TRANSIENCE/RECURRENCE AND THE SPEED OF A ONE-DIMENSIONAL RANDOM WALK IN A "HAVE YOUR COOKIE AND EAT IT" ENVIRONMENT

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Abstract. Consider a variant of the simple random walk on the integers, with the following transition mechanism. At each site x, the probability of jumping to the right is $\omega(x) \in [\frac{1}{2}, 1)$, until the first time the process jumps to the left from site x, from which time onward the probability of jumping to the right is $\frac{1}{2}$. We investigate the transience/recurrence properties of this process in both deterministic and stationary, ergodic environments $\{\omega(x)\}_{x\in Z}$. In deterministic environments, we also study the speed of the process.

Résumé. Considerons une variante de la marche aléatoire simple et symétrique sur les entiers, avec le mécanisme de transition suivant: A chaque site x, la probabilité de sauter á droite est $\omega(x) \in [\frac{1}{2}, 1)$, jusqu'a la première fois que le processus saute á gauche du site x, aprés lequel la probabilité de sauter á droite est $\frac{1}{2}$. Nous examinons les propriétés de transience/recurrence pour ce processus, dans les environnements déterministes et aussi dans les environnements stationnaires et ergodiques $\{\omega(x)\}_{x\in Z}$. Dans les environnements deterministes, nous étudions aussi la vitesse du processus.

1. Introduction and Statement of Results

A simple random walk on the integers in a cookie environment $\{\omega(x,i)\}_{x\in Z, i\in\mathcal{N}}$ is a stochastic process $\{X_n\}_{n=0}^{\infty}$ defined on the integers with the following transition mechanism:

$$P(X_{n+1} = X_n + 1|\mathcal{F}_n) = 1 - P(X_{n+1} = X_n - 1|\mathcal{F}_n) = \omega(X_n, V(X_n; n)),$$

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where $\mathcal{F}_n = \sigma(X_0, \cdots, X_n)$ is the filtration of the process up to time n and $V(x;n) = \sum_{k=0}^{n} \chi_x(X_k)$. A graphic description can be given as follows. Place an infinite sequence of cookies at each site x. The i-th time the process reaches the site x, the process eats the i-th cookie at site x, thereby empowering it to jump to the right with probability $\omega(x,i)$ and to the left with probability $1-\omega(x,i)$. If $\omega(x,i) = \frac{1}{2}$, the corresponding cookie is a placebo; that is, its effect on the direction of the random walk is neutral. Thus, when for each $x \in Z$ one has $\omega(x,i) = \frac{1}{2}$, for all sufficiently large i, one can consider the process as a simple, symmetric random walk with a finite number of symmetry-breaking cookies per site.

Letting P_x denote probabilities for the process conditioned on $X_0 = x$, and letting Define the process to be transient or recurrent respectively according to whether $P_y(X_n = x \text{ i.o.})$ is equal to 0 or 1 for all x and all y. Of course from this definition, it is not a priori clear that the process is either transient or recurrent. When there is exactly one cookie per site and $\omega(x,1) = p \in (\frac{1}{2}, 1)$, for all $x \in Z$, the random walk has been called an excited random walk. It is not hard to show that this random walk is recurrent [3], [2]. Zerner showed [6] that when multiple cookies are allowed per site, then it is possible to have transience. For a stationary, ergodic cookie environment $\{\omega(x,i)\}_{x \in Z, i \in \mathcal{N}}$, where the cookie values are in $[\frac{1}{2}, 1)$, he gave necessary and sufficient conditions for transience or recurrence. In particular, in the deterministic case where there are $k \ge 2$ cookies per site and $\omega(x,i) = p$, for $i = 1, \cdots, k$ and all $x \in Z$, a phase transition with regard to transience/recurrence occurs at $p = \frac{1}{2} + \frac{1}{2k}$: the process is recurrent if $p \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{2k}]$ and transient if $p > \frac{1}{2} + \frac{1}{2k}$. Zerner also showed in the above ergodic setting that if there is no excitement after the second visit—that is, with probability one, $\omega(0,i) = \frac{1}{2}$, for $i \ge 3$—then the random walk is non-ballistic; that is, it has 0 speed: $\lim_{n\to\infty} \frac{X_n}{n} = 0$ a.s.. He left as an open problem the question of whether it is possible to have positive speed when there are a bounded number of cookies per site. Recently, Basdevant and

Singh [1] have given an explicit criterion for determining whether the speed is positive or zero in the case of a deterministic, spatially homogeneous environment with a finite number of cookies. In particular, if there are $k \geq 3$ cookies per site and $\omega(x,i) = p$, for $i = 1, \cdots, k$ and all $x \in Z$, then a phase transition with regard to ballisticity occurs at $p = \frac{1}{2} + \frac{1}{k}$: one has $\lim_{n\to\infty} \frac{X_n}{n} = 0$ a.s. if $p \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{k}]$ and $\lim_{n\to\infty} \frac{X_n}{n} > 0$ a.s. if $p > \frac{1}{2} + \frac{1}{k}$. In particular, this shows that it is possible to have positive speed with three cookies per site. Kosygina and Zerner [5] have considered cookie environments that can take values in $(0, 1)$, so that the cookie bias can be to the right at certain sites and to the left at other sites.

In this paper we consider one-sided cookie bias in what we dub a "have your cookie and eat it environment." There is one cookie at each site x and its value is $\omega(x) \in [\frac{1}{2}, 1)$. If the random walk moves to the right from site x, then the cookie at site x remains in place unconsumed, so that the next time the random walk reaches site x, the cookie environment is still in effect. The cookie at site x only gets consumed when the random walk moves leftward from x. After this occurs, the transition mechanism at site x becomes that of the simple, symmetric random walk. We study the transience/recurrence and ballisticity of such processes. We have several motivations for studying this process. One of the motivations for studying the cookie random walks described above is to understand processes with self-interactions. The random walk in the "have your cookie and eat it" environment adds an extra level of self-interaction—in previous works, the number of times cookies are employed at a given site is an external parameter, whereas here it depends on the behavior of the process. Another novelty is that we are able to calculate certain quantities explicitly, quantities that have not been calculated explicitly in previous models. For certain sub-ranges of the appropriate parameter, we calculate explicitly the speed in the ballistic case and the probability of ever reaching 0 from 1 in the transient case. Finally, although we use some of the ideas and methods of [6] and [1] at the beginning of our proofs, we introduce some new techniques, which may well be useful in other cookie problems also.

When considering a deterministic environment ω, we will denote probabilities and expectations for the process starting at x by P_x and E_x respectively, suppressing the dependence on the environment, except in Proposition 1 below, where we need to compare probabilities for two different environments. When considering a stationary, ergodic environment $\{\omega(x)\}_{x\in Z}$, we will denote the probability measure on the environment by $\mathcal{P}^S$ and the corresponding expectation by $\mathcal{E}^S$. Probabilities for the process in a fixed environment ω starting from x will be denoted by P_x^ω (the *quenched* measure).

Before stating the results, we present a lemma concerning transience and recurrence. Let

$$T_x = \inf\{n \ge 0 : X_n = x\}. \tag{1.1}$$

Lemma 1. *Let $\{X_n\}_{n=0}^\infty$ be a random walk in a "have your cookie and eat it" environment $\{\omega(x)\}_{x\in Z}$, where $\omega(x) \in [\frac{1}{2}, 1)$, for all $x \in Z$.*
1. Assume that the environment ω is deterministic and periodic: for some $N \ge 1$, $\omega(x+N) = \omega(x)$, for all $x \in Z$.

i. If $P_1(T_0 = \infty) = 0$, then the process is recurrent.

ii. If $P_1(T_0 = \infty) > 0$, then the process is transient and $\lim_{n\to\infty} X_n = \infty$ a.s..

2. Assume that the environment ω is stationary and ergodic. Then either $P_1^\omega(T_0 = \infty) = 0$ for $\mathcal{P}^S$-almost every ω or $P_1^\omega(T_0 = \infty) > 0$ for $\mathcal{P}^S$-almost every ω.

i. If $P_1^\omega(T_0 = \infty) = 0$ for $\mathcal{P}^S$-almost every ω, then for $\mathcal{P}^S$-almost every ω the process is recurrent.

ii. If $P_1^\omega(T_0 = \infty) > 0$ for $\mathcal{P}^S$-almost every ω, then for $\mathcal{P}^S$-almost every ω the process is transient and $\lim_{n\to\infty} X_n = \infty$ a.s..

The following theorem concerns transience/recurrence in deterministic environments.

Theorem 1. *Let $\{X_n\}_{n=0}^{\infty}$ be a random walk in a deterministic "have your cookie and eat it" environment $\{\omega(x)\}_{x\in Z}$, where $\omega(x)\in[\frac{1}{2},1)$ for all $x\in Z$.*

i. *Let $\omega(x)=p\in(\frac{1}{2},1)$, for all $x\in Z$. Then*

$$\begin{cases} P_1(T_0=\infty)=0, & \text{if } p\le\frac{2}{3};\\ \frac{3p-2}{p}\le P_1(T_0=\infty)\le\frac{1}{p}\frac{3p-2}{2p-1}, & \text{if } p\in(\frac{2}{3},1).\end{cases}$$

In particular the process is recurrent if $p\le\frac{2}{3}$ and transient if $p>\frac{2}{3}$.

ii. *Let ω be periodic with period $N>1$. Then the process is recurrent if $\frac{1}{N}\sum_{x=1}^{N}\frac{\omega(x)}{1-\omega(x)}\le 2$ and transient if $\frac{1}{N}\sum_{x=1}^{N}\frac{\omega(x)}{1-\omega(x)}>2$.*

Remark 1. We note the following heuristic connection between part (i) of the theorem and Zerner's result that was described above. Zerner proved that when there are k cookies per site, all with strength $p\in(\frac{1}{2},1)$, then the process is recurrent if $p\le\frac{1}{2}+\frac{1}{2k}$ and transient if $p>\frac{1}{2}+\frac{1}{2k}$. For the process in a "have your cookie and eat it" environment, if it is recurrent so that it continually returns to every site, then for each site, the expected number of times it will jump from that site according to the probabilities p and $1-p$ is $\sum_{m=1}^{\infty}mp^{m-1}(1-p)=\frac{1}{1-p}$. Therefore, on the average, the process in a "have your cookie and eat it" environment with parameter p behaves like a k cookies per site process with $k=\frac{1}{1-p}$ (which of course is usually not an integer). If one substitutes this value of k in the equation $p=\frac{1}{2}+\frac{1}{2k}$ and solves for p, one obtains $p=\frac{2}{3}$.

Remark 2. In the case of a periodic environment, it follows from Jensen's inequality that if the average cookie strength $\frac{1}{N}\sum_{x=1}^{N}\omega(x)>\frac{2}{3}$, then $\frac{1}{N}\sum_{x=1}^{N}\frac{\omega(x)}{1-\omega(x)}>2$, and thus the process is transient. Indeed, it is possible to have transience with the average cookie strength arbitrarily close to $\frac{1}{2}$: let $\omega(x)=\frac{1}{2}$, for $x=1,\cdots,N-1$, and $\omega(N)>\frac{N+1}{N+2}$.

Remark 3. Part (i) of Theorem 1 gives two-sided bounds on $P_1(T_0=\infty)$. In Proposition 2 in section 4, we will prove that $P_1(T_0=\infty)\ge\frac{3p-2}{2p-1}$, for $p>\frac{2}{3}$, with equality for $p\ge\frac{3}{4}$. We believe that equality holds for all $p>\frac{2}{3}$.

The next theorem treats transience/recurrence in stationary, ergodic environments $\{\omega(x)\}_{x\in Z}$.

Theorem 2. *Let $\{X_n\}_{n=0}^{\infty}$ be a random walk in a stationary, ergodic "have your cookie and eat it" environment $\{\omega(x)\}_{x\in Z}$. Then for $\mathcal{P}^S$-almost every environment ω, the process is P_x^ω-recurrent if $\mathcal{E}^S\frac{\omega(0)}{1-\omega(0)}\le 2$ and P_x^ω-transient if $\mathcal{E}^S\frac{\omega(0)}{1-\omega(0)}>2$.*

We now turn to the speed of the random walk.

Theorem 3. *Let $\{X_n\}_{n=0}^{\infty}$ be a random walk in a deterministic "have your cookie and eat it" environment with $\omega(x)=p$, for all x.*
If $p<\frac{3}{4}$, then the process is non-ballistic; one has $\lim_{n\to\infty}\frac{X_n}{n}=0$ a.s..
If $p>\frac{3}{4}$, then the process is ballistic; one has $\lim_{n\to\infty}\frac{X_n}{n}>0$ a.s..
In fact,

$$\lim_{n\to\infty}\frac{X_n}{n}\ge 4p-3, \text{ with equality if } \frac{10}{11}\le p<1. \tag{1.2}$$

Remark 1. We believe that equality holds in (1.2) for all $p\in(\frac{3}{4},1)$. In particular, this would prove non-ballisticity in the borderline case $p=\frac{3}{4}$.

Remark 2. We note the following heuristic connection between Theorem 3 and the result of Basdevant and Singh that was described above. That result states that if there are k cookies per site, all with strength $p\in(\frac{1}{2},1)$, then the process has 0 speed if $p\le\frac{1}{2}+\frac{1}{k}$ and positive speed if $p>\frac{1}{2}+\frac{1}{k}$. As noted in the remark following Theorem 1, on the average the process in a "have your cookie and eat it" environment behaves like a k cookie per site process with $k=\frac{1}{1-p}$. If one substitutes this value of k in the equation $p=\frac{1}{2}+\frac{1}{k}$ and solves for p, one obtains $p=\frac{3}{4}$.

In section 2 we prove a monotonicity result with respect to environments and the use it to prove Lemma 1. The proofs of Theorems 1 and 2 are given in section 3, and the proof of Theorem 3 is given in section 4.

2. A monotonicity Result and the Proof of Lemma 1

A standard coupling [6, Lemma 1] shows that $\{X_n\}_{n=0}^{\infty}$ under P_y^{ω} stochastically dominates the simple, symmetric random walk starting from y. However, standard couplings don't work for comparing random walks in two different "have your cookie and eat it" environments (or multiple cookie environments)—see [6]. Whereas standard couplings are "space-time" couplings, we will construct a "space only" coupling, which allows the time parameter of the two processes to differ.

Proposition 1. *Let ω_1 and ω_2 be environments with $\omega_1(z) \leq \omega_2(z)$, for all $z \in Z$. Denote probabilities for the random walk in the "have your cookie and eat it" environment ω_i by $P_{\cdot}^{\omega_i}$, $i = 1, 2$. Then*

$$P_y^{\omega_1}(T_z \leq T_x \wedge N) \leq P_y^{\omega_2}(T_z \leq T_x \wedge N), \text{ for } x < y < z \text{ and } N > 0. \tag{2.1}$$

In particular then, $P_y^{\omega_1}(T_z < T_x) \leq P_y^{\omega_2}(T_z < T_x)$ and $P_y^{\omega_1}(T_x = \infty) \leq P_y^{\omega_2}(T_x = \infty)$.

Remark. The same result was proven in [6] for one-side multiple cookies, where one assumes that the two environments satisfy $\omega_1(x, i) \leq \omega_2(x, i)$, for all $x \in Z$ and all $i \geq 1$. The proof does not use coupling. Our proof extends to that setting if one assumes that the environments satisfy $\omega_1(x, j) \leq \omega_2(x, i)$, for all $x \in Z$ and all $j \geq i \geq 1$.

Proof. On a probability space $(\Omega, \mathcal{P}, \mathcal{G})$, define a sequence $\{U_k\}_{k=1}^{\infty}$ of IID random variables uniformly distributed on $[0, 1]$. Let $\mathcal{G}_k = \sigma(U_j : j = 1, \cdots, k)$. We now couple two processes, $\{X_n^1\}$ and $\{X_n^2\}$, on $(\Omega, \mathcal{P}, \mathcal{G})$, insuring that $\{X_n^i\}$ has the distribution of the $P_y^{\omega_i}$ process, $i = 1, 2$. The coupling will be for all times n, without reference to T_x or T_z or N in the statement of the proposition. We begin with $X_0^1 = X_0^2 = y$. For $i = 1, 2$, we let $X_1^i = y + 1$ or $X_1^i = y - 1$ respectively depending on whether $U_1 \leq \omega_i(y)$ or $U_1 > \omega_i(y)$. If $U_1 \leq \omega_1(y)$ or $U_1 > \omega_2(y)$, then both processes moved together. In this case, use U_2 to continue the coupling another step. Continue like this until the first time $n_1 \ge 1$ that the processes differ. (Assume $n_1 < \infty$, since otherwise there is nothing to prove. We will continue to make this kind of assumption below without further remark.) Necessarily, $X^1_{n_1} = X^2_{n_1} - 2$. Up until this point the coupling has used the random variables $\{U_k\}_{k=1}^{n_1}$. Now leave the process $\{X^2_n\}$ alone and continue moving the process $\{X^1_n\}$, starting with the random variable U_{n_1+1}. By comparison with the simple, symmetric random walk, the $\{X^1_n\}$ process will eventually return to the level $X^2_{n_1}$ of the temporarily frozen $\{X^2_n\}$ process. Denote this time by $m_1 > n_1$. So $X^1_{m_1} = X^2_{n_1}$. Now start moving both of the processes together again, starting with the first unused random variable; namely, U_{m_1+1}. Continue the coupling until the processes differ again. This will define an $n_2 \ge 1$ with $X^1_{m_1+n_2} = X^2_{n_1+n_2} - 2$. Then as before, run only the $\{X^1_n\}$ process until it again reaches the newly, temporarily frozen level of the $\{X^2_n\}$ process. This will define an $m_2 > n_2$ with $X^1_{m_1+m_2} = X^2_{n_1+n_2}$. Continuing in this way, we obtain a sequence $\{(n_i, m_i)\}_{i=1}^\infty$. In particular, $X^1_{M_j} = X^2_{N_j}$, where $M_j = \sum_{i=1}^j m_i$ and $N_j = \sum_{i=1}^j n_i$. Note that $\{X^1_i\}_{i=1}^{M_j}$ and $\{X^2_i\}_{i=1}^{N_j}$ are $\mathcal{G}_{M_j}$-measurable.

Clearly, the $\{X^i_n\}$ process has the distribution of the $P^{\omega_i}_y$ process, $i = 1, 2$. Let T^i_r denote the hitting time of r for $\{X^i_n\}$. From the construction it follows that for each $j \ge 1$,

$$\{T^1_z \le T^1_x \wedge N\} \cap \{T^1_z \le M_j\} \subset \{T^2_z \le T^2_x \wedge N\} \cap \{T^2_z \le N_j\}.$$

(The reader should verify this for $j = 1$ after which, the general case will be clear.) Now letting $j \to \infty$ gives $\{T^1_z \le T^1_x \wedge N\} \subset \{T^2_z \le T^2_x \wedge N\}$, from which it follows that $\mathcal{P}(T^1_z \le T^1_x \wedge N) \le \mathcal{P}(T^2_z \le T^2_x \wedge N)$, thereby proving (2.1). The last line of the proposition follows by letting $N \to \infty$ and then letting $z \to \infty$.

□

Proof of Lemma 1. *(1-i).* Since $P_1(T_0 = \infty) = 0$, the process starting at 1 will eventually hit 0. It will then return to 1, by comparison with the simple, symmetric random walk. Upon returning to 1, the current environment, call it ω', satisfies $\omega' \leq \omega$, since a jump to the left along the way at any site x changed the jump mechanism at that site from $\omega(x)$ to $\frac{1}{2}$. By Proposition 1, it then follows that the process will hit 0 again. This process continues indefinitely; thus, $P_1(X_n = 0 \text{ i.o.}) = 1$. For any $x \in Z$, there exists a $\delta_x > 0$ such that regardless of what the current environment is, if the process is at 0 it will go directly to x in $|x|$ steps. Thus from $P_1(X_n = 0 \text{ i.o.}) = 1$ it follows that $P_1(X_n = x \text{ i.o.}) = 1$, for all $x \in Z$. Now let $y > 1$ and let $x < y$. There is a positive probability that the process starting at 1 will make it first $y - 1$ steps to the right, ending up at y at time $y - 1$. The current environment will then still be ω. Since $P_1(X_n = x \text{ i.o.}) = 1$, it then follows that $P_y(X_n = x \text{ i.o.}) = 1$. Now let $y \leq 0$ and $x < y$. By comparison with the simple, symmetric random walk, the process starting at y will eventual hit 1. The current environment now, call it ω', satisfies $\omega' \leq \omega$. Thus since $P_1(X_n = x \text{ i.o,}) = 1$, it follows from Proposition 1 that $P_y(X = x \text{ i.o.}) = 1$.

(1-ii). Let $P_1(T_0 = \infty) > 0$. We first show that $P_y(T_x = \infty) > 0$, for all $x < y$. To do this, we assume that for some $x_0 < y_0$ one has $P_{y_0}(T_{x_0} = \infty) = 0$, and then come to a contradiction. By the argument in (1-i), it follows that $P_{y_0}(X_n = x_0 \text{ i.o}) = 1$, and then that $P_{y_0}(X_n = x \text{ i.o.}) = 1$, for all $x \in Z$. In particular, $P_{y_0}(X_n = 0 \text{ i.o.}) = 1$. If $y_0 = 1$, we have arrived at a contradiction. Consider now the case $y_0 < 1$. Since there is a positive probability of going directly from y_0 to 1 in the first $1 - y_0$ steps, in which case the current environment will still be ω, and since $P_{y_0}(X_n = 0 \text{ i.o.}) = 1$, it follows that $P_1(X_n = 0 \text{ i.o.}) = 1$, a contradiction. Now consider the case $y_0 > 1$. Since $P_1(T_0 = \infty) > 0$, there is a positive probability that starting from 1 the process will hit y_0 without having hit 0, and then after reaching y_0, continue and never ever hit 0. But when this process reached y_0, the current environment, call it ω', satisfied $\omega' \leq \omega$. Since $P_{y_0}(X_n = 0 \text{ i.o.}) = 1$,

it follows from Proposition 1 that after hitting y_0 the process would in fact hit 0 with probability 1, a contradiction.

From the fact that $P_y(T_x = \infty) > 0$, for all $x < y$, and from the periodicity of the environment, it follows that $\gamma \equiv \inf_{x\in Z} P_x(T_{x-1} = \infty) = \inf_{x\in\{0,1,\cdots,N-1\}} P_x(T_{x-1} = \infty) > 0$. From this we can prove transience. Indeed, fix x and y. By comparison with the simple, symmetric random walk, with P_y-probability one, the process will attain a new maximum infinitely often. The current environment at and to the right of any new maximum coincides with ω. Thus, with probability at least γ the process will never again go below that maximum. From this it follows that $P_y(X_n = x \text{ i.o}) = 0$ and that $\lim_{n\to\infty} X_n = \infty$ a.s..

(2). It follows from the proof of part (1) that if $P_1^\omega(T_0 = \infty) = 0$, then $P_{x+1}^\omega(T_x = \infty) = 0$ for all $x \ge 1$, and if $P_1^\omega(T_0 = \infty) > 0$, then $P_{x+1}^\omega(T_x = \infty) > 0$ for all $x \ge 1$. From this and the stationarity and ergodicity assumption, it follows that either $P_1^\omega(T_0 = \infty) = 0$ for $\mathcal{P}^S$-almost every ω or $P_1^\omega(T_0 = \infty) > 0$ for $\mathcal{P}^S$-almost every ω. For an ω such that $P_1^\omega(T_0 = \infty) = 0$, the proof of recurrence is the same as the corresponding proof in 1-i. For an ω such that $P_1^\omega(T_0 = \infty) > 0$, the proof of transience and the proof that $\lim_{n\to\infty} X_n = \infty$ a.s. are similar to the corresponding proofs in 1-ii. In 1-ii we used the fact that $\inf_{x\in Z} P_{x+1}(T_x = \infty) > 0$. However, it is easy to see that all we needed there is $\limsup_{x\to\infty} P_{x+1}(T_x = \infty) > 0$. By the stationarity and ergodicity, this condition holds for a.e. ω.

□

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. For $x \in Z$, define $\sigma_x = \inf\{n \ge 1 : X_{n-1} = x, X_n = x - 1\}$. Let $D_0^x = 0$ and for $n \ge 1$, let

$$D_n^x = \#\{m < n : X_m = x, \sigma_x > m\}. \tag{3.1}$$

Define

$$D_n = \sum_{x \in Z} (2\omega(x) - 1) D_n^x. \tag{3.2}$$

It follows easily from the definition of the random walk in a "have your cookie and eat in" environment that $X_n - D_n$ is a martingale. Doob's optional stopping theorem gives

$$nP_1(T_n < T_0) = 1 + E_1 D_{T_0 \wedge T_n} = 1 + \sum_{x=1}^{n-1} (2\omega(x) - 1) E_1 D_{T_0 \wedge T_n}^x. \tag{3.3}$$

One has $\lim_{n \to \infty} P_1(T_n < T_0) = P_1(T_0 = \infty)$. Let $\gamma_x = P_{x+1}(T_x = \infty)$.

Assume for now that the process is transient. For the meantime, we do not distinguish between parts (i) and (ii) of the theorem; that is, between $N = 1$ and $N > 1$. By the transience and Lemma 1, $\gamma_x > 0$ for all x. Consider the random variable $D_{T_0}^x$ under P_1, for $x \geq 1$. We now construct a random variable which dominates $D_{T_0}^x$. If $T_0 < T_x$, then $D_n^x = 0$. If $T_x < T_0$, and at time T_x the process jumps to $x - 1$, which occurs with probability $1 - \omega(x)$, or jumps to $x + 1$ and then never returns to x, which occurs with probability $\omega(x)\gamma_x$, then one has $D_{T_0}^x = 1$. Otherwise, $D_{T_0}^x > 1$. In this latter case, the process will return to x from the right. Upon returning, the process jumps to $x - 1$ with probability $1 - \omega(x)$, in which case $D_{T_0}^x = 2$. Whenever the process jumps again to $x + 1$, it may not ever return to x, but for the domination, we ignore this possibility and assume that it returns to x with probability 1. Taking the above into consideration, it follows that under P_1, the random variable $D_{T_0}^x$ is stochastically dominated by $I_x Z_x$, where I_x and Z_x are independent random variables satisfying

$$\begin{aligned} &P(I_x = 1) = 1 - P(I_x = 0) = P_1(T_x < T_0), \\ &\begin{cases} P(Z_x = 1) = 1 - \omega(x) + \omega(x)\gamma_x; \\ P(Z_x = k) = (\omega(x) - \omega(x)\gamma_x)\omega^{k-2}(x)(1 - \omega(x)), \ k \geq 2. \end{cases} \end{aligned} \tag{3.4}$$

Thus, $E_1 D^x_{T_0} \le E I_x Z_x = P_1(T_x < T_0) E Z_x$. One has $EZ_x = \frac{1-\omega(x)\gamma_x}{1-\omega(x)}$. Thus,

$$E_1 D^x_{T_0} \le \frac{1-\omega(x)\gamma_x}{1-\omega(x)} P_1(T_x < T_0). \tag{3.5}$$

Substituting (3.5) into (3.3) and using the monotonicity of D_n gives

$$P_1(T_n < T_0) \le \frac{1}{n} + \frac{1}{n}\sum_{x=1}^{n-1}(2\omega(x)-1)\frac{1-\omega(x)\gamma_x}{1-\omega(x)} P_1(T_x < T_0). \tag{3.6}$$

Consider now part (i) of the theorem. In this case $\omega(x) = p$ for all x, and thus $\gamma \equiv \gamma_x$ is independent of x. Letting $n \to \infty$ in (3.6) and using the transience assumption, $\gamma > 0$, one obtains

$$\frac{(2p-1)(1-p\gamma)}{1-p} \ge 1,$$

or equivalently,

$$\gamma = P_1(T_0 = \infty) \le \frac{1}{p}\frac{3p-2}{2p-1}. \tag{3.7}$$

Since (3.7) was derived under the assumption that $\gamma > 0$, it follows that $p > \frac{2}{3}$ is a necessary condition for transience. Note also that (3.7) gives the upper bound on $P_1(T_0 = \infty)$ in part (i) of the theorem.

Now consider part (ii) of the theorem. In this case $\omega(x)$ and thus also γ_x are periodic with period $N > 1$. By the transience assumption and Lemma 1, $\gamma_x > 0$, for all x. Thus, letting $n \to \infty$ in (3.6), we obtain

$$\frac{1}{N}\sum_{x=1}^{N}(2\omega(x)-1)\frac{1-\omega(x)\gamma_x}{1-\omega(x)} \ge 1. \tag{3.8}$$

Since (3.8) was derived under the assumption that $\gamma_x > 0$, for all x, it follows from (3.8) that $\frac{1}{N}\sum_{x=1}^{N}\frac{2\omega(x)-1}{1-\omega(x)} > 1$ is a necessary condition for transience. This is equivalent to $\frac{1}{N}\sum_{x=1}^{N}\frac{\omega(x)}{1-\omega(x)} > 2$.

The analysis above has given necessary conditions for transience in parts (i) and (ii), and has also given the upper bound on $P_1(T_0 = \infty)$ in part (i). We now turn to necessary conditions for recurrence in parts (i) and (ii), and to the lower bound on $P_1(T_0 = \infty)$ in part (i). For the time being, we make no assumption of transience or recurrence. Under P_1, consider the random variable $D^x_{T_0 \wedge T_n}$ appearing on the right hand side of (3.3). If

$T_0 < T_x$, then $D^x_{T_0\wedge T_n} = 0$. Let $\epsilon_{x,n}$ denote the probability that after the first time the process jumps rightward from x to $x+1$, it does not return to x before reaching n. Similarly, for each $k>1$, let $\epsilon^{(k)}_{x,n}$ denote the probability, conditioned on the process returning to x from $x+1$ at least $k-1$ times before hitting n, that after the k-th time the process jumps rightward from x to $x+1$, it does not return to x before reaching n. Each time the process jumps from x to $x+1$, the current environment, call it ω'', will satisfy $\omega'' \le \omega'$, where ω' was the environment that was in effect the previous time the process jumped from x to $x+1$. Thus, by Proposition 1, it follows that $\epsilon^{(k)}_{x,n} \le \epsilon_{x,n}$, for $k>1$. In light of these facts, it follows that the random variable $D^x_{T_0\wedge T_n}$ stochastically dominates I_xY_x, where I_x and Y_x are independent random variables, with I_x as in (3.4) and with Y_x satisfying

$$P(Y_x = k) = (1-\omega(x)+\omega(x)\epsilon_{x,n})(\omega(x)-\omega(x)\epsilon_{x,n})^{k-1},\ k\ge 1. \tag{3.9}$$

Thus,

$$\begin{aligned} E_1 D^x_{T_n\wedge T_0} &\ge EI_xY_x = \frac{1}{1-\omega(x)+\omega(x)\epsilon_{x,n}} P_1(T_x < T_0) \\ &\ge \frac{1}{1-\omega(x)+\omega(x)\epsilon_{x,n}} P_1(T_n<T_0),\ x=1,\cdots,n-1. \end{aligned} \tag{3.10}$$

From (3.3) and (3.10) it follows that for any n,

$$\frac{1}{n}\sum_{x=1}^{n-1} \frac{2\omega(x)-1}{1-\omega(x)+\omega(x)\epsilon_{x,n}} < 1. \tag{3.11}$$

Consider now part (i) of the theorem. In this case $\omega(x) = p$ for all x, and it follows that $\epsilon_{x,n}$ depends only on $n-x$. It also follows that $\lim_{(n-x)\to\infty} \epsilon_{x,n} = \gamma \equiv P_1(T_0=\infty)$. If the process is recurrent, then $\gamma = 0$. Using these facts and letting $n\to\infty$ in (3.11), we conclude that $\frac{2p-1}{1-p} \le 1$ is a necessary condition for recurrence. This is equivalent to $p \le \frac{2}{3}$. We also conclude that in the transient case, $\frac{2p-1}{1-p+p\gamma} \le 1$ or equivalently,

$$\gamma = P_1(T_0=\infty) \ge \frac{3p-2}{p}. \tag{3.12}$$

This gives the lower bound on $P_1(T_0=\infty)$ in part (i).

Consider now part (ii) of the theorem and assume recurrence. Then $\lim_{n\to\infty}\epsilon_{x,n}=0$. Since $\omega(x)$ is periodic with period $N>1$, the environment to the right of x is the same as the environment to the right of $x+N$. Thus, $\epsilon_{x,n}=\epsilon_{x+N,n+N}$, and therefore $\lim_{n-x\to\infty}\epsilon_{x,n}=0$. Using these facts and letting $n\to\infty$ in (3.11), it follows that a necessary condition for recurrence is $\frac{1}{N}\sum_{x=1}^{N}\frac{2\omega(x)-1}{1-\omega(x)}\le 1$. This is equivalent to $\frac{1}{N}\sum_{x=1}^{N}\frac{\omega(x)}{1-\omega(x)}\le 2$.

□

Proof of Theorem 2. By Lemma 1, the process is either recurrent for $\mathcal{P}^S$-almost every ω or transient for $\mathcal{P}^S$-almost every ω. Assume for now almost sure transience, and let ω be an environment for which the process is transient. For the fixed environment ω, (3.6) holds; that is,

$$P_1^\omega(T_n<T_0)\le\frac{1}{n}+\frac{1}{n}\sum_{x=1}^{n-1}(2\omega(x)-1)\frac{1-\omega(x)\gamma_x(\omega)}{1-\omega(x)}P_1^\omega(T_x<T_0), \tag{3.13}$$

where $\gamma_x=\gamma_x(\omega)=P_{x+1}^\omega(T_x=\infty)$. Note that $(2\omega(x)-1)\frac{1-\omega(x)\gamma_x(\omega)}{1-\omega(x)}$ can be expressed in the form $F(\theta^x\omega)$, where θ^x is the shift operator defined by $\theta^x(w)(z)=\omega(x+z)$. For each ω, we have $\lim_{z\to\infty}P_1^\omega(T_z<T_0)=\gamma_1(\omega)$. Thus, letting $n\to\infty$ in (3.13) and using the stationarity and ergodicity, it follows that

$$\mathcal{E}^S(2\omega(0)-1)\frac{1-\omega(0)\gamma_0(\omega)}{1-\omega(0)}\ge 1. \tag{3.14}$$

By the transience assumption and Lemma 1, $\gamma_0>0$ a.s. $\mathcal{P}^S$. Thus it follows from (3.14) that $\mathcal{E}^S\frac{2\omega(0)-1}{1-\omega(0)}>1$ is a necessary condition for transience. This is equivalent to $\mathcal{E}^S\frac{\omega(0)}{1-\omega(0)}>2$.

Now assume almost sure recurrence. For any fixed environment ω, (3.11) holds; that is,

$$\frac{1}{n}\sum_{x=1}^{n}\frac{2\omega(x)-1}{1-\omega(x)+\omega(x)\epsilon_{x,n}(\omega)}<1,$$

where $\epsilon_{x,n}(\omega)$ is the P_1^ω-probability that after the first time the process jumps rightward from x to $x+1$, it does not return to x before reaching n.

Since $\epsilon_{x,n}(\omega) \le \epsilon_{x,x+M}(\omega)$, for $x \le n-M$, we have

$$\frac{1}{n}\sum_{x=1}^{n-M}\frac{2\omega(x)-1}{1-\omega(x)+\omega(x)\epsilon_{x,x+M}(\omega)} < 1. \tag{3.15}$$

By the stationarity and ergodicity,

$$\lim_{r\to\infty}\frac{1}{r}\sum_{x=1}^{r}\epsilon_{x,x+M}(\omega) = \mathcal{E}^S\epsilon_{0,M} \quad \text{a.s.,} \tag{3.16}$$

and by the recurrence assumption,

$$\lim_{M\to\infty}\mathcal{E}^S\epsilon_{0,M} = 0. \tag{3.17}$$

From (3.16) and (3.17) it follows that for each $\delta > 0$ there exist r_δ, M_δ such that

(3.18)

$$\mathcal{P}^S(\frac{1}{r}\#\{x \in \{1,\cdots,r\} : \epsilon_{x,x+M} < \delta\} > 1-\delta) > 1-\delta, \text{ for } r > r_\delta, M > M_\delta.$$

By the stationarity and ergodicity again,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{x=1}^{n-M}\frac{2\omega(x)-1}{1-\omega(x)} = \mathcal{E}^S\frac{2\omega(0)-1}{1-\omega(0)} \quad \text{a.s..} \tag{3.19}$$

From (3.15), (3.18) and (3.19), we deduce that a necessary condition for recurrence is $\mathcal{E}^S\frac{2\omega(0)-1}{1-\omega(0)} \le 1$. This is equivalent to $\mathcal{E}^S\frac{\omega(0)}{1-\omega(0)} \le 2$. □

4. Proof of Theorem 3

Since the random walk $\{X_n\}_{n=0}^\infty$ is recurrent if $p \in (\frac{1}{2}, \frac{2}{3}]$, we may assume that $p > \frac{2}{3}$. The proof begins like the proof of the corresponding result in [1]. Let

$$U_m^n = \#\{0 \le k < T_n : X_k = m, X_{k+1} = m-1\} \tag{4.1}$$

denote the number of times the process jumps from m to $m-1$ before reaching n, and let

$$K_n = \#\{0 \le k \le T_n : X_k < 0\}. \tag{4.2}$$

An easy combinatorial argument yields

$$T_n = K_n - U_0^n + n + 2\sum_{m=0}^{n} U_m^n. \tag{4.3}$$

Since $p > \frac{2}{3}$, by Theorem 1 the process is transient, and by Lemma 1 one has $\lim_{n\to\infty} X_n = \infty$ a.s.. Thus, K_n and U_0^n are almost surely bounded, and we conclude that almost surely $T_n \sim n + 2\sum_{m=0}^n U_m^n$ as $n \to \infty$. We will show below that there is a time-homogeneous irreducible Markov process $\{Y_0, Y_1, \cdots\}$ such that for each n, the distribution of $\{U_n^n, \cdots, U_0^n\}$ under P_0 is the same as the distribution of $\{Y_0, \cdots, Y_n\}$. By the transience of $\{X_n\}_{n=0}^\infty$, U_0^n converges to a limiting distribution as $n \to \infty$. Thus, the same is true of Y_n, which means that this Markov process is positive recurrent. Denote its stationary distribution on Z^+ by μ. By the ergodic theorem for irreducible Markov chains, we conclude that

$$\lim_{n\to\infty} \frac{T_n}{n} = 1 + 2\sum_{k=0}^{\infty} k\mu(k) \le \infty. \tag{4.4}$$

Now as is well-known [6, p. 113], for any $v \in [0, \infty)$, one has $\lim_{n\to\infty} \frac{X_n}{n} = v$ a.s. if and only if $\lim_{n\to\infty} \frac{T_n}{n} = \frac{1}{v}$. Thus in light of (4.4), the process $\{X_n\}_{n=0}^\infty$ is ballistic if and only if $\sum_{k=0}^\infty k\mu(k) < \infty$, and the speed of the process is given by $\frac{1}{1+2\sum_{k=0}^\infty k\mu(k)}$.

We now show that there is a time-homogeneous, irreducible Markov process $\{Y_0, Y_1, \cdots\}$ such that for each n, the distribution of $\{U_n^n, \cdots, U_0^n\}$ under P_0 is the same as the distribution of $\{Y_0, \cdots, Y_n\}$, and we calculate its transition probabilities. For this we need to define an IID sequence as follows. Let P denote probabilities for an IID sequence $\{\zeta_n\}_{n=1}^\infty$ of geometric random variables satisfying $P(\zeta_1 = m) = \frac{1}{2}^{m+1}$, $m = 0, 1, \cdots$. Consider now the distribution under P_0 of the random variable U_m^n, conditioned on $U_{m+1}^n = j$. First consider the case $j = 0$. The process starts at 0 and eventually reaches m for the first time. From m, it jumps right to $m+1$ with probability p, and jumps left to $m-1$ with probability $1-p$. If the process jumps right to $m+1$, then it will never again find itself at

m. On the other hand, if it jumps left to $m-1$, then it will eventually return to m. When it returns to m, it jumps left or right with probability $\frac{1}{2}$. If it jumps right, then it will never again return to m, but if it jumps left, then it will eventually return to m again, where it again jumps left or right with probability $\frac{1}{2}$. This situation is repeated until the process finally jumps right to $m+1$, from which it will never return to m. From the above description, it follows that $P_0(U_m^n = 0|U_{m+1}^n = 0) = p$ and $P_0(U_m^n = k|U_{m+1}^n = 0) = (1-p)P(\zeta_1 = k-1)$, for $k \geq 1$.

Now consider $j \geq 1$. As above, the process starts at 0 and eventually reaches m for the first time. From m, it jumps right to $m+1$ with probability p, and jumps left to $m-1$ with probability $1-p$. If it jumps to the right, then since $j \geq 1$ it will eventually return to m. Upon returning to m it will again jump right with probability p and left with probability $1-p$. By the conditioning, it will jump back to m from $m+1$ a total of j times. As long as the process keeps jumping to the right when it is at m, the probability of jumping to the right the next time it is at m will still be p. But as soon as it jumps left from m, then whenever it again reaches m it will jump right with probability $\frac{1}{2}$. From this description, it follows that $P_0(U_m^n = 0|U_{m+1}^n = j) = p^{j+1}$ and $P_0(U_m^n = k|U_{m+1}^n = j) = (1-p)\sum_{l=0}^{j} p^l P(\sum_{i=1}^{j+1-l} \zeta_i = k-1)$.

We have thus shown that there exists such a time homogeneous Markov process $\{Y_0, Y_1, \cdots\}$. Denote probabilities and expectations for this process starting from $m \in \{0, 1, \cdots\}$ by $\mathcal{P}_m$ and $\mathcal{E}_m$. We have also shown that the transition probabilities of this Markov process are given by

(4.5)

$$p_{jk} \equiv \mathcal{P}(Y_1 = k|Y_0 = j) = \begin{cases} p^{j+1}, \text{ if } k = 0; \\ (1-p)\sum_{l=0}^{j} p^l P(\sum_{i=1}^{j+1-l} \zeta_i = k-1), \text{ if } k \geq 1. \end{cases}$$

As noted above, since $\{X_n\}_{n=0}^{\infty}$ is transient, the process $\{Y_n\}_{n=0}^{\infty}$ is positive recurrent. Furthermore, denoting its invariant measure by μ, the process is ballistic if and only if $\sum_{k=0}^{\infty} k\mu(k) < \infty$.

We now analyze $\sum_{k=0}^{\infty} k\mu(k)$. Let $\sigma_m = \inf\{n \geq 1 : Y_n = m\}$. By Khasminskii's representation [4, chapter 5, section 4] of invariant probability measures, one has

$$\mu_k = \frac{\mathcal{E}_1 \sum_{m=1}^{\sigma_1} \chi_k(Y_m)}{\mathcal{E}_1 \sigma_1}. \tag{4.6}$$

From (4.6), it follows that $\sum_{k=0}^{\infty} k\mu(k) < \infty$ if and only if

$$\mathcal{E}_1 \sum_{m=1}^{\sigma_1} Y_m < \infty. \tag{4.7}$$

The generator of $\{Y_n\}_{n=0}^{\infty}$ is $P - I$, where $Pu(j) = \sum_{k=0}^{\infty} p_{jk} u(k)$. Thus, letting

$$M_0^u = u(Y_0), \quad M_n^u = u(Y_n) - \sum_{m=0}^{n-1} (Pu - u)(Y_m), \ n \geq 1, \tag{4.8}$$

it follows that $\{M_n^u\}_{n=0}^{\infty}$ is a martingale for all functions u which satisfy

$$\sum_{k=0}^{\infty} p_{jk} |u(k)| < \infty, \quad \text{for all } j. \tag{4.9}$$

Consider the function $u(k) = (k-1)^2$. The ensuing calculation shows that u satisfies (4.9). From (4.9) and the fact that $E\zeta_1 = 1$ and $E\zeta_1^2 = 3$, it follows that

$$\begin{aligned} &Pu(j) = p^{j+1} u(0) + (1-p) \sum_{l=0}^{j} p^l E(\sum_{i=1}^{j+1-l} \zeta_i)^2 = \\ &p^{j+1} + (1-p) \sum_{l=0}^{j} p^l \left((j+1-l)^2 + 2(j+1-l)\right). \end{aligned} \tag{4.10}$$

From (4.10) and the fact that $(1-p)\sum_{l=0}^{\infty} l p^l = \frac{p}{1-p}$, it follows that

$$(Pu - u)(j) = (6 - 2\frac{p}{1-p}) j + 0(1), \ \text{as } j \to \infty. \tag{4.11}$$

From Doob's optional sampling theorem,

$$\mathcal{E}_1 M_{\sigma_1 \wedge n}^u = \mathcal{E}_1 M_0^u = 0, \ n \geq 0, \tag{4.12}$$

and from (4.11) and (4.8), one has

(4.13)

$$\mathcal{E}_1 M^u_{\sigma_1\wedge n} = \mathcal{E}_1(Y_{\sigma_1\wedge n}-1)^2-(6-2\frac{p}{1-p})\mathcal{E}_1\sum_{m=0}^{\sigma_1\wedge n-1} Y_m+O(\mathcal{E}_1\sigma_1\wedge n) \text{ as } n\to\infty.$$

By the positive recurrence of $\{Y_n\}_{n=0}^\infty$, which holds for all $p>\frac{2}{3}$, one has $\mathcal{E}_1\sigma_1\wedge n\to\mathcal{E}_1\sigma_1<\infty$ as $n\to\infty$. Thus (4.12) and (4.13) yield

$$\begin{aligned}&\mathcal{E}_1(Y_{\sigma_1\wedge n}-1)^2-(6-2\frac{p}{1-p})\mathcal{E}_1\sum_{m=0}^{\sigma_1\wedge n-1} Y_m = C_n, \text{ for all } n,\\ &\text{where } C_n = O(1), \text{ as } n\to\infty.\end{aligned} \tag{4.14}$$

If $p>\frac{3}{4}$, one has $6-2\frac{p}{1-p}<0$. Letting $n\to\infty$ in (4.14), we conclude that (4.7) holds if $p>\frac{3}{4}$, and thus the process $\{X_n\}_{n=0}^\infty$ is ballistic.

To calculate the speed of the process we will need to build on some calculations used in the proof of Proposition 2 below. Thus we defer the proof of (1.2) until after the proof of Proposition 2.

For the case $p\in(\frac{2}{3},\frac{3}{4})$, we will need the following proposition.

Proposition 2. *Let $\{X_n\}_{n=0}^\infty$ be a random walk in a deterministic "have your cookie and eat it" environment with $\omega(x)=p$, for all x. Then*

$$\begin{aligned}P_1(T_0=\infty)&\geq\frac{3p-2}{2p-1}, \text{ for } p\in(\frac{2}{3},\frac{3}{4});\\ P_1(T_0=\infty)&=\frac{3p-2}{2p-1}, \text{ for } p\in[\frac{3}{4},1).\end{aligned} \tag{4.15}$$

Remark. We believe that equality holds in Proposition 2 for all $p\in(\frac{2}{3},1)$.

We will also need the following corollary, whose proof will be pointed out during the proof of Proposition 2.

Corollary 1. *Let $p\in(\frac{2}{3},1)$. Let $\sigma_1=\inf\{m\geq1: Y_m=1\}$. Then $l\equiv\lim_{n\to\infty}E_1Y_{\sigma_1\wedge n}$ exists, and $l\geq1$, with equality if and only if $P_1(T_0=\infty)=\frac{3p-2}{2p-1}$.*

We now prove non-ballisticity in the case $p\in(\frac{2}{3},\frac{3}{4})$. Then we will prove Proposition 2. We break the proof into two cases—$P_1(T_0=\infty)>\frac{3p-2}{2p-1}$ and

$P_1(T_0=\infty)=\frac{3p-2}{2p-1}$. The first case we can dispose of easily. (By the remark above, we expect that this case is actually vacuous.) By Corollary 1, we have $\lim_{n\to\infty} E_1 Y_{\sigma_1\wedge n}>1$. Using this along with the fact that $\lim_{n\to\infty} Y_{\sigma_1\wedge n}=Y_{\sigma_1}=1$ a.s., it follows that

$$\lim_{n\to\infty} E_1(Y_{\sigma_1\wedge n}-1)^2=\infty. \tag{4.16}$$

Now letting $n\to\infty$ in (4.14), and using (4.16) and the fact that $6-2\frac{p}{1-p}>0$, it follows that (4.7) does not hold.

We now treat the case $P_1(T_0=\infty)=\frac{3p-2}{2p-1}$. We take a completely different tack from the above. In [6], it was shown that for the random walks with cookies in that paper, one has

$$\lim_{n\to\infty}\frac{X_n}{n}=\frac{1}{u},\ P_0-\text{a.s., where } u\equiv\sum_{j=1}^\infty P_0(T_{j+1}-T_j\ge j).$$

The reader can check that the same proof works for the random walk in a "have your cookie and eat it" environment. Thus, to prove nonballisticity in the case $p\in(\frac{2}{3},\frac{3}{4})$, it suffices to show that $u=\infty$

For $m<j$, let $\nu_{m;j}=\inf\{k\ge T_j: X_k=m\}$. From the definition of the "have your cookie and eat it" environment, on the event $\{\nu_{j-[\sqrt{j}\,];j}<T_{j+1}\}$, after hitting $j-[\sqrt{j}]$ at time $\nu_{j-[\sqrt{j}];j}$, the process will eventually find itself at $j-[\frac{1}{2}\sqrt{j}]$, with the environment up to a distance $[\frac{1}{2}\sqrt{j}]$ on either side of it being a simple, symmetric random walk environment. From $j-[\frac{1}{2}\sqrt{j}]$, the process must exit the interval $[j-[\sqrt{j}],j]$ before time T_{j+1}. By the scaling properties of the simple, symmetric random walk, there is a positive probability, independent of j, that it will take at least j steps to exit this interval, starting from its mid-point $j-[\frac{1}{2}\sqrt{j}]$. Thus, we conclude that for some $c>0$,

$$P_0(T_{j+1}-T_j\ge j)\ge cP_0(\nu_{j-[\sqrt{j}\,];j}<T_{j+1}). \tag{4.17}$$

Therefore to complete the proof it suffices to show that

$$\sum_{j=1}^\infty P_0(\nu_{j-[\sqrt{j}\,];j}<T_{j+1})=\infty. \tag{4.18}$$

From the beginning of the proof of Theorem 1, recall that $\sigma_x = \inf\{n \geq 1 : X_{n-1} = x, X_n = x-1\}$. Define

$$\rho_x^{(j)} = \begin{cases} p, \text{ if } \sigma_x > T_j, \\ \frac{1}{2}, \text{ if } \sigma_x < T_j, \end{cases} \qquad \rho_x^{(\infty)} = \begin{cases} p, \text{ if } \sigma_x = \infty, \\ \frac{1}{2}, \text{ if } \sigma_x < \infty. \end{cases} \tag{4.19}$$

We introduce the notation $P_0^{\{\rho_x^{(j)}\}}(\cdot) \equiv P_0(\cdot|\{\rho_x^{(j)}\}_{x=1}^{j-1})$. We can write

$$\begin{aligned} &P_0^{\{\rho_x^{(j)}\}}(\nu_{j-[\sqrt{j}];j} < T_{j+1}) = \\ &P_0^{\{\rho_x^{(j)}\}}(\nu_{j-1;j} < T_{j+1}) \prod_{r=1}^{[\sqrt{j}]-1} P_0^{\{\rho_x^{(j)}\}}(\nu_{j-r-1;j} < T_{j+1}|\nu_{j-r;j} < T_{j+1}). \end{aligned} \tag{4.20}$$

Clearly,

$$P_0^{\{\rho_x^{(j)}\}}(\nu_{j-1;j} < T_{j+1}) = 1-p. \tag{4.21}$$

The conditional probability $P_0^{\{\rho_x^{(j)}\}}(\nu_{j-r-1;j} < T_{j+1}|\nu_{j-r;j} < T_{j+1})$ is the probability that a simple random walk starting from $j-r$ will hit $j-r-1$ before it hits $j+1$, where the probability of jumping to the right is equal to $\frac{1}{2}$ at the sites $x > j-r$ and is equal to $\rho_{j-r}^{(j)}$ at the site $j-r$. This probability can be calculated explicitly using a standard difference equation with boundary conditions. We leave it to the reader to check that one obtains

$$P_0^{\{\rho_x^{(j)}\}}(\nu_{j-r-1;j} < T_{j+1}|\nu_{j-r;j} < T_{j+1}) = 1 - \frac{\rho_{j-r}^{(j)}}{(r+1)(1-\rho_{j-r}^{(j)}) + \rho_{j-r}^{(j)}}. \tag{4.22}$$

For every $\epsilon > 0$, there exists an $x_\epsilon > 0$ such that $\log(1-x) > -(1+\epsilon)x$, for $0 < x < x_\epsilon$. Thus, from (4.19)-(4.22), it follows that there exists an r_ϵ such that

(4.23)

$$P_0^{\{\rho_x^{(j)}\}}(\nu_{j-[\sqrt{j}];j} < T_{j+1}) \ge$$

$$(1-p)\left(\prod_{r=1}^{r_\epsilon} 1 - \frac{\rho_{j-r}^{(j)}}{r(1-\rho_{j-r}^{(j)}) + \rho_{j-r}^{(j)}}\right)\exp\left(-(1+\epsilon)\sum_{r=r_\epsilon+1}^{[\sqrt{j}]-1}\frac{\rho_{j-r}^{(j)}}{r(1-\rho_{j-r}^{(j)}) + \rho_{j-r}^{(j)}}\right) \ge$$

$$C_{\epsilon,p}\exp\left(-(1+\epsilon)\sum_{r=r_\epsilon+1}^{[\sqrt{j}]-1}\frac{\rho_{j-r}^{(j)}}{r(1-\rho_{j-r}^{(j)})}\right).$$

Using (4.23) and Jensen's inequality, we obtain

$$\begin{aligned}
&P_0(\nu_{j-[\sqrt{j}];j} < T_{j+1}) = E_0 P_0^{\{\rho_x^{(j)}\}}(\nu_{j-[\sqrt{j}];j} < T_{j+1}) \ge \\
&C_{\epsilon,p}E_0\exp\left(-(1+\epsilon)\sum_{r=r_\epsilon+1}^{[\sqrt{j}]-1}\frac{\rho_{j-r}^{(j)}}{r(1-\rho_{j-r}^{(j)})}\right) \ge \\
&C_{\epsilon,p}\exp\left(-(1+\epsilon)\sum_{r=r_\epsilon+1}^{[\sqrt{j}]-1}(E_0\frac{\rho_{j-r}^{(j)}}{1-\rho_{j-r}^{(j)}})\frac{1}{r}\right).
\end{aligned} \tag{4.24}$$

Note that $\lim_{j\to\infty}\rho_x^{(j)} = \rho_x^{(\infty)}$. The distribution of $\rho_x^{(\infty)}$ is independent of x and is given by $P_0(\rho_x^{(\infty)} = p) = 1 - P_0(\rho_x^{(\infty)} = \frac{1}{2}) = \gamma$, where $\gamma \equiv P_1(T_0 = \infty)$. Note that in fact the distribution of $\rho_{j-r}^{(j)}$ depends only on r and converges weakly to the distribution of $\rho_x^{(\infty)}$ as $r \to \infty$. We have $E_0\frac{\rho_x^{(\infty)}}{1-\rho_x^{(\infty)}} = \gamma\frac{p}{1-p} + 1 - \gamma$. In light of these facts, and increasing r_ϵ if necessary, we have

$$\begin{aligned}
&\sum_{r=r_\epsilon+1}^{[\sqrt{j}]-1}(E_0\frac{\rho_{j-r}^{(j)}}{1-\rho_{j-r}^{(j)}})\frac{1}{r} \le \\
&(\gamma\frac{p}{1-p} + 1 - \gamma + \epsilon)\log\sqrt{j}.
\end{aligned} \tag{4.25}$$

From (4.24) and (4.25), we obtain

$$P_0(\nu_{j-[\sqrt{j}];j} < T_{j+1}) \ge \frac{C_{\epsilon,p}}{j^{\frac{1}{2}(1+\epsilon)(\gamma\frac{p}{1-p}+1-\gamma+\epsilon)}}. \tag{4.26}$$

Since ϵ can be chosen arbitrarily small, it follows from (4.26) that (4.18) holds if $\frac{1}{2}(\gamma\frac{p}{1-p}+1-\gamma)<1$, or equivalently, if $\gamma<\frac{1-p}{2p-1}$. But we have assumed that $\gamma=\frac{3p-2}{2p-1}$. Thus, (4.18) holds if $\frac{3p-2}{2p-1}<\frac{1-p}{2p-1}$, which is equivalent to $p<\frac{3}{4}$. This completes the proof of non-ballisticity in the case $p\in(\frac{2}{3},\frac{3}{4})$.

We now prove Proposition 2, after which we will prove (1.2), which as already noted requires some of the calculations in the proof of the proposition.

Proof of Proposition 2. Since for each n, $\{U_n^n,\cdots U_1^n\}$ under P_0 is a Markov process with transition probabilities p_{jk} as above in (4.5) and with stationary distribution μ, it follows that $\mu_0=\lim_{n\to\infty}P_0(U_1^n=0)$. On the other hand $P_1(T_0=\infty)=\lim_{n\to\infty}P_0(U_1^n=0)$. Thus, $P_1(T_0=\infty)=\mu_0$. By the invariance of μ and (4.5), we have $\mu_0=\sum_{j=0}^{\infty}p_{j0}\mu_j=\sum_{j=0}^{\infty}p^{j+1}\mu_j$. Thus,

$$P_1(T_0=\infty)=\sum_{j=0}^{\infty}p^{j+1}\mu_j. \tag{4.27}$$

To prove the proposition, we will evaluate the right hand side of (4.27).

Consider the martingale $\{M_n^u\}_{n=0}^{\infty}$ defined in (4.8), with $u(k)=k-1$. Using the fact that $E\zeta_1=1$, one has from (4.5)

$$\begin{aligned}Pu(j)=\sum_{k=0}^{\infty}p_{jk}u(k)=-p^{j+1}+(1-p)\sum_{l=0}^{j}p^lE(\sum_{i=1}^{j+1-l}\zeta_i)=\\ -p^{j+1}+(1-p)\sum_{l=0}^{j}p^l(j+1-l)=\\ -p^{j+1}+(j+1)(1-p^{j+1})-\frac{p}{1-p}(1-(j+1)p^j+jp^{j+1}).\end{aligned} \tag{4.28}$$

After regrouping terms, one obtains

$$(Pu-u)(j)=\frac{1}{1-p}\left(2-3p+p^{j+1}(2p-1)\right). \tag{4.29}$$

By Doob's optional stopping theorem, $\{M_n^u\}_{n=0}^{\infty}$ satisfies (4.12), where $\sigma_1=\inf\{m\geq 1: Y_m=1\}$. Using this along with (4.29) and the definition of

$\{M^u_n\}_{n=0}^\infty$, we obtain

$$\mathcal{E}_1 Y_{\sigma_1\wedge n} = 1 + \mathcal{E}_1 \sum_{m=0}^{\sigma_1\wedge n-1} \frac{1}{1-p}\left(2-3p+p^{Y_m+1}(2p-1)\right). \tag{4.30}$$

By Khasminskii's representation of invariant probability measures [4, chapter 5, section 4], one has

$$\begin{aligned} &\lim_{n\to\infty} \mathcal{E}_1 \sum_{m=0}^{\sigma_1\wedge n-1} \frac{1}{1-p}\left(2-3p+p^{Y_m+1}(2p-1)\right) = \\ &(\mathcal{E}_1\sigma_1)\sum_{j=0}^{\infty} \frac{1}{1-p}\left(2-3p+p^{j+1}(2p-1)\right)\mu_j. \end{aligned} \tag{4.31}$$

This shows that the right hand side of (4.30) converges as $n\to\infty$; thus, so does the left hand side. Since $\sigma_1<\infty$ a.s., and since $Y_{\sigma_1\wedge n}$ is bounded from below and $Y_{\sigma_1}=1$, it follows that the left hand side of (4.30) converges to a limit that is greater than or equal to 1 as $n\to\infty$. Thus, the right hand side of (4.31) is nonnegative; equivalently, $\sum_{j=0}^\infty p^{j+1}\mu_j \ge \frac{3p-2}{2p-1}$. From this and (4.27), it follows that $P_1(T_0=\infty)\ge\frac{3p-2}{2p-1}$. The above calculation also shows that Corollary 1 holds. For the proof of non-ballisticity given above in the case $p\in(\frac23,\frac34)$, this is all we need. We now continue and show that $P_1(T_0=\infty)=\frac{3p-2}{2p-1}$, if $p\ge\frac34$. This will be needed for the proof of (1.2).

First consider the case $p=\frac34$. If on the contrary, one had $P_1(T_0=\infty)>\frac{3p-2}{2p-1}$, then it would follow from Corollary 1 that $\lim_{n\to\infty}E_1Y_{\sigma_1\wedge n}>1$, and as noted in the proof of Theorem 3 (see (4.16)), this would imply that $\lim_{n\to\infty}E_1(Y_{\sigma_1\wedge n}-1)^2=\infty$. Using this and letting $n\to\infty$ in (4.14) would lead to a contradiction since $6-2\frac{p}{1-p}=0$.

Now consider the case $p>\frac34$. Let $u(k)=\exp(-\gamma(k-1))$, where $\gamma>0$. Since μ is the invariant probability measure, we have

$$\sum_{j=0}^\infty Pu(j)\mu_j=\sum_{j=0}^\infty u(j)\mu_j=\sum_{j=0}^\infty \exp(-\gamma(j-1))\mu_j. \tag{4.32}$$

Using the fact that $E\exp(-\gamma\zeta_1) = (2-\exp(-\gamma))^{-1}$, we have from (4.5)

$$
\begin{aligned}
& Pu(j) = \sum_{k=0}^{\infty} p_{jk}\exp(-\gamma(k-1)) = p^{j+1}\exp(\gamma) + \\
& \sum_{k=1}^{\infty}(1-p)\sum_{l=0}^{j} p^l P(\sum_{i=1}^{j+1-l}\zeta_i = k-1)\exp(-\gamma(k-1)) = \\
& p^{j+1}\exp(\gamma) + (1-p)\sum_{l=0}^{j} p^l E\exp(-\gamma(\sum_{i=1}^{j+1-l}\zeta_i)) = \\
& p^{j+1}\exp(\gamma) + (1-p)\sum_{l=0}^{j} p^l (\frac{1}{2-\exp(-\gamma)})^{j+1-l} = \\
& p^{j+1}\exp(\gamma) + (\frac{1}{2-\exp(-\gamma)})^{j+1}(1-p)\frac{1-(p(2-\exp(-\gamma)))^{j+1}}{1-p(2-\exp(-\gamma))} = \\
& p^{j+1}\exp(\gamma) + \frac{1-p}{p(2-\exp(-\gamma))-1}(p^{j+1} - (\frac{1}{2-\exp(-\gamma)})^{j+1}).
\end{aligned} \tag{4.33}
$$

Substituting for Pu from the right hand side of (4.33) into the left hand side of (4.32), and then multiplying both sides of the resulting equation by $p(2-\exp(-\gamma))-1$, we obtain

$$
\begin{aligned}
& (2p-1)(\exp(\gamma)-1)\sum_{j=0}^{\infty} p^{j+1}\mu_j = (1-p)\sum_{j=0}^{\infty}(\frac{1}{2-\exp(-\gamma)})^{j+1}\mu_j + \\
& ((2p-1)\exp(\gamma)-p)\sum_{j=0}^{\infty}\exp(-\gamma j)\mu_j.
\end{aligned} \tag{4.34}
$$

Differentiating (4.34) with respect to γ gives

(4.35)

$$(2p-1)\exp(\gamma)\sum_{j=0}^{\infty}p^{j+1}\mu_j=(p-(2p-1)\exp(\gamma))\sum_{j=0}^{\infty}\exp(-\gamma j)j\mu_j-$$

$$(1-p)\exp(-\gamma)(2-\exp(-\gamma))^{-2}\sum_{j=0}^{\infty}(\frac{1}{2-\exp(-\gamma)})^j j\mu_j+$$

$$(2p-1)\exp(\gamma)\sum_{j=0}^{\infty}\exp(-\gamma j)\mu_j-$$

$$(1-p)\exp(-\gamma)(2-\exp(-\gamma))^{-2}\sum_{j=0}^{\infty}(\frac{1}{2-\exp(-\gamma)})^j\mu_j=$$

$$I-II+III-IV.$$

Letting $\gamma\to 0$, the left hand side of (4.35) converges to $(2p-1)\sum_{j=0}^{\infty}p^{j+1}\mu_j$ and III−IV converges to $3p-2$. Since $p>\frac{3}{4}$, it follows from Theorem 3 that $\sum_{j=0}^{\infty}j\mu_j<\infty$. Thus, I−II converges to 0 as $\gamma\to 0$, and consequently one obtains $\sum_{j=0}^{\infty}p^{j+1}\mu_j=\frac{3p-2}{2p-1}$. From this and (4.27) it follows that $P_1(T_0=\infty)=\frac{3p-2}{2p-1}$. This concludes the proof of Proposition 2. (If one could show that I−II is non-positive for small $\gamma>0$, then one could conclude from the above calculation that $P_1(T_0=\infty)\le\frac{3p-2}{2p-1}$, which in conjunction with the reverse inequality proved above, would prove that $P_1(T_0=\infty)=\frac{3p-2}{2p-1}$. However our analysis of I−II proved inconclusive.)

□

We now turn to the proof of (1.2). As was noted in the paragraph following (4.4), the speed of the process is given by $\frac{1}{1+2\sum_{k=0}^{\infty}k\mu_k}$. Thus, to prove (1.2), we need to show that $\sum_{k=0}^{\infty}k\mu_k\le\frac{2-2p}{4p-3}$, with equality if $p\in[\frac{10}{11},1)$. The calculations are in the spirit of several of the previous calculations, except that they are more tedious. We skip some of the intermediate steps. First we show that

$$\sum_{k=0}^{\infty}k^2\mu_k<\infty,\ \text{if } p\in(\frac{10}{11},1). \tag{4.36}$$

Let $u(k) = (k-1)^3$. From (4.5) we have

$$Pu(j) = -p^{j+1} + (1-p)\sum_{l=0}^{j} p^l E(\sum_{i=1}^{j+1-l} \zeta_i)^3. \tag{4.37}$$

We have already noted that $E\zeta_1 = 1$ and $E\zeta_1^2 = 3$. One can check that $E\zeta_1^3 = 13$. Also, one has $(1-p)\sum_{l=0}^{\infty} lp^l = \frac{p}{1-p}$ and $(1-p)\sum_{l=0}^{\infty} l^2 p^l = \frac{p}{1-p} + 2(\frac{p}{1-p})^2$. Using these facts, one calculates that

(4.38)

$$(Pu-u)(j) = j^2\left(30 - \frac{3p}{1-p}\right) + j\left(34 - 49\frac{p}{1-p} + 2(\frac{p}{1-p})^2\right) + O(1), \text{ as } j \to \infty.$$

Similar to (4.7), one has from (4.6) that $\sum_{k=0}^{\infty} k^2 \mu(k) < \infty$ if and only if

$$\mathcal{E}_1 \sum_{m=1}^{\sigma_1} Y_m^2 < \infty. \tag{4.39}$$

Applying the argument from (4.8) to (4.14), but using the function $u(k) = (k-1)^3$ instead of the function $u(k) = (k-1)^2$, using (4.38) instead of (4.11), and noting that $30 - \frac{3p}{1-p} < 0$ if $p > \frac{10}{11}$, one concludes that (4.39) holds if $p > \frac{10}{11}$. This proves (4.36)

We now prove the equality in (1.2) for $p \in [\frac{10}{11}, 1)$. The case $p = \frac{10}{11}$ uses the same type of argument as that used to prove that $P_1(T_0 = \infty) = \frac{3p-2}{2p-1}$ in the case $p = \frac{3}{4}$. One uses the function $u(k) = (k-1)^3$ and the calculations in the previous paragraph rather than the function $u(k) = (k-1)^2$. We leave the details to the reader. For the case $p > \frac{10}{11}$, we differentiate (4.35) in γ and set $\gamma = 0$. Two of the terms that result in this tedious calculation are $\pm(1-p)\sum_{k=0}^{\infty} k^2 \mu_k$. By (4.36), these terms pose no problem. One obtains

$$(2p-1)\sum_{j=0}^{\infty} p^{j+1}\mu_j = 2 - p + (-8p+6)\sum_{j=0}^{\infty} j\mu_j. \tag{4.40}$$

In the proof of Proposition 2, it was shown that $\sum_{j=0}^{\infty} p^{j+1}\mu_j = \frac{3p-2}{2p-1}$. Thus we conclude from (4.40) that $\sum_{j=0}^{\infty} j\mu_j = \frac{2-2p}{4p-3}$.

We now turn to the case $p \in (\frac{3}{4}, \frac{10}{11})$. The argument from (4.29) until the end of that paragraph, applied to $u(k) = (k-1)^2$ instead of to $u(k) = k-1$, shows that

$$\sum_{j=0}^{\infty}(Pu-u)(j)\mu_j \geq 0. \tag{4.41}$$

A very tedious calculation starting from the formula for $Pu(j)$ on the second line of (4.10) reveals that

$$(Pu-u)(j) = j(6-\frac{2p}{1-p})+\frac{1}{1-p}(-5p+\frac{2p^2}{1-p}-\frac{2p^{j+3}}{1-p}+2+p^{j+1}). \tag{4.42}$$

Substituting (4.42) into (4.41) and again using the fact that $\sum_{j=0}^{\infty} p^{j+1}\mu_j = \frac{3p-2}{2p-1}$, one concludes after a bit of algebra that $\sum_{j=0}^{\infty} j\mu_j \leq \frac{2-2p}{4p-3}$. □

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