarrow \infty} W_n \stackrel{dist}{=} \frac{1}{\theta} D_\theta, \quad \text{where } \theta = \frac{c_p}{a_0}.$$

Otherwise,  $\lim_{n \rightarrow \infty} W_n \stackrel{dist}{=} 1$ .

$\mathbf{J}_p = \mathbf{1}$ . We have

$$p_n \sim \frac{c_p}{n^{b_0} (\log n)^{b_1}}, \quad b_1 \neq 0, \quad \mu_n \sim c_\mu n^{a_0} \prod_{j=1}^{J_\mu} (\log^{(j)} n)^{a_j}.$$

In order that (1.8) hold, we require either  $b_0 = 0$  and  $b_1 > 0$ , or  $0 < b_0 < 1$ , or  $b_0 = 1$  and  $b_1 \leq 1$ . We also require either:  $a_0 - b_0 > -1$ ; or  $a_0 - b_0 = -1$  and  $a_1 - b_1 > -1$ ; or  $a_0 - b_0 = a_1 - b_1 = -1$  and  $a_2 > -1$ ; etc.

If  $J_\mu \geq 1$ ,  $b_0 = b_1 = 1$ ,  $a_0 = 0$  and  $a_1 > 0$ , then

$$\lim_{n \rightarrow \infty} W_n \stackrel{dist}{=} \frac{1}{\theta} D_\theta, \quad \text{where } \theta = \frac{c_p}{a_1}.$$

If  $b_0 = 1$  and  $a_0 > 0$ , then  $\lim_{n \rightarrow \infty} W_n \stackrel{dist}{=} 0$ .

Otherwise,  $\lim_{n \rightarrow \infty} W_n \stackrel{dist}{=} 1$ .

**Remark.** In [3] and [7], where the GD(1) distribution arises, one has  $b_0 = b_1 = 1, a_0 = 0, a_1 = 1, c_p = c_\mu = 1$ .

The organization of the rest of the paper is as follows. In section 2 we use Theorems 1 and 2 to investigate a question raised in [5] concerning the statistics of the number of inversions in certain random shuffling schemes. In sections 3 and 4 respectively we prove Theorems 1 and 2. Finally, in section 5 we prove the basic facts about the Dickman distribution and its density, as was promised earlier in this section.

As mentioned above, we end this section with a little background concerning the Dickman function  $\rho \equiv \rho_1$ . The Dickman function arises in probabilistic number theory in the context of so-called *smooth* numbers; that is, numbers all of whose prime divisors are “small.” Let  $\Psi(x, y)$  denote the number of positive integers less than or equal to  $x$  with no prime divisors greater than  $y$ . Numbers with no prime divisors greater than  $y$  are called *y-smooth* numbers. Then for  $s \geq 1$ ,  $\Psi(N, N^{\frac{1}{s}}) \sim N\rho(s)$ , as  $N \rightarrow \infty$ . This result was first proved by Dickman in 1930 [4], whence the name of the function, with later refinements by de Bruijn [2]. See also [6] or [9]. Let  $[n] = \{1, \dots, n\}$  and let  $p^+(n)$  denote the largest prime divisor of  $n$ . Then Dickman’s result states that the random variable  $\frac{\log p^+(j)}{\log n}$ ,  $j \in [n]$ , on the probability space  $[n] = \{1, \dots, n\}$  with the uniform distribution converges in distribution as  $n \rightarrow \infty$  to the distribution whose distribution function is  $\rho(\frac{1}{x})$ ,  $x \in [0, 1]$ , and whose density is  $-\frac{\rho'(\frac{1}{x})}{x^2} = \frac{\rho(\frac{1}{x}-1)}{x}$ ,  $x \in [0, 1]$ . It is easy to see that an equivalent statement of Dickman’s result is that the random variable  $\frac{\log p^+(j)}{\log j}$ ,  $j \in [n]$ , on the probability space  $[n]$  with the uniform distribution converges in distribution as  $n \rightarrow \infty$  to the distribution whose distribution function is  $\rho(\frac{1}{x})$ ,  $x \in [0, 1]$ . We note that the length of the longest cycle of a uniformly random permutation of  $[n]$ , normalized by dividing by  $n$ , also converges to a limiting distribution whose distribution function is  $\rho(\frac{1}{x})$ . If instead of using the uniform measure on  $S_n$ , the set of permutations of  $[n]$ , one uses the Ewens sampling distribution on  $S_n$ , obtained by giving each permutation  $\sigma \in S_n$  the probability proportional to  $\theta^{C(\sigma)}$ , where  $C(\sigma)$  denotes the number of cycles in  $\sigma$ , then the length of the longest cycle of such a random permutation of  $[n]$ , normalized by dividing by  $n$ , converges to a limiting distribution whose distribution function is  $\rho_\theta(\frac{1}{x})$ ,  $x \in [0, 1]$ . This distribution is also the distribution of the first coordinate of the Poisson-Dirichlet distribution  $\text{PD}(\theta)$  (see [1]).

The examples in the above paragraph lead to limiting distributions where the Dickman function arises as a distribution function, not as a density as is the case with the  $\text{GD}(\theta)$  distributions discussed in this paper. The  $\text{GD}(\theta)$

distribution arises as a normalized limit in the context of certain natural probability measures that one can place on  $\mathbb{N}$ ; see [3], [7].

## 2. AN APPLICATION TO RANDOM PERMUTATIONS

We consider a setup that appeared in [5], and which in the terminology of this paper can be described as follows. For each  $k \in \mathbb{N}$ , let  $E_k \subset \{1, \dots, k-1\}$ . Let  $X_k$  be uniformly distributed on  $E_k$ , and let  $B_k \stackrel{\text{dist}}{=} \text{Ber}(\frac{|E_k|}{k})$ . So

$$\mu_k = \frac{1}{|E_k|} \sum_{l \in E_k} l, \quad p_k = \frac{|E_k|}{k}.$$

Define

$$I_n = \sum_{k=1}^n B_k X_k.$$

We allow  $E_k = \emptyset$ , in which case  $B_k = 0$  and  $X_k$  is not defined. In such a case, we define  $B_k X_k = 0$  and  $\mu_k = 0$ . We always have  $E_1 = \emptyset$ .

Consider first the case that  $E_k = \{1, \dots, k-1\}$ . Then  $B_1 X_1 = 0$  and for  $2 \leq k \leq n$ ,  $B_k X_k$  is uniformly distributed over  $\{0, 1, \dots, k-1\}$ . In this case,  $I_n$  has the distribution of the number of inversions in a uniformly random permutation from  $S_n$ . (The authors in [5] have a typo and wrote  $E_k = \{1, \dots, k\}$  instead.) To see this, consider the following shuffling procedure for  $n$  cards, numbered from 1 to  $n$ . The cards are to be inserted in a row, one by one, in order of their numbers. At step one, card number 1 is set down. The number of inversions created by this step is zero, which is given by  $B_1 X_1$ . At step  $k$ , for  $k \in \{2, \dots, n\}$ , card number  $k$  is randomly inserted in the current row of cards, numbered 1 to  $k-1$ . Thus, for any  $j \in \{0, 1, \dots, k-1\}$ , card number  $k$  has probability  $\frac{1}{k}$  of being placed in the position with  $j$  cards to its right (and  $k-1-j$  cards to its left), in which case this step will have created  $j$  new inversions, and this is represented by  $B_k X_k$ . It is clear from the construction that the random variables  $\{B_k X_k\}_{k=1}^n$  are independent. Thus,  $I_n$  indeed gives the number of inversions in a uniformly random permutation from  $S_n$ . It is well-known that the law of large numbers and the central limit theorem hold for  $I_n$  in this case.

Consider now the general case that  $E_k \subset \{1, \dots, k-1\}$ . Then  $I_n$  gives the number of inversions in a random permutation created by a shuffling procedure in the same spirit as the above one. At step  $k$ , with probability  $1 - \frac{|E_k|}{k}$ , card number  $k$  is inserted at the right end of the row, thereby creating no new inversions, and for each  $j \in E_k$ , with probability  $\frac{1}{k}$  it is inserted in the position with  $j$  cards to its right, thereby creating  $j$  new inversions.

In particular, as a warmup consider the cases  $E_k = \{1\}$  and  $E_k = \{k-1\}$ ,  $2 \leq k \leq n$ . In each of these two cases, at step  $k$ ,  $2 \leq k \leq n$ , card number  $k$  is inserted at the right end of the row with probability  $1 - \frac{1}{k}$ . In the first case, with probability  $\frac{1}{k}$  card number  $k$  is inserted immediately to the left of the right most card, thereby creating one new inversion, while in the second case, with probability  $\frac{1}{k}$  card number  $k$  is inserted at the left end of the row, thereby creating  $k-1$  new inversions. In both cases  $\frac{X_n}{\mu_n} \stackrel{\text{dist}}{=} 1$  for all  $n$ , and in both cases,  $p_k = \frac{1}{k}$ . In the first case,  $\mu_k = 1$  while in the second case,  $\mu_k = k-1$ . Thus, in the first case,  $M_n = \sum_{k=1}^n p_k \mu_k \sim \log n$ , and in the second case,  $M_n \sim n$ . Therefore, it follows from Theorem 1 or 2 that in the first case  $\frac{I_n}{\log n}$  converge in distribution to 1, while in the second case,  $\frac{I_n}{n}$  converges in distribution to GD(1).

The authors of [5] ask which choices of  $\{E_k\}_{k=1}^\infty$  lead to the Dickman distribution and which choices lead to the central limit theorem. Of course, the law of large numbers is a prerequisite for the central limit theorem. The following theorem gives sufficient conditions for the law of large numbers to hold and sufficient conditions for convergence to a distribution from the family  $\text{GD}^{(\mathcal{X})}(\theta)$ . In order to avoid trivialities, we need to assume that (1.8) holds. Recalling that  $\mu_k = 0$  when  $|E_k| = 0$ , and that  $\mu_k \geq 1$  otherwise, note that

$$M_n = EI_n = \sum_{k=1}^{\infty} \frac{|E_k|}{k} \mu_k \geq \sum_{k=1}^{\infty} \frac{|E_k|}{k} = \sum_{k=1}^{\infty} p_k.$$

Thus, in the present context the requirement (1.8) is

$$(2.1) \quad \sum_{k=1}^{\infty} \frac{|E_k|}{k} = \infty,$$

which holds in particular if  $E_k \neq \emptyset$  for all sufficiently large  $k$ .

**Theorem 3.** *Assume that (2.1) holds.*

*i. Assume that at least one of the following conditions holds:*

- a.  $\lim_{k \rightarrow \infty} |E_k| = \infty$  and  $\{\frac{\mu_n}{\sum_{k=1}^n \frac{\mu_k}{k}}\}_{n=1}^{\infty}$  is bounded;*
- b.  $\lim_{n \rightarrow \infty} \frac{\mu_n}{\sum_{k=1}^n \frac{\mu_k}{k}} = 0$ .*

*Then  $\frac{I_n}{EI_n} \xrightarrow{\text{dist}} 1$ .*

*ii. Assume that  $|E_k| = N \geq 1$ , for all large  $k$ , and that  $\frac{X_k}{\mu_k} \xrightarrow{\text{dist}} \mathcal{X}$ . Also assume that  $\mu_k \sim \mu(k)$  and  $\mu_{k+1} - \mu_k \sim \mu'(k)$ , where  $\mu(x) = c_\mu x^{a_0} \prod_{j=1}^{J_\mu} (\log^{(j)} x)^{a_j}$ , with  $a_0 > 0$ .*

*Then  $\frac{I_n}{EI_n} \xrightarrow{\text{dist}} \frac{1}{\theta} D_\theta^{(\mathcal{X})}$ , with  $\theta = \frac{N}{a_0}$ , where  $D_\theta^{(\mathcal{X})}$  is a random variable with the  $GD^{(\mathcal{X})}(\theta)$  distribution.*

**Remark 1.** The condition on  $\{\mu_k\}$  in part (i-a) is just a very weak regularity requirement on its growth rate (recall that  $1 \leq \mu_k < k - 1$ ). The condition in part (i-b) is fulfilled if  $\mu_k \sim \mu(k)$  and  $j_{k+1} - j_k \sim \mu'(k)$ , where  $\mu(x) = c_\mu \prod_{j=1}^{J_\mu} (\log^{(j)} x)^{a_j}$  with  $J_\mu \geq 0$ .

**Remark 2.** Note that the random variable  $\mathcal{X}$  in part (ii) takes on no more than  $N$  distinct values.

*Proof.* Assume first that the condition in part (i-a) holds. We claim that since  $\{\frac{\mu_n}{\sum_{k=1}^n \frac{\mu_k}{k}}\}_{n=1}^{\infty}$  is bounded, there exists a sequence of positive integers  $\{\gamma_n\}_{n=1}^{\infty}$  satisfying  $\lim_{n \rightarrow \infty} \gamma_n = \infty$  and such that  $\{\frac{\mu_n}{\sum_{k=\gamma_n+1}^n \frac{\mu_k}{k}}\}_{n=1}^{\infty}$  is also bounded. Indeed, assume to the contrary. Then, in particular,  $\{\mu_n\}_{n=1}^{\infty}$  is unbounded. Also, since  $\mu_k < k$ , we have  $\sum_{k=1}^n \frac{\mu_k}{k} < \gamma_n + \sum_{k=\gamma_n+1}^n \frac{\mu_k}{k}$ , and it would follow that  $\{\frac{\mu_n}{\gamma_n}\}_{n=1}^{\infty}$  is bounded for all sequences  $\{\gamma_n\}_{n=1}^{\infty}$  satisfying  $\lim_{n \rightarrow \infty} \gamma_n = \infty$ , which is a contradiction.

Let  $\{\gamma_n\}_{n=1}^{\infty}$  be such a sequence. Then

$$M_n = \sum_{k=1}^n \frac{|E_k|}{k} \mu_k \geq (\min_{k > \gamma_n} |E_k|) \sum_{k=\gamma_n}^n \frac{\mu_k}{k}.$$

Thus, the condition in (i-a) guarantees that (1.9) holds.

Now assume that the condition in part (i-b) holds. Since  $M_n \geq \sum_{k=1}^n \frac{\mu_k}{k}$ , it follows again that (1.9) holds.

Thus, assuming either (i-a) or (i-b), it follows from part (i-a) of Theorem 1 that  $\frac{I_n}{EI_n} \xrightarrow{\text{dist}} 1$ .

Now assume that the condition in part (ii) holds. Then  $p_k = \frac{N}{k}$ , for large  $k$ , and  $\mu_k \sim c_\mu k^{a_0} \prod_{j=1}^{J_\mu} (\log^{(j)} k)^{a_j}$ , with  $a_0 > 0$ . Thus,

$$M_n = \sum_{k=1}^n \frac{|E_k|}{k} \mu_k \sim \frac{N c_\mu}{a_0} n^{a_0} \prod_{j=1}^{J_\mu} (\log^{(j)} n)^{a_j},$$

and  $\lim_{k \rightarrow \infty} \frac{\mu_k}{M_k} = \frac{a_0}{N}$ . Also, if the condition in part (ii) holds, then  $\mu_{k+1} - \mu_k \sim a_0 c_\mu k^{a_0-1} \prod_{j=1}^{J_\mu} (\log^{(j)} k)^{a_j}$ . Thus,  $\lim_{k \rightarrow \infty} \frac{p_k \mu_k}{\mu_{k+1} - \mu_k} = \frac{N}{a_0}$ . We conclude from part (ii) of Theorem 1 that  $\frac{I_n}{EI_n} \xrightarrow{\text{dist}} \frac{1}{\theta} \text{GD}^{(\mathcal{X})}(\theta)$ , where  $\theta = \frac{N}{a_0}$ .  $\square$

### 3. PROOF OF THEOREM 1

Since  $EW_n = 1$ , for all  $n$ , the distributions of  $\{W_n\}_{n=1}^\infty$  are tight. Thus, since the random variables are nonnegative, it suffices to show that their Laplace transforms  $E \exp(-\lambda W_n)$  converge under the conditions of part (i) to  $\exp(-\lambda c)$ , for the specified value of  $c$ , and under the conditions of part (ii) to  $\exp(\theta \int_0^1 \frac{E e^{-L\lambda x} - 1}{x} dx)$ , which is the Laplace transform of  $LD^{(\mathcal{X})}(\theta)$ .

*Proof of part (i).* Note that part (i-a) is the particular case of part (i-b) in which one can choose  $K_n = n$ , and then (1.12) holds with  $c = 1$ . Thus, it suffices to consider part (i-b). We have for  $\lambda > 0$ ,

$$\begin{aligned} (3.1) \quad E \exp(-\lambda W_n) &= \\ &= \prod_{k=1}^n E \exp\left(-\frac{\lambda}{M_n} B_k X_k\right) = \prod_{k=1}^n \left(1 - p_k \left(1 - E \exp\left(-\frac{\lambda}{M_n} X_k\right)\right)\right) = \\ &= \prod_{k=1}^{K_n} \left(1 - p_k \left(1 - E \exp\left(-\frac{\lambda}{M_n} X_k\right)\right)\right) \prod_{k=K_n+1}^n \left(1 - p_k \left(1 - E \exp\left(-\frac{\lambda}{M_n} X_k\right)\right)\right). \end{aligned}$$

Since

$$\prod_{k=K_n+1}^n (1 - p_k) \leq \prod_{k=K_n+1}^n \left( 1 - p_k \left( 1 - E \exp\left(-\frac{\lambda}{M_n} X_k\right) \right) \right) \leq 1,$$

it follows from assumption (1.10) that

$$(3.2) \quad \lim_{n \rightarrow \infty} \prod_{k=K_n+1}^n \left( 1 - p_k \left( 1 - E \exp\left(-\frac{\lambda}{M_n} X_k\right) \right) \right) = 1.$$

Applying the mean value theorem to  $E \exp\left(-\frac{\lambda}{M_n} X_k\right)$  as a function of  $\lambda$ , and recalling that  $\mu_k = EX_k$ , we have

$$(3.3) \quad \frac{\lambda}{M_n} EX_k \exp\left(-\frac{\lambda}{M_n} X_k\right) \leq 1 - E \exp\left(-\frac{\lambda}{M_n} X_k\right) \leq \lambda \frac{\mu_k}{M_n}.$$

In light of (1.11) and the assumption that  $\{\frac{X_k}{\mu_k}\}_{k=1}^{\infty}$  is uniformly integrable, it follows that for all  $\epsilon > 0$ , there exists an  $n_\epsilon$  such that

$$(3.4) \quad \begin{aligned} \frac{\lambda}{M_n} EX_k \exp\left(-\frac{\lambda}{M_n} X_k\right) &= \lambda \frac{\mu_k}{M_n} E \frac{X_k}{\mu_k} \exp\left(-\lambda \frac{\mu_k}{M_n} \frac{X_k}{\mu_k}\right) \geq (1 - \epsilon) \lambda \frac{\mu_k}{M_n}, \\ 1 \leq k \leq K_n, \quad n \geq n_\epsilon. \end{aligned}$$

Thus, (3.3) and (3.4) yield

$$(3.5) \quad (1 - \epsilon) \lambda \frac{\mu_k}{M_n} \leq 1 - E \exp\left(-\frac{\lambda}{M_n} X_k\right) \leq \lambda \frac{\mu_k}{M_n}, \quad 1 \leq k \leq K_n, \quad n \geq n_\epsilon.$$

Since for any  $\epsilon > 0$ , there exists an  $x_\epsilon > 0$  such that  $-(1 + \epsilon)x \leq \log(1 - x) \leq -x$ , for  $0 < x < x_\epsilon$ , it follows from (3.5) and (1.11) that there exists an  $n'_\epsilon$  such that

$$(3.6) \quad \begin{aligned} -(1 + \epsilon) \lambda p_k \frac{\mu_k}{M_n} &\leq \log \left( 1 - p_k \left( 1 - E \exp\left(-\frac{\lambda}{M_n} X_k\right) \right) \right) \leq -(1 - \epsilon) \lambda p_k \frac{\mu_k}{M_n}, \\ 1 \leq k \leq K_n, \quad n \geq n'_\epsilon. \end{aligned}$$

From (3.6) we have

$$(3.7) \quad \begin{aligned} -(1 + \epsilon) \lambda \frac{\sum_{k=k_\epsilon}^{K_n} p_k \mu_k}{M_n} &\leq \log \prod_{k=1}^{K_n} \left( 1 - p_k \left( 1 - E \exp\left(-\frac{\lambda}{M_n} X_k\right) \right) \right) \leq \\ &-(1 - \epsilon) \lambda \frac{\sum_{k=k_\epsilon}^{K_n} p_k \mu_k}{M_n}, \quad n \geq n'_\epsilon. \end{aligned}$$

If

$$(3.8) \quad c \equiv \lim_{n \rightarrow \infty} \frac{M_{K_n}}{M_n} = \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^{K_n} p_k \mu_k}{M_n}$$

exists, then from (3.1), (3.2), (3.7) and (3.8), along with the fact that  $\epsilon > 0$  is arbitrary, we conclude that

$$\lim_{n \rightarrow \infty} E \exp(-\lambda W_n) = \exp(-\lambda c),$$

which proves that  $\lim_{n \rightarrow \infty} W_n \stackrel{\text{dist}}{=} c$ . The rest of the results in part (i-b), concerning accumulation points, follow in the same manner.

*Proof of part (ii).* From (3.1), we have

$$(3.9) \quad \log E \exp(-\lambda W_n) = \sum_{k=1}^n \log \left( 1 - p_k \left( 1 - E \exp\left(-\frac{\lambda}{M_n} X_k\right) \right) \right).$$

Since by assumption  $\lim_{k \rightarrow \infty} p_k = 0$ , for any  $\epsilon > 0$  there exists a  $k_\epsilon$  such that

$$(3.10) \quad \begin{aligned} & - (1 + \epsilon) p_k \left( 1 - E \exp\left(-\frac{\lambda}{M_n} X_k\right) \right) \leq \log \left( 1 - p_k \left( 1 - E \exp\left(-\frac{\lambda}{M_n} X_k\right) \right) \right) \leq \\ & - p_k \left( 1 - E \exp\left(-\frac{\lambda}{M_n} X_k\right) \right), \quad k \geq k_\epsilon. \end{aligned}$$

We now show that for any  $\epsilon > 0$  there exists a  $k'_\epsilon$  such that

$$(3.11) \quad (1 - \epsilon) E \exp\left(-\lambda \frac{\mu_k}{M_n} \mathcal{X}\right) \leq E \exp\left(-\frac{\lambda}{M_n} X_k\right) \leq (1 + \epsilon) E \exp\left(-\lambda \frac{\mu_k}{M_n} \mathcal{X}\right), \quad k \geq k'_\epsilon.$$

By assumption (1.14) and the assumption that  $\{\mu_n\}_{n=1}^\infty$  is increasing, there exists a  $C$  such that  $\frac{\mu_k}{M_n} \leq C$ , for  $1 \leq k \leq n$  and  $n \geq 1$ . By assumption,  $\frac{X_k}{\mu_k} \stackrel{\text{dist}}{\rightarrow} \mathcal{X}$ . Without loss of generality, we assume that all of these random variables are defined on the same space and that  $\frac{X_k}{\mu_k} \rightarrow \mathcal{X}$  a.s. For  $\delta > 0$ , let

$$A_{k;\delta} = \left\{ \sup_{l \geq k} \left| \frac{X_l}{\mu_l} - \mathcal{X} \right| \leq \delta \right\}.$$

Then  $A_{k;\delta}$  is increasing in  $k$  and  $\lim_{k \rightarrow \infty} P(A_{k;\delta}) = 1$ . We have

$$(3.12) \quad \int_{A_{k;\delta}^c} \exp\left(-\frac{\lambda}{M_n} X_k\right) dP \leq P(A_{k;\delta}^c),$$



and

$$(3.13) \quad \begin{aligned} & \exp(-\lambda C\delta) \int_{A_{k;\delta}} \exp(-\lambda \frac{\mu_k}{M_n} \mathcal{X}) dP \leq \int_{A_{k;\delta}} \exp(-\lambda \frac{\mu_k}{M_n} \frac{X_k}{\mu_k}) dP \leq \\ & \exp(\lambda C\delta) \int_{A_{k;\delta}} \exp(-\lambda \frac{\mu_k}{M_n} \mathcal{X}) dP. \end{aligned}$$

Now (3.11) follows from (3.12) and (3.13).

$$(3.14) \quad \begin{aligned} & \text{Letting } k''_\epsilon = \max(k_\epsilon, k'_\epsilon), \text{ it follows from (3.10) and (3.11) that} \\ & -(1+\epsilon)p_k \left(1 - (1-\epsilon)E \exp(-\lambda \frac{\mu_k}{M_n} \mathcal{X})\right) \leq \log \left(1 - p_k \left(1 - E \exp(-\lambda \frac{\lambda}{M_n} X_k)\right)\right) \leq \\ & -p_k \left(1 - (1+\epsilon)E \exp(-\lambda \frac{\mu_k}{M_n} \mathcal{X})\right), \quad k \geq k''_\epsilon. \end{aligned}$$

From (3.9) and (3.14) we have

$$(3.15) \quad \begin{aligned} & - \sum_{k=k''_\epsilon}^n p_k (1+\epsilon) \left(1 - (1-\epsilon)E \exp(-\lambda \frac{\mu_k}{M_n} \mathcal{X})\right) + o(1) \leq \log E \exp(-\lambda W_n) \leq \\ & - \sum_{k=k''_\epsilon}^n p_k \left(1 - (1+\epsilon)E \exp(-\lambda \frac{\mu_k}{M_n} \mathcal{X})\right), \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Define  $x_k^{(n)} = \frac{\mu_k}{M_n}$ ,  $k''_\epsilon \leq k \leq n$ , and  $\Delta_k^{(n)} = x_{k+1}^{(n)} - x_k^{(n)} = \frac{\mu_{k+1} - \mu_k}{M_n}$ ,  $k''_\epsilon \leq k \leq n-1$ . Then we have

$$(3.16) \quad \begin{aligned} & \sum_{k=k''_\epsilon}^n p_k \left(1 - (1 \pm \epsilon)E \exp(-\lambda \frac{\mu_k}{M_n} \mathcal{X})\right) = \\ & \sum_{k=k''_\epsilon}^n \frac{1 - (1 \pm \epsilon)E \exp(-\lambda x_k^{(n)} \mathcal{X})}{x_k^{(n)}} \Delta_k^{(n)} \left(p_k \frac{\mu_k}{\mu_{k+1} - \mu_k}\right). \end{aligned}$$

By assumption,  $\{\mu_k\}_{k=1}^\infty$  is increasing; thus  $\{x_k^{(n)}\}_{k=k''_\epsilon}^n$  is a partition of  $[\frac{\mu_{k''_\epsilon}}{M_n}, \frac{\mu_n}{M_n}]$ . By assumption,  $\lim_{n \rightarrow \infty} \frac{\mu_{k''_\epsilon}}{M_n} = 0$  and  $\lim_{n \rightarrow \infty} \frac{\mu_n}{M_n} = L$ . We now show that the mesh,  $\max_{k''_\epsilon \leq k \leq n-1} \Delta_k^{(n)}$ , of the partition converges to 0 as  $n \rightarrow \infty$ . Let  $\Delta_{j_n}^{(n)} = \max_{k''_\epsilon \leq k \leq n-1} \Delta_k^{(n)}$ , where  $k''_\epsilon \leq j_n \leq n$ . Without loss of generality, assume either that  $\{j_n\}$  is bounded or that  $\lim_{n \rightarrow \infty} j_n = \infty$ . In the former case it is clear that  $\max_{k''_\epsilon \leq k \leq n-1} \Delta_k^{(n)} = \Delta_{j_n}^{(n)} = \frac{\mu_{j_n+1} - \mu_{j_n}}{M_n} \xrightarrow{n \rightarrow \infty}$

0. Now consider the latter case. From assumption (1.14) and the assumption that  $\lim_{k \rightarrow \infty} p_k = 0$ , it follows that  $\lim_{n \rightarrow \infty} \frac{\mu_{n+1} - \mu_n}{M_n} = 0$ . Then we have

$$\max_{k'' \leq k \leq n-1} \Delta_k^{(n)} = \Delta_{j_n}^{(n)} = \frac{\mu_{j_n+1} - \mu_{j_n}}{M_n} = \frac{\mu_{j_n+1} - \mu_{j_n}}{M_{j_n}} \frac{M_{j_n}}{M_n} \leq \frac{\mu_{j_n+1} - \mu_{j_n}}{M_{j_n}} \xrightarrow{n \rightarrow \infty} 0.$$

Finally, we note that from (1.14) we have  $\lim_{k \rightarrow \infty} p_k \frac{\mu_k}{\mu_{k+1} - \mu_k} = \theta$ . In light of these facts, along with (3.15), (3.16) and the fact that  $\epsilon > 0$  is arbitrary, it follows that

$$(3.17) \quad \lim_{n \rightarrow \infty} \log E \exp(-\lambda W_n) = \theta \int_0^L \frac{E \exp(-\lambda x \mathcal{X}) - 1}{x} dx = \theta \int_0^1 \frac{E \exp(-\lambda L x \mathcal{X}) - 1}{x} dx,$$

□

#### 4. PROOF OF THEOREM 2

We will assume that  $J_p, J_\mu \geq 1$  so that we can use a uniform notation, leaving it to the reader to verify that the proof also goes through if  $J_p$  or  $J_\mu$  is equal to zero.

First assume that (1.15) holds. Then by the assumptions in the theorem,

$$1 \leq J_p \leq J_\mu;$$

$$\mu_k \sim c_\mu \prod_{j=J_p}^{J_\mu} (\log^{(j)} k)^{a_j}, \quad a_{J_p} > 0;$$

$$p_k \sim c_p \left( x \prod_{j=1}^{J_p} \log^{(j)} k \right)^{-1};$$

$$\mu_{k+1} - \mu_k \sim c_\mu a_{J_p} \frac{(\log^{(J_p)} k)^{a_{J_p}-1}}{x \prod_{j=1}^{J_p-1} \log^{(j)} k} \prod_{j=J_p+1}^{J_\mu} (\log^{(j)} k)^{a_j}.$$

Thus,

$$M_n = \sum_{k=1}^n p_k \mu_k \sim c_\mu c_p \frac{(\log^{(J_p)} n)^{a_{J_p}}}{a_{J_p}} \prod_{j=J_p+1}^{J_\mu} (\log^{(j)} n)^{a_j}.$$

Consequently,

$$(4.1) \quad \lim_{k \rightarrow \infty} \frac{\mu_k}{M_k} = \frac{a_{J_p}}{c_p} \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{p_k \mu_k}{\mu_{k+1} - \mu_k} = \frac{c_p}{a_{J_p}}.$$

Thus, from part (ii) of Theorem 1 it follows that  $\lim_{n \rightarrow \infty} W_n \stackrel{\text{dist}}{=} \frac{1}{\theta} D_\theta$ , where  $\theta = \frac{c_p}{a_{J_p}}$ .

Now assume that (1.15) does not hold. We need to show that  $\{K_n\}_{n=1}^\infty$  can be defined so that (1.10) and (1.11) hold, and so that (1.12) holds with  $c \in \{0, 1\}$ . We also have to show when  $c = 0$  and when  $c = 1$ . Recall the definitions in (1.16). If  $\{0 \leq j \leq J_\mu : a_j \neq 0\}$  is empty, or if it is not empty and  $a_{\kappa_\mu} < 0$ , then  $\{\mu_k\}_{k=1}^\infty$  is bounded. Therefore, (1.10) and (1.11) hold with  $K_n = n$  and it follows from part (i-a) of Theorem 1 that  $\lim_{n \rightarrow \infty} W_n \stackrel{\text{dist}}{=} 1$ . Thus, from now on we assume that  $\{0 \leq j \leq J_\mu : a_j \neq 0\}$  is not empty and that  $a_{\kappa_\mu} > 0$ . In order to use uniform notation, we will assume that  $\kappa_\mu > 0$ , leaving the reader to verify that the proof goes through if  $\kappa_\mu = 0$ . Thus, we have

$$(4.2) \quad \mu_k \sim \prod_{j=\kappa_\mu}^{J_\mu} (\log^{(j)} k)^{a_j}, \quad \kappa_\mu \geq 1, \quad a_{\kappa_\mu} > 0.$$

In order to simplify notation, for the rest of this proof, we will let  $\mathcal{L}_l(k)$  denote a positive constant multiplied by a product of powers (possibly of varying sign) of iterated logarithms  $\log^{(j)} k$ , where the smallest  $j$  is strictly larger than  $l$ . The exact form of this expression may vary from line to line. Sometimes we will need to distinguish between two such expressions in the same formula, in which case we will use the notation  $\mathcal{L}_l^{(1)}(k), \mathcal{L}_l^{(2)}(k)$ . Thus, we rewrite (4.2) as

$$(4.3) \quad \mu_k \sim (\log^{(\kappa_\mu)} k)^{a_{\kappa_\mu}} \mathcal{L}_{\kappa_\mu}(k), \quad \kappa_\mu \geq 1, \quad a_{\kappa_\mu} > 0.$$

If  $\{0 \leq j \leq J_p : b_j \neq 1\}$  is empty, then the second condition in (1.15) is fulfilled and we have

$$(4.4) \quad p_k \sim c_p \left( x \prod_{j=1}^{J_p} \log^{(j)} k \right)^{-1}.$$

Since we are assuming that (1.15) does not hold, at least one of the other two conditions in (1.15) must fail. This forces  $\kappa_\mu \neq J_p$ . (Recall that we are assuming that  $\{0 \leq j \leq J_\mu : a_j \neq 0\}$  is not empty and that  $a_{\kappa_\mu} > 0$ .)

Consider first the case that  $\kappa_\mu > J_p$ . Then from (4.3) and (4.4) we have

$$(4.5) \quad M_n = \sum_{k=1}^n p_k \mu_k \sim (\log^{(J_p+1)} n) (\log^{(\kappa_\mu)} n)^{a_{\kappa_\mu}} \mathcal{L}_{\kappa_\mu}(n), \text{ where } \kappa_\mu \geq J_p + 1.$$

From (4.3) and (4.5) it follows that (1.10) and (1.11) hold by choosing  $K_n = n$ . Thus, from part (i-a) of Theorem 1,  $\lim_{n \rightarrow \infty} W_n \stackrel{\text{dist}}{=} 1$ .

Now consider the case  $\kappa_\mu < J_p$ . Then from (4.3) and (4.4) we have

$$(4.6) \quad M_n = \sum_{k=1}^n p_k \mu_k \sim (\log^{(\kappa_\mu)} n)^{a_{\kappa_\mu}} \mathcal{L}_{\kappa_\mu}(n), \text{ where } \kappa_\mu \leq J_p - 1,$$

and for any  $K_n$  satisfying  $K_n \rightarrow \infty$  and  $K_n \leq n$ , we have

$$(4.7) \quad \sum_{k=K_n}^n p_k \sim c_p (\log^{(J_p+1)} n - \log^{(J_p+1)} K_n) = c_p \log \frac{\log^{(J_p)} n}{\log^{(J_p)} K_n}.$$

From (4.3) and (4.6) we have

$$(4.8) \quad \begin{aligned} \frac{\mu_{K_n}}{M_n} &\sim \left( \frac{\log^{(\kappa_\mu)} K_n}{\log^{(\kappa_\mu)} n} \right)^{a_{\kappa_\mu}} \frac{\mathcal{L}_{\kappa_\mu}^{(1)}(K_n)}{\mathcal{L}_{\kappa_\mu}^{(2)}(n)}, \quad \kappa_\mu \leq J_p - 1, \quad a_{\kappa_\mu} > 0; \\ \frac{M_{K_n}}{M_n} &\sim \left( \frac{\log^{(\kappa_\mu)} K_n}{\log^{(\kappa_\mu)} n} \right)^{a_{\kappa_\mu}} \frac{\mathcal{L}_{\kappa_\mu}^{(1)}(K_n)}{\mathcal{L}_{\kappa_\mu}^{(2)}(n)}, \quad \kappa_\mu \leq J_p - 1, \quad a_{\kappa_\mu} > 0; \end{aligned}$$

As we explain in some detail below, since  $\kappa_\mu < J_p$ , we can choose  $\{K_n\}_{n=1}^\infty$  so that

$$(4.9) \quad \lim_{n \rightarrow \infty} \frac{\log^{(J_p)} K_n}{\log^{(J_p)} n} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \left( \frac{\log^{(\kappa_\mu)} K_n}{\log^{(\kappa_\mu)} n} \right)^{a_{\kappa_\mu}} \frac{\mathcal{L}_{\kappa_\mu}^{(1)}(K_n)}{\mathcal{L}_{\kappa_\mu}^{(2)}(n)} = 0.$$

From (4.3) and (4.7)-(4.9), we conclude that  $\{K_n\}$  can be defined so that (1.10) and (1.11) hold, and so that (1.12) holds with  $c = 0$ . This proves that  $\lim_{n \rightarrow \infty} W_n \stackrel{\text{dist}}{=} 0$ .

To explain (4.9), note that  $\frac{\mathcal{L}_{\kappa_\mu}^{(1)}(K_n)}{\mathcal{L}_{\kappa_\mu}^{(2)}(n)} \leq (\log^{(\kappa_\mu+1)} n)^A$ , for some  $A > 0$  and all large  $n$ . (Recall that the powers of the iterated logarithms in  $\mathcal{L}_{\kappa_\mu}^{(2)}$  can be negative.) Thus, in place of the second limit in (4.9), it suffices to show that  $\delta_n \equiv \left( \frac{\log^{(\kappa_\mu)} K_n}{\log^{(\kappa_\mu)} n} \right)^{a_{\kappa_\mu}} (\log^{(\kappa_\mu+1)} n)^A \xrightarrow{n \rightarrow \infty} 0$ . We have

$$\log^{(\kappa_\mu)} K_n = (\delta_n)^{\frac{1}{a_{\kappa_\mu}}} (\log^{(\kappa_\mu+1)} n)^{-\frac{A}{a_{\kappa_\mu}}} \log^{(\kappa_\mu)} n;$$

thus,

$$(4.10) \quad \frac{\log^{(\kappa_\mu+1)} K_n}{\log^{(\kappa_\mu+1)} n} = \frac{\log \delta_n}{a_{\kappa_\mu} \log^{(\kappa_\mu+1)} n} - \frac{A \log^{(\kappa_\mu+2)} n}{a_{\kappa_\mu} \log^{(\kappa_\mu+1)} n} + 1.$$

Defining  $K_n$  by choosing  $\delta_n = (\log^{(\kappa_\mu+1)} n)^{-1}$ , it follows from (4.10) and the fact that  $J_p \geq \kappa_\mu + 1$  that the two equalities in (4.9) hold.

We now consider the case that  $\{0 \leq j \leq J_p : b_j \neq 1\}$  is not empty. Then in order to fulfill the second condition in (1.8), we have  $b_{\kappa_p} < 1$ . We write

$$(4.11) \quad p_k \sim c_p \left( x \prod_{j=1}^{\kappa_p-1} \log^{(j)} k \right)^{-1} (\log^{(\kappa_p)} k)^{-b_{\kappa_p}} \left( \prod_{j=\kappa_p+1}^{J_p} \log^{(j)} k \right)^{-b_j}.$$

From (4.3) and (4.11) it follows that  $M_n = \sum_{k=1}^n p_k \mu_k$  satisfies

$$(4.12) \quad M_n \sim \begin{cases} (\log^{(\kappa_\mu)} n)^{a_{\kappa_\mu}} \mathcal{L}_{\kappa_\mu}(n), & \kappa_\mu < \kappa_p; \\ (\log^{(\kappa_p)} n)^{a_{\kappa_p} - b_{\kappa_p} + 1} \mathcal{L}_{\kappa_p}(n), & \kappa_\mu = \kappa_p; \\ (\log^{(\kappa_p)} n)^{1 - b_{\kappa_p}} \mathcal{L}_{\kappa_p}(n), & \kappa_\mu > \kappa_p, \end{cases}$$

and from (4.11) it follows that for any  $K_n$  satisfying  $K_n \rightarrow \infty$  and  $K_n \leq n$ ,

$$(4.13) \quad \sum_{k=K_n}^n p_k \sim \frac{c_p}{1 - b_{\kappa_p}} \left[ (\log^{(\kappa_p)} n)^{1 - b_{\kappa_p}} \left( \prod_{j=\kappa_p+1}^{J_p} \log^{(j)} n \right)^{-b_j} - (\log^{(\kappa_p)} K_n)^{1 - b_{\kappa_p}} \left( \prod_{j=\kappa_p+1}^{J_p} \log^{(j)} K_n \right)^{-b_j} \right].$$

From (4.3) and (4.12) we have

$$(4.14) \quad \frac{\mu K_n}{M_n} \sim \begin{cases} \left( \frac{\log^{(\kappa_\mu)} K_n}{\log^{(\kappa_\mu)} n} \right)^{a_{\kappa_\mu}} \frac{\mathcal{L}_{\kappa_\mu}^{(1)}(K_n)}{\mathcal{L}_{\kappa_\mu}^{(2)}(n)}, & \kappa_\mu < \kappa_p; \\ \frac{(\log^{(\kappa_p)} K_n)^{a_{\kappa_p}}}{(\log^{(\kappa_p)} n)^{a_{\kappa_p} - b_{\kappa_p} + 1}} \frac{\mathcal{L}_{\kappa_p}^{(1)}(K_n)}{\mathcal{L}_{\kappa_p}^{(2)}(n)}, & \kappa_\mu = \kappa_p; \\ \left( \frac{\log^{(\kappa_\mu)} K_n}{\log^{(\kappa_p)} n} \right)^{a_{\kappa_\mu}} \frac{\mathcal{L}_{\kappa_\mu}^{(1)}(K_n)}{\mathcal{L}_{\kappa_p}^{(2)}(n)}, & \kappa_\mu > \kappa_p. \end{cases}$$

It is immediate (4.3) and (4.14) that if  $\kappa_\mu \geq \kappa_p$ , then (1.10) and (1.11) hold by choosing  $K_n = n$ . (For the case  $\kappa_\mu = \kappa_p$ , recall that  $b_{\kappa_p} \in (0, 1)$ .) Thus, from part (i-a) of Theorem 1,  $\lim_{n \rightarrow \infty} W_n \stackrel{\text{dist}}{=} 1$ .

Now consider the case  $\kappa_\mu < \kappa_p$ . For simplicity, we will assume that the higher order iterated logarithmic terms do not appear; that is, we will assume from (4.12)-(4.14) that

$$(4.15) \quad \begin{aligned} \sum_{k=K_n}^n p_k &\sim \frac{c_p}{1-b_{\kappa_p}} \left[ (\log^{(\kappa_p)} n)^{1-b_{\kappa_p}} - (\log^{(\kappa_p)} K_n)^{1-b_{\kappa_p}} \right]; \\ \frac{\mu_{K_n}}{M_n} &\sim \left( \frac{\log^{(\kappa_\mu)} K_n}{\log^{(\kappa_\mu)} n} \right)^{a_{\kappa_\mu}}; \\ \frac{M_{K_n}}{M_n} &\sim \left( \frac{\log^{(\kappa_\mu)} K_n}{\log^{(\kappa_\mu)} n} \right)^{a_{\kappa_\mu}}. \end{aligned}$$

The additional logarithmic terms can be dealt with similarly to the way they were dealt with for (4.9), as explained in the paragraph following (4.9). Applying the mean value theorem to the function  $x^{1-b_{\kappa_p}}$ , we obtain

$$(4.16) \quad (\log^{(\kappa_p)} n)^{1-b_{\kappa_p}} - (\log^{(\kappa_p)} K_n)^{1-b_{\kappa_p}} = \frac{(1-b_{\kappa_p}) \log^{(\kappa_p)} \frac{n}{K_n}}{(\log^{(\kappa_p)} n^*)^{b_{\kappa_p}}},$$

where  $n^* \in (K_n, n)$ . Since  $\kappa_\mu < \kappa_p$ , we can choose  $K_n \rightarrow \infty$  such that  $\lim_{n \rightarrow \infty} \frac{\log^{(\kappa_\mu)} K_n}{\log^{(\kappa_\mu)} n} = 0$ , but  $\lim_{n \rightarrow \infty} \log^{(\kappa_p)} \frac{K_n}{n} = 1$ . For such a choice of  $\{K_n\}$ , it follows from (4.3), (4.15) and (4.16) that (1.10) and (1.11) hold, and that (1.12) holds with  $c = 0$ ; thus,  $\lim_{n \rightarrow \infty} W_n \stackrel{\text{dist}}{=} 0$ .  $\square$

## 5. BASIC FACTS CONCERNING GENERALIZED DICKMAN DISTRIBUTIONS

We proved in Theorem 1 that  $\exp(\theta \int_0^1 \frac{e^{-\lambda x} - 1}{x} dx)$  is in fact the Laplace transform of a probability distribution, which we have denoted by  $GD(\theta)$ . In particular, if we let  $X_k = \mu_k = k$  and  $p_k = \frac{\theta}{k}$ , in which case  $M_n = \sum_{k=1}^n p_k \mu_k = \theta n$ , then it follows from Theorem 1 that

$$(5.1) \quad \hat{W}_n \equiv \theta W_n = \frac{1}{n} \sum_{k=1}^n k B_k \stackrel{\text{dist}}{\rightarrow} D_\theta,$$

where  $D_\theta \stackrel{\text{dist}}{\sim} GD(\theta)$ .

We now prove (1.2). Let

$$J_n^+ = \max\{k \leq n : B_k \neq 0\},$$

with  $\max \emptyset \equiv 0$ . We write

$$(5.2) \quad \hat{W}_n \equiv \frac{1}{n} \sum_{k=1}^n kB_k = \frac{J_n^+ - 1}{n} \left( \frac{1}{J_n^+ - 1} \sum_{k=1}^{J_n^+ - 1} kB_k \right) + \frac{J_n^+}{n},$$

where the first of the two summands on the right hand side above is interpreted as equal to 0 if  $J_n^+ \leq 1$ . We have

$$(5.3) \quad P\left(\frac{J_n^+}{n} \leq x\right) = \prod_{k=[xn+1]}^n \left(1 - \frac{\theta}{k}\right) \sim x^\theta, \quad x \in (0, 1).$$

Also, by the independence of  $\{B_k\}_{k=1}^\infty$ , we have

$$(5.4) \quad \frac{1}{J_n^+ - 1} \sum_{k=1}^{J_n^+ - 1} kB_k \mid \{J_n^+ = k_0\} \stackrel{\text{dist}}{=} \frac{1}{k_0 - 1} \sum_{k=1}^{k_0 - 1} kB_k = \hat{W}_{k_0 - 1}, \quad k_0 \geq 2.$$

Letting  $n \rightarrow \infty$  in (5.2) and using (5.1), (5.3) and (5.4), we conclude that (1.2) holds, where  $U$  is a distributed according to the uniform distribution on  $[0, 1]$ ,  $D_\theta \stackrel{\text{dist}}{\sim} \text{GD}(\theta)$  and  $U$  and  $D_\theta$  on the right hand side are independent.

We now use (1.2) to show that the  $\text{GD}(\theta)$  distribution has a density function  $p_\theta$  satisfying  $p_\theta = c_\theta \rho_\theta$ , for some  $c_\theta > 0$ , where  $\rho_\theta$  satisfies (1.1). Let  $F_\theta(x) = P(D_\theta \leq x)$  denote the distribution function for the  $\text{GD}(\theta)$  distribution. Then from (1.2) we have

$$(5.5) \quad \begin{aligned} F_\theta(x) &= P(D_\theta \leq x) = P(U^{\frac{1}{\theta}}(D_\theta + 1) \leq x) = \int_0^1 P(D_\theta + 1 \leq xy^{-\frac{1}{\theta}}) dy = \\ &= \int_0^1 F_\theta(xy^{-\frac{1}{\theta}} - 1) dy. \end{aligned}$$

For  $x > 0$ , making the change of variables,  $v = xy^{-\frac{1}{\theta}} - 1$ , we can rewrite (5.5) as

$$(5.6) \quad F_\theta(x) = \theta x^\theta \int_{x-1}^\infty F_{D_\theta}(v)(1+v)^{-1-\theta} dv, \quad x > 0.$$

From (5.6) and the fact that  $F_\theta(x) = 0$ , for  $x \leq 0$ , it follows that  $F_\theta$  is continuous on  $\mathbb{R}$ . Also, since  $F_\theta(x) = 0$ , for  $x \leq 0$ , we have

$$\int_{x-1}^\infty F_{D_\theta}(v)(1+v)^{-1-\theta} dv = \int_0^\infty F_{D_\theta}(v)(1+v)^{-1-\theta} dv, \quad x \leq 1.$$

Consequently, it follows from (5.6) that  $F_\theta(x) = C_\theta x^\theta$ , for  $x \in [0, 1]$ , where  $C_\theta = \theta \int_0^\infty F_{D_\theta}(v)(1+v)^{-1-\theta} dv$ . From this and (5.6) it follows that  $F$  is differentiable on  $(0, 1)$  and on  $(1, \infty)$ , and that, letting  $p_\theta = F'_\theta$ ,

$$(5.7) \quad p_\theta = c_\theta x^{\theta-1}, \quad 0 < x < 1, \quad c_\theta = \theta^2 \int_0^\infty F_{D_\theta}(v)(1+v)^{-1-\theta} dv,$$

and

$$(5.8) \quad p_\theta(x) = \theta^2 x^{\theta-1} \int_{x-1}^\infty F_{D_\theta}(v)(1+v)^{-1-\theta} dv - \theta x^{-1} F_\theta(x-1) = \frac{\theta}{x} (F_\theta(x) - F_\theta(x-1)), \quad x > 1.$$

From (5.8), it follows that  $p_\theta$  is differentiable on  $x > 1$ , and that  $(xp_\theta(x))' = \theta(p_\theta(x) - p_\theta(x-1))$ , for  $x > 1$ , or equivalently,

$$(5.9) \quad xp'_\theta(x) + (1-\theta)p_\theta(x) + \theta p_\theta(x-1) = 0, \quad x > 1.$$

From (5.7) and (5.9) we conclude that  $p_\theta(x) = c_\theta \rho_\theta$ , where  $\rho_\theta$  satisfies (1.1).

Integrating by parts in the formula for  $c_\theta$  in (5.7) shows that

$$c_\theta = \theta \int_0^\infty (1+v)^{-\theta} p_\theta(v) dv = \theta E(1 + D_\theta)^{-\theta}.$$

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