THE FUJITA EXPONENT FOR SEMILINEAR HEAT EQUATIONS WITH QUADRATICALLY DECAYING POTENTIAL OR IN AN EXTERIOR DOMAIN

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ABSTRACT. Consider the equation

$$(0.1)\qquad \begin{aligned} &u_t = \Delta u - Vu + au^p \text{ in } R^n \times (0,T);\\ &u(x,0) = \phi(x) \gneq 0, \text{ in } R^n, \end{aligned}$$

where $p > 1$, $n \geq 2$, $T \in (0,\infty]$, $V(x) \sim \frac{\omega}{|x|^2}$ as $|x| \to \infty$, for some $\omega \neq 0$, and $a(x)$ is on the order $|x|^m$ as $|x| \to \infty$, for some $m \in (-\infty,\infty)$. A solution to the above equation is called global if $T = \infty$. Under some additional technical conditions, we calculate a critical exponent p^* such that global solutions exist for $p > p^*$, while for $1 < p \leq p^*$, all solutions blow up in finite time. We also show that when $V \equiv 0$, the blow-up/global solution dichotomy for (0.1) coincides with that for the corresponding problem in an exterior domain with the Dirichlet boundary condition, including the case in which p is equal to the critical exponent.

1. INTRODUCTION AND STATEMENT OF RESULTS

Consider the semilinear heat equation

$$(1.1)\qquad \begin{aligned} &u_t = \Delta u - Vu + u^p \text{ in } R^n \times (0,T);\\ &u(x,0) = \phi(x) \gneq 0 \text{ in } R^n, \end{aligned}$$

where $p > 1$, $n \geq 1$ and $T \in (0,\infty]$. In this paper, when we speak of a *solution* to the above equation, or to any of the other equations appearing later on, we mean *a classical solution u satisfying* $||u(\cdot,t)||_\infty < \infty$*, for*

2000 *Mathematics Subject Classification.* 35K55, 35B33, 35B40 .

Key words and phrases. critical exponent, blow-up, global solution, Fujita exponent, exterior domain.

$0 < t < T$. This allows us to employ comparison principles. A solution to (1.1) is called global if $T = \infty$. In the case that $V \equiv 0$, $p^* \equiv 1 + \frac{2}{n}$ is the critical exponent, the so-called *Fujita exponent*, and one has the following dichotomy: if $p > p^*$, then for sufficiently small initial data ϕ, the solution to (1.1) is global, whereas if $1 < p \le p^*$, then (1.1) has no global solution—every solution blows up in finite time. This result goes back to Fujita [3] in the case $p \ne p^*$. Various proofs of blow-up in the borderline case $p = p^*$ can be found in [1], [8], [12].

More recently, Zhang [14] considered (1.1) with $n \ge 3$ for potentials V behaving like $\frac{\omega}{1+|x|^b}$, for $b > 0$ and $\omega \ne 0$. He proved the following result.

Theorem (Zhang). *Let $n \ge 3$.*
i. If $0 \le V(x) \le \frac{\omega}{1+|x|^b}$, for some $b > 2$ and $\omega > 0$, then $p^* = 1 + \frac{2}{n}$ and consequently the potential does not affect the critical exponent;
ii. If $V(x) \ge \frac{\omega}{1+|x|^b}$, for some $b \in (0, 2)$ and $\omega > 0$, then $p^* = 1$ and there exist global solutions for all $p > 1$;
iii. If $\frac{\omega}{1+|x|^b} \le V(x) \le 0$, for some $b > 2$ and $\omega < 0$ with $|\omega|$ sufficiently small, then $p^* = 1 + \frac{2}{n}$ and consequently the potential does not affect the critical exponent;
iv. If $V(x) \le \frac{\omega}{1+|x|^b}$, for some $b \in (0, 2)$ and $\omega < 0$, then $p^* = \infty$ and there are no global solutions for any $p > 1$.

Note that wherever the statement of the result is that there exist global solutions, Zhang either does not allow for negative V or else requires that $|V|$ be sufficiently small. The reason for this will become clear from Theorem 2 below.

Zhang noted that it seemed difficult to specify the exact value of the critical exponent in the case of quadratic decay; that is in the case that $V(x) \sim \frac{\omega}{|x|^2}$ as $|x| \to \infty$. He also noted that it is unclear whether or not p^* is finite in the case that $V(x) \sim \frac{\omega}{|x|^2}$, with $\omega < 0$.

Very recently, Ishige [6] treated (1.1) for $n \geq 3$ in the case $V(x) \sim \frac{\omega}{|x|^2}$ with $\omega > 0$. Let $\alpha = \alpha(\omega, n)$ denote the larger root of the equation $\alpha(\alpha+n-2) = \omega$; that is

$$\alpha(\omega, n) = \frac{2 - n + \sqrt{(n-2)^2 + 4\omega}}{2}. \tag{1.2}$$

Since we are assuming here that $\omega > 0$, one has $\alpha(\omega, n) > 0$. Define

$$p^*(\omega) = 1 + \frac{2}{n + \alpha(\omega, n)}. \tag{1.3}$$

Theorem (Ishige). *Let* $n \geq 3$ *and assume that* $V \geq 0$. *Let* $\omega > 0$.
i. If $V(x) \geq \frac{\omega}{|x|^2}$ *for large* $|x|$, *then for* $p > p^*(\omega)$ *there exist global solutions to* (1.1)*;*
ii. If $V(x) \leq \frac{\omega}{|x|^2}$ *for large* $|x|$, *then for* $1 < p \leq p^*(\omega)$ *every solution to* (1.1) *blows up in finite time.*

Note that Ishige assumes from the outset that $V \geq 0$. The delicacy between having global solutions and allowing V to take negative values will be explained by Theorem 2 below.

Ishige's proof involved comparison with a solution to the radially symmetric linear equation $v_t = \Delta v - \hat{V}(|x|)v$, where $\hat{V}(r) \sim \frac{\omega}{r^2}$ as $r \to \infty$. The large time behavior of this linear equation, which is needed for the comparison, was recently obtained by Ishige and Kawakami [7].

In this paper, our main focus is the study of the remaining case, $V(x) \sim \frac{\omega}{|x|^2}$, with $\omega < 0$. In fact we treat the following more general problem:

$$\begin{aligned} &u_t = \Delta u - Vu + au^p \text{ in } R^n \times (0, T); \\ &u(x, 0) = \phi(x) \gneq 0, \text{ in } R^n, \end{aligned} \tag{1.4}$$

where $p > 1$, $n \geq 2$, $T \in (0, \infty]$, ϕ is bounded and continuous, $0 \lneq a \in C^\alpha(R^n)$ and $V \in C^\alpha(R^n - \{0\})$ $\alpha \in (0, 1]$. We also require that $\liminf_{x \to 0} V(x) > -\infty$ so that V is locally bounded from below. Our methods, which are completely different from the method employed by Ishige, also allow one to obtain weaker versions of Ishige's results for the case $\omega > 0$, but in the more general context of equation (1.4) with $n \geq 2$. The method of proof also leads naturally to a study of the critical exponent in an exterior domain with the Dirichlet boundary condition in the case $V \equiv 0$.

In the case that $V \equiv 0$ and that a satisfies

(1.5)
$$c_1|x|^m \leq a(x) \leq c_2|x|^m, \text{ for sufficiently large } |x| \text{ and some } m \in (-\infty,\infty),\ c_1, c_2 > 0,$$

the critical exponent p^* for (1.4) was calculated in [12]; it is given by

$$p^* = 1 + \frac{(2+m)^+}{n}. \tag{1.6}$$

In (1.2) we defined $\alpha(\omega,n)$ for $\omega > 0$. We now extend the definition of $\alpha(\omega,n)$ in (1.2) to $\omega \geq -\frac{1}{4}(n-2)^2$. Note that $\alpha(\omega,n) < 0$ for $-\frac{1}{4}(n-2)^2 \leq \omega < 0$. Now define

$$p^*(\omega,m) = 1 + \frac{(2+m)^+}{n+\alpha(\omega,n)}. \tag{1.7}$$

We will prove the following theorem.

Theorem 1. *Let $n \geq 3$ and let $-\frac{1}{4}(n-2)^2 \leq \omega < 0$. Consider* (1.4) *with $a(x)$ satisfying* (1.5)*. Assume that $V \in C^\alpha(R^n - \{0\})$ and that $\liminf_{x\to 0} V(x) > -\infty$. Let $p^*(\omega,m)$ be as in* (1.7)*.*
i. If $V(x) \geq \frac{\omega}{|x|^2}$, then there exist global solutions to (1.4) *for $p > p^*(\omega,m)$;*
ii. If $V(x) \leq \frac{\omega}{|x|^2}$, for sufficiently large $|x|$, then there are no global solutions to (1.4) *for $1 < p \leq p^*(\omega,m)$.*

Remark. Note that in the case of the existence of global solutions, we allow V to be negative up to a precise globally specified size. The reason for this will become clear in Theorem 2.

We now consider what happens when $\omega < -\frac{1}{4}(n-2)^2$, $n \geq 2$. We will show that $p^* = \infty$ under a certain general condition on the operator $-\Delta + V$, and that this condition holds if $V(x) \leq \frac{\omega}{|x|^2}$, for $|x| > \epsilon$, with sufficiently small $\epsilon > 0$.

Let $D \subsetneqq R^n$ be a domain. Then $-\Delta + V$ on D with the Dirichlet boundary condition on ∂D can be realized as a self-adjoint operator on $L^2(D)$. Denoting its spectrum by $\sigma(-\Delta + V; D)$, let

$$\lambda_{0;D}(-\Delta + V) \equiv \inf \sigma(-\Delta + V; D).$$

Theorem 2. *If there exists a domain $D \subsetneqq R^n$ for which $\inf_{x \in D} a(x) > 0$ and $\lambda_{0;D}(-\Delta + V) < 0$, then there are no global solutions to* (1.4) *for any $p > 1$; that is, $p^* = \infty$.*

We can use Theorem 2 to prove the following corollary.

Corollary 1. *Consider* (1.4) *with $a > 0$ on R^n, $n \geq 2$. Let $\omega < -\frac{1}{4}(n-2)^2$. There exists an $\epsilon > 0$ such that if $V(x) \leq \frac{\omega}{|x|^2}$, for $|x| > \epsilon$, then there are no global solutions to* (1.4) *for any $p > 1$; that is, $p^* = \infty$.*

Remark 1. Note that there is a discontinuity in the critical exponent at $\omega = -\frac{1}{4}(n-2)^2$. By Theorem 1, if $V(x) = -\frac{(n-2)^2}{4|x|^2}$, for sufficiently large $|x|$, and $V(x) \geq -\frac{(n-2)^2}{4|x|^2}$, for all x, then the critical exponent is equal to $p^*(-\frac{1}{4}(n-2)^2, m) = 1 + \frac{2(2+m)^+}{n+2}$. However, if $V(x) = \frac{\omega}{|x|^2}$, for some $\omega < -\frac{1}{4}(n-2)^2$ and $|x| > \epsilon$, for sufficiently small $\epsilon > 0$, then the critical exponent is ∞.

Remark 2. Theorem 2 makes it clear why in the theorems of Zhang and of Ishige and in Theorem 1, one needed to be careful with regard to stating the existence of global solutions and allowing V to take negative values. For example, part (iii) of the theorem of Zhang states that if $\frac{\omega}{1+|x|^b} \leq V(x) \leq 0$ for some $b > 2$ and $\omega < 0$, with $|\omega|$ sufficiently small, then the critical exponent for (1.1) is $1 + \frac{2}{n}$. The requirement that $|\omega|$ be sufficiently small is mandatory in light of Theorem 2. Indeed, for any $D \subsetneqq R^n$, if $\omega < 0$ and $|\omega|$ is sufficiently large, then $\lambda_{0;D}(-\Delta + \frac{\omega}{1+|x|^b}) < 0$ and thus, by Theorem 2, one has $p^* = \infty$.

The method of proof in Theorem 1 also yields the following result for the case $\omega > 0$.

Theorem 3. *Let $n \geq 2$ and $\omega > 0$. Consider* (1.4) *with $a(x)$ satisfying* (1.5)*. Assume that $V \in C^\alpha(R^n - \{0\})$. Let $p^*(\omega, m)$ be as in* (1.7)*.*
i. If $V(x) \geq \frac{\omega}{|x|^2}$, then there exist global solutions to (1.4) *for $p > p^*(\omega, m)$.*
ii. If $V(x) \leq \frac{\omega}{|x|^2}$, for sufficiently large $|x|$, then there are no global solutions to (1.4) *for $1 \leq p \leq p^*(\omega, m)$.*

Remark. Note that part (i) requires that V approach ∞ as $|x| \to 0$. In fact, as Ishige has proven in the case that $a \equiv 1$ and $n \geq 3$, the result should hold as long as $V(x) \geq \frac{\omega}{|x|^2}$, for sufficiently large $|x|$, and $V \geq 0$ for all x. However, our method of proof does not seem to be extendable to this situation.

As will be seen below, the method of proof we employ for the blow-up case in Theorems 1 and 3 will lead naturally to a consideration of the critical exponent for the semilinear heat equation in an exterior domain with the Dirichlet boundary condition and with $V \equiv 0$. Let $B_r = \{x \in R^n : |x| < r\}$. Consider the following problem:

$$\begin{aligned} &u_t = \Delta u + a u^p \text{ in } (R^n - \bar{B}_{r_0}) \times (0, T);\\ &u(x,t) = 0, \text{ for } |x| = r_0,\ t \geq 0; \qquad (1.8)\\ &u(x,0) = \phi(x) \gneq 0 \text{ in } R^n - \bar{B}_{r_0}, \end{aligned}$$

where

(1.9)

$$c_1|x|^m \leq a(x) \leq c_2|x|^m, \text{ for sufficiently large } |x| \text{ and some } m \in (-\infty, \infty),\ c_1, c_2 > 0.$$

We prove that restricting to an exterior domain does not affect the blow-up/global solution dichotomy.

Theorem 4. *Let $n \geq 2$. Consider* (1.8) *with $a(x)$ satisfying* (1.9)*. Let*

$$p^* = 1 + \frac{(2+m)^+}{n}$$

as in (1.6)*.*
i. If $1 \leq p \leq p^*$, then there exist global solutions to (1.8)*;*

ii. If $p > p^*$, then there are no global solutions to (1.8).

Remark. In the case $a \equiv 1$, $n \geq 3$ and $p \neq p^*$, the result in Theorem 4 was proven in [2]. For some other works that treat the critical exponent in exterior domains, see [9] and [15]. Most of the results in these papers do not cover the case in which p is equal to the critical exponent.

We end this section with an outline of the methods used to prove Theorems 1 and 3, concentrating on the case of nonexistence of global solutions, which is where our method is novel, and leads to a consideration of the critical exponent in the case of a semilinear heat equation in an exterior domain with the Dirichlet boundary condition. By standard comparison techniques, it suffices to treat the radially symmetric case. Thus, instead of considering solutions $u(x,t)$ of (1.4) with a satisfying (1.5), we may consider solutions $u(r,t)$ of the equation

$$\begin{aligned} &u_t = u_{rr} + \frac{n-1}{r}u_r - V(r)u + a(r)u^p \text{ in } (0,\infty)\times(0,T); \\ &u(r,0) = \phi(r) \gneq 0 \text{ in } [0,\infty), \end{aligned} \tag{1.10}$$

where $p > 1$, $T \in (0,\infty]$, ϕ is bounded and continuous, $V \in C^\alpha((0,\infty))$ and $\liminf_{r\to 0} V(r) > -\infty$, $0 \lneq a \in C^\alpha([0,\infty))$, $\alpha \in (0,1]$, with a satisfying

$$c_1 r^m \leq a(r) \leq c_2 r^m, \text{ for sufficiently large } r \text{ and some } m \in (-\infty,\infty),\ c_1, c_2 > 0. \tag{1.11}$$

For the existence of global solutions when $p > p^*(\omega, m)$ in part (i) of Theorems 1 and 3, we construct a global super-solution to (1.10). Note that in general it is much more difficult to use the method of super/sub-solutions to prove blow-up, since the construction of an appropriate sub-solution would probably require a reasonable knowledge of the blow-up profile.

We now turn to the nonexistence of global solutions when $1 < p \leq p^*(\omega, m)$ in part (ii) of Theorems 1 and 3, We may assume without loss of generality that the initial data ϕ in (1.10) satisfy $\phi(r) > 0$ for all $r > 0$. Indeed, if this is not the case, then for $\delta > 0$ sufficiently small, we can consider $\bar{u}(r,t) \equiv u(r,t+\delta)$, which also satisfies (1.10) and is strictly positive at

$t = 0$. We apply a transformation as follows. Let u be a solution to (1.10) and define $v(r,t) = r^{-\alpha}u(r,t)$. Let $\psi(r) = r^{-\alpha}\phi(r)$. Then one calculates that

(1.12)
$$v_t = v_{rr} + \frac{n-1+2\alpha}{r}v_r + \left(\frac{\alpha(\alpha+n-2)}{r^2} - V(r)\right)v + r^{\alpha(p-1)}a(r)v^p$$
$$\text{in } (0,\infty)\times(0,T);$$
$$v(r,0) = \psi(r) > 0 \text{ in } (0,\infty).$$

There will be global solutions of v if and only if there are global solutions of u; thus it suffices to study (1.12). In part (ii) of Theorems 1 and 3, we are assuming that $V(r) \le \frac{\omega}{r^2}$, for sufficiently large r, say for $r \ge r_0$, where $\omega \ge -\frac{1}{4}(n-2)^2$. If one now chooses $\alpha = \alpha(\omega, n)$ as in (1.2), then the coefficient of v in (1.12) is nonnegative for $r \ge r_0$. By the comparison principle, the solution to that equation dominates the solution to the equation

$$\begin{aligned} &w_t = w_{rr} + \frac{N-1}{r}w_r + \hat{a}(r)w^p \text{ in } (r_0,\infty)\times(0,T);\\ &w(r,0) = \psi(r) > 0 \text{ in } [r_0,\infty);\\ &w(r_0,t) = 0,\ t > 0, \end{aligned} \tag{1.13}$$

where

$$N \equiv n + 2\alpha(\omega, n) \tag{1.14}$$

and $\hat{a}(r) = r^{\alpha(p-1)}a(r)$. In terms of $\hat{a}$, the assumption (1.11) on a is

$$c_1 r^M \le \hat{a}(r) \le c_2 r^M,\ \text{for sufficiently large } r,\ c_1, c_2 > 0, \tag{1.15}$$

where

$$M \equiv \alpha(\omega, n)(p-1) + m,\ m \in (-\infty, \infty). \tag{1.16}$$

(The reason we insisted on $\phi(r) > 0$ for all $r > 0$, and thus also $\psi(r) > 0$ for all $r > 0$, is that otherwise we could have ended up with $\psi \equiv 0$ in (1.13).) Thus, it suffices to show that there are no global solutions to (1.13)-(1.16).

Now (1.13)-(1.16) is the radial version of (1.4)-(1.5) in the case $V \equiv 0$, except that we have placed the Dirichlet boundary condition at $r = r_0$ instead of considering the problem for all $r > 0$, and except that m in (1.5) is replaced by M and the dimension n is replaced by the "dimension" N. (Note from the definition of $\alpha(\omega, n)$ that one always has $N \geq 2$.) The critical exponent p^* for (1.4) with $V \equiv 0$ and with a satisfying (1.5) was given in (1.6). Substituting N and M for n and m in (1.6), it is not unreasonable to suspect that no global solutions will exist if

$$1 < p \leq 1 + \frac{(2+M)^+}{N} = 1 + \frac{(2+\alpha(\omega,n)(p-1)+m)^+}{n+2\alpha(\omega,n)}. \tag{1.17}$$

We now solve (1.17) for p. Consider first the case $\omega < 0$, in which case $\alpha(\omega, n) < 0$. Since we are assuming that $p > 1$, (1.17) will never hold if $2 + \alpha(\omega, n)(p-1) + m \leq 0$; that is, if

$$p \geq 1 - \frac{2+m}{\alpha(\omega,n)}. \tag{1.18}$$

On the other hand, if $2 + \alpha(\omega, n)(p-1) + m > 0$, then solving (1.17) for p gives

$$1 < p \leq 1 + \frac{2+m}{n+\alpha(\omega,n)}. \tag{1.19}$$

One can check that for $m > -2$, the right hand side of (1.19) is strictly less than the right hand side of (1.18). From this fact along with (1.18) and (1.19), we conclude that (1.17) holds if and only if $1 < p \leq p^*(\omega, m)$, where $p^*(\omega, m)$ is as in (1.7).

Now consider the case $\omega > 0$, in which case $\alpha(\omega, n) > 0$. If $2 + \alpha(\omega, n)(p-1) + m > 0$, then solving (1.17) as we did above gives (1.19). On the other hand, if $2 + \alpha(\omega, n)(p-1) + m \leq 0$ (which implies that $m < -2$), then (1.17) does not hold. Putting these facts together leads again to (1.17) holding if and only if $1 < p \leq p^*(\omega, m)$.

To turn the above argument into a rigorous proof, we need to show that indeed no global solutions exist for (1.13)-(1.16) when $1 < p \leq 1 + \frac{(2+M)^+}{N}$. That is we need to show that the proof in [12], which treated the operator Δ

in R^n (whose radial part is $\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}$), can accommodate two changes: (1) operators of the form $\frac{d^2}{dr^2}+\frac{N-1}{r}\frac{d}{dr}$ with fractional N and (2) the Dirichlet boundary condition at $r=r_0$, which serves to make solutions smaller. The proof in [12] made rather heavy use of the explicit form of the heat kernel $p(t,x,y)=(4\pi t)^{-\frac{n}{2}}\exp(-\frac{|y-x|^2}{4t})$ for the corresponding linear operator $\Delta-\frac{\partial}{\partial t}$ in R^n. In the present case, the corresponding linear operator is $\frac{\partial^2}{\partial r^2}+\frac{N-1}{r}\frac{\partial}{\partial r}-\frac{\partial}{\partial t}$ with the Dirichlet boundary condition at $r=r_0$. It turns out that if $N>2$ (equivalently, $\omega>-\frac{1}{4}(n-2)^2$), then the heat kernel for this operator is comparable in an appropriate sense to the heat kernel for $\frac{\partial^2}{\partial r^2}+\frac{N-1}{r}\frac{\partial}{\partial r}-\frac{\partial}{\partial t}$ on the entire space $r>0$; thus, we will be able to use this latter heat kernel, which we can exhibit explicitly. However, this latter heat kernel is a much less convenient object than the Gaussian heat kernel. In fact, this obstacle prevented us from using the method of proof in [12] to prove the existence of global solutions above the critical exponent; hence the use of super-solutions. However, we were able to use this heat kernel and amend the nonexistence proof in [12] at or below the critical value. When $N=2$ (equivalently, $\omega=-\frac{1}{4}(n-2)^2$), the heat kernel with the Dirichlet boundary condition is not comparable to the heat kernel on the whole space, however an appropriate lower bound is known and sufficient for our needs.

In section 2 we prove the existence of global solutions in part (i) of Theorems 1 and 3. In section 3 we prove the nonexistence of global solutions in part (ii) of Theorems 1 and 3. In section 4 we prove Theorem 4. In section 5 we prove Theorem 2 and Corollary 1.

2. Proofs of Part (i) of Theorems 1 and 3

We assume that $p>p^*(\omega,m)$, where $p^*(\omega,m)$ is as in (1.7). As noted in the first section of the paper, instead of studying (1.4) with a satisfying (1.5), it suffices to study the radial problem (1.10) with a satisfying (1.11). By the standard theory, it suffices to exhibit a global super-solution. We look for such a super-solution in the form

$$v(r,t)=\delta\frac{r^{\alpha}}{(t+1)^{\gamma}}\exp(-\frac{cr^2}{t+1}),$$

for some $\delta, c>0$ and some $\alpha,\gamma\in(-\infty,\infty)$. We have

$$v_r=(\frac{\alpha}{r}-\frac{2cr}{t+1})v; \tag{2.1}$$

$$v_{rr}=(\frac{\alpha^2}{r^2}+\frac{4c^2r^2}{(t+1)^2}-\frac{4c\alpha}{t+1}-\frac{\alpha}{r^2}-\frac{2c}{t+1})v; \tag{2.2}$$

$$v_t=(-\frac{\gamma}{t+1}+\frac{cr^2}{(t+1)^2})v. \tag{2.3}$$

The condition on V in part (i) of Theorems 1 and 3 is that $V(r)\ge\frac{\omega}{r^2}$, with $-\frac{(n-2)^2}{4}\le\omega<0$ in Theorem 1 and $\omega>0$ in Theorem 3. Using this along with (2.1), (2.2) and (2.3), we have

$$\begin{aligned}&v^{-1}(v_{rr}+\frac{n-1}{r}v_r-V(r)v-v_t+a(r)v^p)\le\\ &(4c^2-c)\frac{r^2}{(t+1)^2}+\frac{\alpha^2+(n-2)\alpha-\omega}{r^2}+\frac{\gamma-4c\alpha-2cn}{t+1}\\ &+\delta^{p-1}a(r)\frac{r^{\alpha(p-1)}}{(t+1)^{\gamma(p-1)}}\exp(-\frac{c(p-1)r^2}{t+1}).\end{aligned} \tag{2.4}$$

In order to make the first term on the right hand side of (2.4) vanish, we choose $c=\frac{1}{4}$, and in order to make the second term on the right hand side of (2.4) vanish, we choose $\alpha=\alpha(\omega,n)$ as in (1.2).

If $m\le0$, the assumption on a in (1.11) guarantees that for some $C>0$, $a(r)\le Cr^m$, for all $r>0$. If $m>0$, the assumption on a in (1.11) guarantees that for some $C>0$, $a(r)\le C(r\vee1)^m$, for all $r>0$. This forces us to break up the next part of the proof into two cases. We will continue the proof under the assumption that $m\le0$. After the completion of this case, it will be easy to point out how to handle the case $m>0$.

Since $a(r)\le Cr^m$, the final term on the right hand side of (2.4) (with $c=\frac{1}{4}$ and $\alpha=\alpha(\omega,n)$) is bounded from above by $C\delta^{p-1}\frac{r^{\alpha(\omega,n)(p-1)+m}}{(t+1)^{\gamma(p-1)}}\exp(-\frac{(p-1)r^2}{4(t+1)})$.

Letting $z=\frac{r^2}{t+1}$, this upper bound can be written as

$$\frac{C\delta^{p-1}z^{\frac12\alpha(\omega,n)(p-1)+\frac12 m}\exp(-\frac14(p-1)z)}{(t+1)^{(\gamma-\frac12\alpha(\omega,n))(p-1)-\frac12 m}},$$

which is itself bounded from above by $\frac{C_1C\delta^{p-1}}{(t+1)^{(\gamma-\frac12\alpha(\omega,n))(p-1)-\frac12 m}}$, where $C_1=\sup_{z>0}z^{\frac12\alpha(\omega,n)(p-1)+\frac12 m}\exp(-\frac14(p-1)z)$. In light of the above analysis, it follows from (2.4) that

$$v(r,t)=\delta\frac{r^{\alpha(\omega,n)}}{(t+1)^{\gamma}}\exp(-\frac{r^2}{4(t+1)}) \tag{2.5}$$

satisfies

$$\begin{aligned}&v^{-1}(v_{rr}+\frac{n-1}{r}v_r-V(r)v-v_t+a(r)v^p)\le\\&\frac{\gamma-\alpha(\omega,n)-\frac12 n}{t+1}+\frac{C_1C\delta^{p-1}}{(t+1)^{(\gamma-\frac12\alpha(\omega,n))(p-1)-\frac12 m}}.\end{aligned} \tag{2.6}$$

If

$$\gamma-\alpha(\omega,n)-\frac12 n<0 \tag{2.7}$$

and

$$(\gamma-\frac12\alpha(\omega,n))(p-1)-\frac12 m\ge 1, \tag{2.8}$$

then after choosing $\delta>0$ sufficiently small, the right hand side of (2.6) will be non-positive. The two inequalities (2.7) and (2.8) together are equivalent to

$$\frac12\alpha(\omega,n)+\frac{1+\frac12 m}{p-1}\le\gamma<\alpha(\omega,n)+\frac12 n,$$

and this latter pair of inequalities can be solved for γ if and only if $\alpha(\omega,n)+\frac{2+m}{p-1}<2\alpha(\omega,n)+n$, or equivalently, if and only if $p>1+\frac{2+m}{n+\alpha(\omega,n)}$. Since we have assumed from the outset that $p>1$, we conclude that if $p>1+\frac{(2+m)^+}{n+\alpha(\omega,n)}=p^*(\omega,m)$, then it is possible to choose γ so that (2.7) and (2.8) hold.

In the case $m>0$, we have $a(r)\le Cr^m$, if $r\ge 1$, and $a(r)\le Cr^0$, if $0<r<1$. Thus, in order for the above analysis to go through in this case, we need to have (2.8) hold as it is written and also with m replaced by 0.

However, since $m > 0$, if (2.8) holds as it is written, then it holds a fortiori with m replaced by 0.

In the case $\omega > 0$, the function v given by (2.5) with $\delta > 0$ sufficiently small and γ chosen to satisfy (2.7) and (2.8) serves as an appropriate global super-solution.

In the case $\omega < 0$, there is one technical problem; namely, that $\alpha(\omega, n) < 0$ and thus v is not finite at $r = 0$. This artificial singularity arises from the use of polar coordinates. Unfortunately, if one replaces r by $r + c$ for some $c > 0$, then v will no longer be a super-solution. Thus, we argue as follows. Consider ω and $p > p^*(\omega, n)$ as fixed. Our work so far allows us to conclude that for sufficiently small initial data ϕ, the solution $u(x,t)$ of (1.4) satisfies $u(x,t) \leq v(|x|,t)$ up until some possibly finite blow-up time. Choose $\epsilon > 0$ sufficiently small so that $p > p^*(\omega - \epsilon, n)$. The function v in (2.5) was shown to be a super-solution for (1.4) under the assumption that the potential V satisfies $V(x) \geq \frac{\omega}{|x|^2}$. Recall that in (1.4) we are also assuming that V is locally bounded from below. Therefore, there exists an $r_0 > 0$ such that $V(x) \geq \frac{\omega}{r_0^2}$ for $|x| \leq r_0$. One can check that it is then possible to choose an $x_0 \neq 0$ such that $V(x) \geq \frac{\omega-\epsilon}{|x-x_0|^2}$. Now consider the radial version (1.10) of (1.4) but with the origin shifted to the point x_0. Call the new radial variable $\rho = |x - x_0|$. Since we have $V(\rho) \geq \frac{\omega-\epsilon}{\rho^2}$, the construction above shows that there exists a function $\hat{v}(\rho, t) = \hat{\delta}\frac{\rho^{\alpha(\omega-\epsilon,n)}}{(t+1)^{\hat{\gamma}}}\exp(-\frac{\rho^2}{4(t+1)})$ such that for sufficiently small initial data ϕ, the solution $u(x,t)$ of (1.4) satisfies $u(x,t) \leq \hat{v}(|x - x_0|, t)$ up until some possibly finite blow-up time. We conclude that for sufficiently small initial data ϕ, the solution $u(x,t)$ of (1.4) satisfies $u(x,t) \leq \hat{v}(|x - x_0|, t) \wedge v(|x|, t)$ up until its blow-up time. But the right hand side is finite for all x and t. Thus u is in fact a global solution.

3. Proofs of Part (ii) of Theorems 1 and 3

As was shown at the end of section 1, in order to prove that when $1 < p \le p^*(\omega, m)$ there are no global solutions to (1.4) with a satisfying (1.5), it suffices to show that there are no global solutions for (1.13)-(1.16) when p satisfies (1.17). We will always assume that $M > -2$ since otherwise there is nothing to prove. We wish to employ the method of proof used in [12]. This method requires a fairly explicit knowledge of the heat kernel for the corresponding linear equation. In the present case, the linear equation is $W_t = W_{rr} + \frac{N-1}{r}W_r$ with $(r,t) \in (r_0,\infty) \times (0,\infty)$, for some possibly fractional N with $N \ge 2$, and with the Dirichlet boundary condition at $r = r_0$. Denote the heat kernel for this equation by $\bar{q}_{(N,r_0)}(t,r,\rho)$.

Denote by $q_{(N)}(t,r,\rho)$ the heat kernel for the equation $W_t = W_{rr} + \frac{N-1}{r}W_r$ with $(r,t) \in (0,\infty) \times (0,\infty)$. The kernel $q_{(N)}(t,r,\rho)$ is the transition probability density for the Bessel process of order N, and is given by [5]

$$q_{(N)}(t,r,\rho) = \exp(-\frac{r^2+\rho^2}{4t})\frac{\rho^{N-1}}{2t(r\rho)^{\frac{N}{2}-1}}I_{\frac{N}{2}-1}(\frac{r\rho}{2t}), \tag{3.1}$$

where I_ν is the modified Bessel function of order ν, given by

$$I_\nu(x) = (\frac{x}{2})^\nu \sum_{n=0}^{\infty} \frac{(\frac{x}{2})^{2n}}{n!\Gamma(\nu+n+1)}. \tag{3.2}$$

By the maximum principle, $\bar{q}_{(N,r_0)}(t,r,\rho) \le q_{(N)}(t,r,\rho)$. What we need, however, is an appropriate inequality in the reverse direction.

If $N > 2$ (equivalently, $\omega > -\frac{1}{4}(n-2)^2$), then the Bessel process corresponding to the operator $\frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}$ is transient [11]. Furthermore, as will be explained momentarily, the uniform parabolic Harnack inequality holds for the heat equation $W_t = W_{rr} + \frac{N-1}{r}W_r$ on $r > 0$. Thus, it follows from [4] that there exist constants $K_0, c > 0$ such that

(3.3)

$$\bar{q}_{(N,r_0)}(t,r,\rho) \ge c q_{(N)}(K_0 t,r,\rho), \text{ for } r > r_0+1,\ \rho > r_0+1,\ t > 0 \text{ and } N > 2.$$

(The uniform parabolic Harnack inequality concerns nonnegative solutions W of $W_t = W_{rr} + \frac{N-1}{r}W_r$ on $r > 0$ on a time interval $[\tau, \tau + T]$. See [4, Definition 2.2] for the precise definition. Any such solution can be represented as $W(r, \tau + t) = \int_0^\infty q_{(N)}(t, r, \rho)W(\rho, \tau)d\rho$, $0 \le t \le T$. Using the explicit formula for $q_{(N)}$ in (3.1), one can verify the uniform parabolic Harnack inequality. Indeed, in the case that N is an integer, the above heat equation is just the radial form of the standard heat equation on R^N, and it is well-known that the uniform Harnack inequality holds in this case [10].)

The following key a priori lower bound on solutions to (1.13)-(1.16) in the case that $1 < p \le 1 + \frac{(2+M)^+}{N}$ will be used to prove the theorem. Then we will come back to prove the lemma.

Lemma 1. *Let w be a solution to* (1.13)-(1.16) *on a time interval* $0 < t < T$, *with* $1 < p \le 1 + \frac{(2+M)^+}{N}$ *and* $N > 2$. *Then for some* $K, C > 0$,

$$(3.4) \quad w(r, t) \ge Ct^{-\frac{N}{2}} \log(1 + t) \exp(-\frac{Kr^2}{t}), \text{ for } 2 < t < T, \ r > r_0 + 1.$$

Remark. The proof of Lemma 1 makes use of (3.3). If $N = 2$, a weaker lower bound holds for $\bar{q}_{(N,r_0)}$ in terms of $q_{(N)}$. This weaker bound is enough to prove (3.4) when $N = 2$ with the restriction that $r \ge t^{\frac{1}{2}}$. See Lemma 2 and (4.2). As the proof of Theorem 1 below shows, it is enough to have the estimate (3.4) for $r \ge t^{\frac{1}{2}}$.

In light of the above remark, (3.4) holds for all $N \ge 2$ and $r \ge t^{\frac{1}{2}}$. We now use this to prove the theorem.

Proof of Theorem 1. Assume that $w(r, t)$ is a global solution to (1.13)-(1.16). For $n > r_0 + 1$, define

$$F_n(t) = \int_n^{2n} w(r, t)\phi^{(n)}(r)r^{N-1}dr,$$

where $\phi^{(n)} > 0$, normalized by $\int_n^{2n} \phi^{(n)}(r)r^{N-1}dr = 1$, is the eigenfunction corresponding to the principal eigenvalue $\lambda_n > 0$ for the operator $-(\frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}) = -r^{1-N}\frac{d}{dr}r^{N-1}\frac{d}{dr}$ on $(n, 2n)$ with the Dirichlet boundary condition at the endpoints. For an appropriate value of n, we will show that F_n blows up in finite time, thereby contradicting the assumption that w is a global solution.

From the outset, we assume that n is sufficiently large so that (1.15) holds for $r \geq n$. Simple scaling shows that λ_n is on the order $\frac{1}{n^2}$ as $n \to \infty$. In particular then, there exists a constant $c > 0$ such that $\lambda_n \leq \frac{c}{n^2}$. Since $\phi^{(n)}(n) = \phi^{(n)}(2n) = 0$, one has $(\phi^{(n)})'(n) \geq 0$ and $(\phi^{(n)})'(2n) \leq 0$. Using the facts in this paragraph, integrating by parts and using Jensen's inequality, we have

$$
\begin{aligned}
F_n'(t) &= \int_n^{2n} w_t(r,t)\phi^{(n)}(r)r^{N-1}dr \\
&= \int_n^{2n} \left(w_{rr}(r,t) + \frac{N-1}{r} w_r(r,t) + \hat{a}(r) w^p(r,t) \right) \phi^{(n)}(r) r^{N-1} dr \\
&= \int_n^{2n} (r^{N-1} w_r(r,t))_r \phi^{(n)}(r) dr + \int_n^{2n} \hat{a}(r) w^p(r,t) \phi^{(n)}(r) r^{N-1} dr \\
&\geq \int_n^{2n} (r^{N-1}\phi_r^{(n)}(r))_r w(r,t) dr + c_1 n^M \int_n^{2n} w^p(r,t)\phi^{(n)}(r) r^{N-1} dr \\
&= -\lambda_n F_n(t) + c_1 n^M \int_n^{2n} w^p(r,t)\phi^{(n)}(r) r^{N-1} dr \\
&\geq -\frac{c}{n^2} F_n(t) + c_1 n^M F_n^p(t).
\end{aligned} \tag{3.5}
$$

The function $-\frac{c}{n^2}x + c_1 n^M x^p$ is both positive and increasing for $x > (\frac{c}{c_1})^{\frac{1}{p-1}} n^{-\frac{M+2}{p-1}}$. Therefore, if there exists an n and a T_n for which $F_n(T_n) > (\frac{c}{c_1})^{\frac{1}{p-1}} n^{-\frac{M+2}{p-1}}$, then it follows from (3.5) and the fact that $p > 1$ that $F_n(t)$ will blow up at some finite value of t. From Lemma 1 and the remark following it, we obtain $w(r, n^2) \geq C_1 n^{-N} \log n$, for $n \leq r \leq 2n$ and some $C_1 > 0$. Thus, $F_n(n^2) \geq C_1 n^{-N} \log n$. Since $1 < p \leq 1 + \frac{(2+M)^+}{N}$, one can choose n sufficiently large so that $F_n(n^2) \geq C_1 n^{-N} \log n > (\frac{c}{c_1})^{\frac{1}{p-1}} n^{-\frac{M+2}{p-1}}$. □

Proof of Lemma 1. The solution W to the corresponding linear problem $W_t = W_{rr} + \frac{N-1}{r} W_r$ with the Dirichlet boundary condition at $r = r_0$ and with initial data ψ is given by

$$W(r,t)=\int_{r_0}^{\infty}\bar{q}_{(N,r_0)}(t,r,\rho)\psi(\rho)d\rho. \tag{3.6}$$

By comparison, the solution w to (1.13)-(1.16) satisfies

$$w\geq W. \tag{3.7}$$

On the other hand, the solution w to (1.13)-(1.16) satisfies the inequality

(3.8)

$$w(r,t)\geq\int_{r_0}^{\infty}\bar{q}_{(N,r_0)}(t,r,\rho)\psi(\rho)d\rho+\int_0^t ds\int_{r_0}^{\infty}d\rho\,\bar{q}_{(N,r_0)}(t-s,r,\rho)\hat{a}(\rho)w^p(\rho,s).$$

(See [12] and [13], where it is also shown that under appropriate conditions, (3.8) holds with an equality.) Without loss of generality, we assume that r_0+2 is contained in the support of ψ appearing in (3.6). From (3.1)-(3.3) and (3.6)-(3.8) it then follows that

(3.9)

$$w(r,t)\geq c_1\int_0^t ds\int_{r_0+1}^{\infty}d\rho\,q_{(N)}(K_1(t-s),r,\rho)\hat{a}(\rho)q^p_{(N)}(K_1s,\rho,r_0+2),\ r>r_0+1,$$

for some $K_1,c_1>0$.

In the case that N is an integer, which we denote by N_0, $q_{(N_0)}$ is just the standard N_0-dimensional Gaussian heat kernel in radial coordinates, and (3.9) can be converted back to N_0-dimensional Euclidean coordinates. In [12], the right hand side of (3.9) (converted to Euclidean coordinates and with some other inessential differences) was shown to satisfy the inequality

$$\int_1^{\frac{t}{2}}ds\int_{r_0+1}^{\infty}d\rho\ q_{(N_0)}(K_1(t-s),r,\rho)\hat{a}(\rho)q^p_{(N_0)}(K_1s,\rho,r_0+2)\geq$$

$$\begin{cases} Ct^{1-\frac{N_0}{2}p+\frac{M}{2}}\exp(-\frac{Kr^2}{t}),\ \text{if } p<1+\frac{2+M}{N_0}, \\ Ct^{-\frac{N_0}{2}}\log(1+t)\exp(-\frac{Kr^2}{t}),\ \text{if } p=1+\frac{2+M}{N_0}, \end{cases} \tag{3.10}$$

for $t>2,\ r>r_0+1$,

where $K,C>0$. Recall that we are assuming that $M>-2$. Note that $1-\frac{N_0}{2}p+\frac{M}{2}>-\frac{N_0}{2}$, if $p<1+\frac{2+M}{N_0}$, and $1-\frac{N_0}{2}p+\frac{M}{2}=-\frac{N_0}{2}$, if $p=1+\frac{2+M}{N_0}$. Thus, from (3.9) and (3.10) it follows immediately that (3.4) holds for $N = N_0$. (For (3.10) and (3.4) with $N = N_0$, see the statements and proofs of [12, Lemma 2, Proposition 1 and Lemma 3]. The spatial integral in [12] is over all of R^{N_0}, which would correspond here to $\rho > 0$. But one could have worked just as well with $|x| > r_0 + 1$ in [12], so the restriction here to $\rho > r_0 + 1$ in the spatial integral causes no problem.)

We now proceed to demonstrate that (3.10), and consequently also (3.4), continue to hold in the case that N_0 is replaced by any non-integral $N > 2$. We write $N = N_0 - \beta$, where $N_0 \geq 3$ is an integer and $\beta \in (0,1)$. Let $K_\nu(x) \equiv (\frac{x}{2})^{-\nu} I_\nu(x)$, and note from the definition of I_ν in (3.2) that $K_\nu(x)$ is decreasing in ν. Thus, we have from (3.1),

(3.11)
$$\begin{aligned} q_{(N)}(t,r,\rho) &= \exp(-\frac{r^2+\rho^2}{4t})\frac{\rho^{N-1}}{2t(r\rho)^{\frac{N}{2}-1}}(\frac{r\rho}{2t})^{\frac{N}{2}-1}K_{\frac{N}{2}-1}(\frac{r\rho}{2t}) \\ &= \exp(-\frac{r^2+\rho^2}{4t})\frac{\rho^{N_0-1}}{2t(r\rho)^{\frac{N_0}{2}-1}}(\frac{r\rho}{2t})^{\frac{N_0}{2}-1}K_{\frac{N}{2}-1}(\frac{r\rho}{2t})\left(\frac{\rho^{-\beta}}{(r\rho)^{-\frac{\beta}{2}}}(\frac{r\rho}{2t})^{-\frac{\beta}{2}}\right) \\ &\geq \frac{(2t)^{\frac{\beta}{2}}}{\rho^\beta}q_{(N_0)}(t,r,\rho). \end{aligned}$$

From (3.11) we have

$$\begin{aligned} &q_{(N)}(K_1(t-s),r,\rho)q^p_{(N)}(K_1 s,\rho,r_0+2) \\ &\geq C_1\frac{t^{\frac{\beta}{2}}}{\rho^\beta}s^{\frac{\beta}{2}p}q_{(N_0)}(K_1(t-s),r,\rho)q^p_{(N_0)}(K_1 s,\rho,r_0+2), \\ &\text{for } 1 \leq s \leq \frac{t}{2},\ 0 \leq \rho < \infty, \end{aligned} \tag{3.12}$$

for some $C_1 > 0$. From (4.8) we have

(3.13)
$$\begin{aligned} &\int_1^{\frac{t}{2}} ds \int_{r_0+1}^\infty d\rho\ q_{(N)}(K_1(t-s),r,\rho)\hat{a}(\rho)q^p_{(N)}(K_1 s,\rho,r_0+2) \geq \\ &C_1 t^{\frac{\beta}{2}} \int_1^{\frac{t}{2}} ds \int_{r_0+1}^\infty d\rho\ \rho^{-\beta}s^{\frac{\beta}{2}p}q_{(N_0)}(K_1(t-s),r,\rho)\hat{a}(\rho)q^p_{(N_0)}(K_1 s,\rho,r_0+2). \end{aligned}$$

Note that the only difference between the terms appearing inside the double integral on the right hand side of (3.13) and the terms appearing inside the double integral on the left hand side of (3.10) is the addition of the factors $\rho^{-\beta}$ and $s^{\frac{\beta}{2}}$. Translating the setup and notation in the proof of (3.10) in [12] to the present situation, we note that the integration over ρ introduced a term of the form $((t-s)r(s,t))^{\frac{M}{2}}$, where $r(s,t)=\frac{s}{s+pK_2(t-s)}$, for some $K_2>0$, and the exponent $\frac{M}{2}$ was a consequence of $\hat{a}$ being on the order ρ^M. Since $\hat{a}(\rho)$ is replaced by $\rho^{-\beta}\hat{a}(\rho)$ in (3.13), in the present situation we obtain a term of the form $((t-s)r(s,t))^{\frac{M}{2}-\frac{\beta}{2}}$; see [12, (2.34)-(2.37)]. Thus, whereas in the penultimate step in the proof of (3.10) in [12] we obtained

$$\int_1^{\frac{t}{2}} ds \int_{r_0+1}^{\infty} d\rho\ q_{(N_0)}(K_1(t-s),r,\rho)\hat{a}(\rho)q^p_{(N_0)}(K_1 s,\rho,r_0+2) \geq$$
$$C_2\exp(-\frac{Kr^2}{t})\int_1^{\frac{t}{2}} s^{-\frac{N_0}{2}p}(r(s,t))^{\frac{N_0}{2}+\frac{M}{2}}(t-s)^{\frac{M}{2}}ds,$$

for some $K>0$ (see [12, (2.37)]), we obtain here

(3.14)
$$t^{\frac{\beta}{2}}\int_1^{\frac{t}{2}} ds \int_{r_0+1}^{\infty} d\rho\ \rho^{-\beta}s^{\frac{\beta}{2}p}q_{(N_0)}(K_1(t-s),r,\rho)\hat{a}(\rho)q^p_{(N_0)}(K_1 s,\rho,r_0+2)$$
$$\geq C_2 t^{\frac{\beta}{2}}\exp(-\frac{Kr^2}{t})\int_1^{\frac{t}{2}} s^{-\frac{N_0}{2}p+\frac{\beta}{2}p}(r(s,t))^{\frac{N_0}{2}+\frac{M}{2}-\frac{\beta}{2}}(t-s)^{\frac{M}{2}-\frac{\beta}{2}}ds.$$

Making the change of variables $u=\frac{s}{t}$ and recalling that $N_0-\beta=N$, we have

$$t^{\frac{\beta}{2}}\int_1^{\frac{t}{2}} s^{-\frac{N_0}{2}p+\frac{\beta}{2}p}(r(s,t))^{\frac{N_0}{2}+\frac{M}{2}-\frac{\beta}{2}}(t-s)^{\frac{M}{2}-\frac{\beta}{2}}ds =$$
$$t^{1+\frac{M}{2}-\frac{N}{2}p}\int_{\frac{1}{t}}^{\frac{1}{2}} u^{\frac{N}{2}+\frac{M}{2}-\frac{N}{2}p}(u+pK_2(1-u))^{-\frac{N}{2}-\frac{M}{2}}(1-u)^{\frac{M}{2}-\frac{\beta}{2}}du. \tag{3.15}$$

If $p<1+\frac{2+M}{N}$, then $\frac{N}{2}+\frac{M}{2}-\frac{N}{2}p>-1$ and the integral on the right hand side of (3.15) is bounded in t. However if $p=1+\frac{2+M}{N}$, then $\frac{N}{2}+\frac{M}{2}-\frac{N}{2}p=-1$ and that integral is on the order of $\log t$. Using this fact along with (3.13)-(3.15), we conclude that (3.10) holds with the integer N_0 replaced by non-integral N. From this and (3.9) we then also obtain (3.4) with the integer N_0 replaced by non-integral N. This completes the proof of Lemma 1.

□

4. Proof of Theorem 4

Note that (1.13)-(1.16) with N equal to an integer is the radial version of (1.8)-(1.9) (with N and M identified with n and m). Thus, in fact, Lemma 1 and the proof of Theorem 1 given in section 3 give a proof of Theorem 4 in the case $n \geq 3$. If we prove the equivalent of Lemma 1 for $n = 2$, then we will also have a proof of Theorem 4 for $n = 2$. In fact, as the proof of Theorem 1 showed, it suffices to have the estimate on $w(r,t)$ in Lemma 1 for $r \geq t^{\frac{1}{2}}$. Thus, it suffices to prove the following result.

Lemma 2. *Let w be a solution to* (1.8) *with $n = 2$ on a time interval $0 < t < T$, with $1 < p \leq 1 + \frac{(2+m)^+}{2}$. Then for some $K, C > 0$,*

$$w(x,t) \geq Ct^{-1}\log(1+t)\exp(-\frac{K|x|^2}{t}),\ \text{for } |x| > t^{\frac{1}{2}} \text{ and } 5 < t < T. \tag{4.1}$$

Proof. We assume that $m > -2$ since otherwise there is nothing to prove. Let $p(t,x,y) = (4\pi t)^{-1}\exp(-\frac{|y-x|^2}{4t})$ denote the heat kernel for the Laplacian on R^2, and let $\bar{p}_{r_0}(t,x,y)$ denote the corresponding heat kernel for the Laplacian on $R^2 - \bar{B}_{r_0}$ with the Dirichlet boundary condition at $|x| = r_0$. It was shown in [4] that for appropriate constants $c_0, K_0 > 0$, one has

(4.2)
$$\bar{p}_{r_0}(t,x,y) \geq c_0 \frac{\log(1+|x|)\log(1+|y|)}{\big(\log(1+\sqrt{t}) + \log(1+|x|)\big)\big(\log(1+\sqrt{t}) + \log(1+|y|)\big)} p(K_0 t, x, y),$$
for $|x| > r_0 + 1,\ |y| > r_0 + 1,\ t > 0$.

We now follow to a significant degree the proof of blow-up in [12]. Similar to [12, Lemma 1], we have

(4.3)
$$w(x,t) \geq \int_{R^2-\bar{B}_{r_0}} \bar{p}_{r_0}(t,x,y)\phi(y)dy + \int_0^t \int_{R^2-\bar{B}_{r_0}} \bar{p}_{r_0}(t-s,x,y)a(y)w^p(y,s)dyds.$$

The first term on the right hand side of (4.3), which is the solution of the corresponding linear problem, constitutes a lower bound for w. Thus, using

(4.2), we have similar to [12, Lemma 2],

$$w(x,t) \geq ct^{-1}\exp(-\frac{|x|^2}{2K_0t})\frac{\log(1+|x|)}{\big(\log(1+\sqrt{t})+\log(1+|x|)\big)\big(\log(1+\sqrt{t})\big)}, \tag{4.4}$$

for some $c>0$. Note that for $|x|\geq t^{\frac{1}{2}}$, $|y|\geq t^{\frac{1}{4}}$ and $t\geq 1$, the expression $\frac{\log(1+|x|)\log(1+|y|)}{\big(\log(1+\sqrt{t})+\log(1+|x|)\big)\big(\log(1+\sqrt{t})+\log(1+|y|)\big)}$ is bounded and bounded away from 0. Thus, substituting the estimate (4.4) into the second term on the right hand side of (4.3), and using (4.2) and (1.9), it follows that for some $C>0$,

$$\begin{aligned} &w(x,t)\geq \frac{C}{t}\int_{t^{\frac{1}{2}}}^{\frac{1}{2}t}\int_{|y|>t^{\frac{1}{4}}} s^{-p}|y|^m\exp(-\frac{|y-x|^2}{Ct})\exp(-\frac{|y|^2p}{2K_0s})dyds,\\ &\text{for } |x|\geq t^{\frac{1}{2}} \text{ and large } t. \end{aligned} \tag{4.5}$$

Performing some algebraic manipulations similar to those in [12, p.166], one has for $t,s\geq 1$ and some $c>0$,

$$\exp(-\frac{|y-x|^2}{Ct})\exp(-\frac{|y|^2p}{2K_0s})\geq \exp(-\frac{|x|^2}{ct})\exp(-\frac{|y|^2}{cs}). \tag{4.6}$$

Recalling that $m>-2$ and that $n=2$, it is not hard to show, similar to [12, Lemma 4], that for some $k>0$,

$$\int_{|y|>t^{\frac{1}{4}}}|y|^m\exp(-\frac{|y|^2}{cs})dy\geq ks^{1+\frac{m}{2}},\ \text{for } s\geq t^{\frac{1}{2}}. \tag{4.7}$$

From (4.5)-(4.7), we obtain for some $k_1>0$,

$$w(x,t)\geq \frac{k_1}{t}\exp(-\frac{|x|^2}{ct})\int_{t^{\frac{1}{2}}}^{\frac{1}{2}t}s^{1+\frac{m}{2}-p}ds,\ \text{for } |x|\geq t^{\frac{1}{2}} \text{ and large } t. \tag{4.8}$$

By assumption, $1<p\leq 1+\frac{2+m}{2}=2+\frac{m}{2}$; thus, $1+\frac{m}{2}-p\geq -1$. Consequently, for some $k_2>0$ and $t\geq 5$, we have

$$\int_{t^{\frac{1}{2}}}^{\frac{1}{2}t}s^{1+\frac{m}{2}-p}ds\geq k_2\log t. \tag{4.9}$$

Now (4.1) follows from (4.8) and (4.9). □

5. Proofs of Theorem 2 and Corollary 1

Proof of Theorem 2. It is known that $\lambda_{0;D}(-\Delta+V)$ is non-increasing in D and that $\lambda_{0;D}(-\Delta+V)=\lim_{k\to\infty}\lambda_{0;D_k}(-\Delta+V)$, if $D_k\uparrow D$ [11, chapter 4]. These properties of $\lambda_{0;D}(-\Delta+V)$ allow us to assume without loss of generality that the domain D in the statement of the theorem is bounded and has a smooth boundary. As such, $\lambda_{0;D}(-\Delta+V)<0$ is in fact the principal eigenvalue for $-\Delta+V$ in D with the Dirichlet boundary condition. Let $\psi_0>0$, normalized by $\int_D\psi_0(x)dx=1$, denote the corresponding eigenfunction.

Assume now that $u(r,t)$ is a global solution to (1.4) for some $p>1$. Define

$$F(t)=\int_D u(x,t)\psi_0(x)dx. \tag{5.1}$$

We will show that F blows up at some finite time, thereby contradicting the assumption that u is a global solution. Note that ψ_0 vanishes on ∂D and that $\nabla\psi_0\cdot\nu\le 0$ on ∂D, where ν is the unit outward normal to D at ∂D. Also, by assumption $\inf_{x\in D}a(x)\ge\delta$, for some $\delta>0$. Integrating by parts, and using Jensen's inequality and the facts above, we have

$$\begin{aligned}F'(t)&=\int_D u_t(x,t)\psi_0(x)dx=\int_D(\Delta u-Vv+au^p)(x)\psi_0(x)dx\\&\ge-\lambda_{0;D}(-\Delta+V)F(t)+\delta F^p(t)\ge\delta F^p(t).\end{aligned} \tag{5.2}$$

Although the initial data ϕ of u may vanish identically on D, one certainly has $F(t)>0$ for $t>0$. Thus, it follows from (5.2) and the fact that $p>1$ that F blows up at some finite time. □

Proof of Corollary 1. It is well-known that $\lambda_{0;R^n-\{0\}}(-\Delta+\frac{\gamma}{|x|^2})<0$ if $\gamma>\frac{(n-2)^2}{4}$ [11, pp. 153-154]. Let $B_k=\{x\in R^n:|x|<k\}$. Recalling the facts noted in the first line of the proof of Theorem 2, it follows that $\lambda_{0;B_k-\bar B_\epsilon}(-\Delta+\frac{\gamma}{|x|^2})<0$, for sufficiently large k and sufficiently small $\epsilon>0$. Since a is continuous and positive by assumption, it follows that a is bounded away from 0 on $B_k-\bar B_\epsilon$. Thus, the corollary follows from Theorem 2. □

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