

Transience, Recurrence and the Speed of a Random Walk in a Site-Based Feedback Environment

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Abstract

We study a random walk on \mathbb{Z} which evolves in a dynamic environment determined by its own trajectory. Sites flip back and forth between two modes, p and q . R consecutive right jumps from a site in the q -mode are required to switch it to the p -mode, and L consecutive left jumps from a site in the p -mode are required to switch it to the q -mode. From a site in the p -mode the walk jumps right with probability p and left with probability $1 - p$, while from a site in the q -mode these probabilities are q and $1 - q$.

We prove a sharp cutoff for right/left transience of the random walk in terms of an explicit function of the parameters $\alpha = \alpha(p, q, R, L)$. For $\alpha > 1/2$ the walk is transient to $+\infty$ for any initial environment, whereas for $\alpha < 1/2$ the walk is transient to $-\infty$ for any initial environment. In the critical case, $\alpha = 1/2$, the situation is more complicated and the behavior of the walk depends on the initial environment. Nevertheless, we are able to give a characterization of transience/recurrence in many instances, including when either $R = 1$ or $L = 1$ and when $R = L = 2$. In the noncritical case, we also show that the walk has positive speed, and in some situations are able to give an explicit formula for this speed.

1 Introduction and Statement of Results

In this paper we introduce a process we call a site-based feedback random walk on \mathbb{Z} . The process $(X_n)_{n \geq 0}$ is a nearest neighbor random walk governed by four parameters: $p, q \in (0, 1)$ and $R, L \in \mathbb{N}$. An informal description is as follows.

Initially each site $x \in \mathbb{Z}$ is set to either the p -mode or the q -mode. From a site in the p -mode the walk jumps right with probability p and left with probability $1 - p$, whereas from a site in the q -mode these probabilities are q and $1 - q$, respectively. A site x switches from the q -mode to the p -mode after the walk jumps right from x on R consecutive visits to x , and a site x switches from the p -mode to the q -mode after the walk jumps left from x on L consecutive visits to x .

In light of this description, we say the random walk (X_n) has *positive feedback* if $q < p$ and *negative feedback* if $q > p$. Of course, if $q = p$ the situation is trivial; we just have a simple random walk of bias p .

We now give the formal description and set some notation.

- $\Lambda = \{(p, 0), \dots, (p, L - 1), (q, 0), \dots, (q, R - 1)\}$ is the set of *single site configurations*. A typical configuration is denoted by $\lambda = (r, i)$, where $r \in \{p, q\}$ is the *mode* and i is the number of *charges* in favor of the alternative mode.
- $\Lambda_p = \{(p, 0), \dots, (p, L - 1)\}$ is the set of p -configurations, and $\Lambda_q = \{(q, 0), \dots, (q, R - 1)\}$ is the set of q -configurations.
- $\omega = \{\omega(x)\}_{x \in \mathbb{Z}} \in \Lambda^{\mathbb{Z}}$ is the *initial environment*. ω_n is the (random) environment at time $n \geq 0$, $\omega_0 = \omega$.

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- At each step the walk (X_n) jumps right or left according to the following rules:

$$\text{If } \omega_n(X_n) \in \Lambda_p, \text{ then } \begin{cases} \mathbb{P}(X_{n+1} = X_n + 1) = p, \\ \mathbb{P}(X_{n+1} = X_n - 1) = 1 - p. \end{cases}$$

$$\text{If } \omega_n(X_n) \in \Lambda_q, \text{ then } \begin{cases} \mathbb{P}(X_{n+1} = X_n + 1) = q, \\ \mathbb{P}(X_{n+1} = X_n - 1) = 1 - q. \end{cases}$$

- For all $x \neq X_n$, $\omega_{n+1}(x) = \omega_n(x)$. The configuration at the current position of the walk X_n is updated as follows, depending on the direction of the next jump:

$$\text{If } \omega_n(X_n) \in \Lambda_p \cup \{(q, R - 1)\} \text{ and } X_{n+1} = X_n + 1, \text{ then } \omega_{n+1}(X_n) = (p, 0).$$

$$\text{If } \omega_n(X_n) = (q, i), 0 \leq i \leq R - 2, \text{ and } X_{n+1} = X_n + 1, \text{ then } \omega_{n+1}(X_n) = (q, i + 1).$$

$$\text{If } \omega_n(X_n) \in \Lambda_q \cup \{(p, L - 1)\} \text{ and } X_{n+1} = X_n - 1, \text{ then } \omega_{n+1}(X_n) = (q, 0).$$

$$\text{If } \omega_n(X_n) = (p, i), 0 \leq i \leq L - 2, \text{ and } X_{n+1} = X_n - 1, \text{ then } \omega_{n+1}(X_n) = (p, i + 1).$$

This site-based feedback random walk is motivated by so-called cookie random walks and shares certain fundamental properties of two outgrowths of the basic cookie random walk. A basic cookie random walk on \mathbb{Z} is defined as follows. Let $M \geq 1$ be a positive integer. At each site $x \in \mathbb{Z}$, place a pile of M “cookies” with values $\omega(x, k) \in [0, 1]$, $k = 1, \dots, M$. For $k \leq M$, the k -th time the process reaches site x , it eats the k -th cookie at that site, whose value is $\omega(x, k)$, and this empowers the process to jump to the right with probability $\omega(x, k)$ and to the left with probability $1 - \omega(x, k)$. After the site x has been visited M times, whenever the process visits that site again, it behaves like an ordinary simple, symmetric random walk, jumping left or right with equal probability. Cookie random walks were first introduced by Benjamini and Wilson [1]; see the survey paper of Kosygina and Zerner [2] for more on cookie random walks and a bibliography.

We now describe two outgrowths of the basic cookie random walk. Kozma, Orenshtein, and Shinkar [3] recently considered a *periodic cookie* random walk. Instead of having a cookie only the first M times the process is at a given site, consider periodic cookies with period M , and assume that these cookies are identical at each $x \in \mathbb{Z}$. Thus, one defines $\omega(k)$, $k \in \mathbb{N}$, with $\omega(k + M) = \omega(k)$. For each $x \in \mathbb{Z}$, the k th time the process is at x it jumps right or left with probabilities $\omega(k)$ and $1 - \omega(k)$ respectively. In particular, the process never reverts to a simple, symmetric random walk at any site. Another outgrowth of the basic cookie random walk is the “have your cookie and eat it” random walk [4]. Now there is only one cookie at each site; call it $\omega(x)$, $x \in \mathbb{Z}$, and assume $\omega(x) > 1/2$. When the process first reaches x , it jumps right with probability $\omega(x)$ and left with probability $1 - \omega(x)$. For each site x , as long as the process continues to jump to the right from x , it continues to use this right-biased cookie; but after the first time the process jumps to the left from x , the cookie at x is removed. From then on, whenever the process is at x , it behaves like a simple, symmetric random walk, jumping left or right with equal probability.

The site-based feedback random walk has something in common with each of the two above processes. In particular, the sequence of configurations encountered on repeated visits to a given site in the site-based feedback case is a finite-state Markov chain, and thus “roughly periodic” on long time scales, while the jump mechanism at a given site in the site-based feedback case depends not only on the number of visits to that site but also on the direction of the jumps on these visits, as in the “have your cookie and eat it” random walk. However, the site-based feedback random walk is also fundamentally different from both of the above processes in that it itself has *both* of these properties, and also in that it has persistent interactions with its environment, whereas in the “have your cookie and eat it” case the interactions at a given site x terminate after the first leftward jump.

In this paper we study the transience/recurrence properties of the site-based feedback random walk, and in the transient case we study the speed of the process. Some new features occur that were

not present in other cookie random walk models. In particular, the initial environment can have a dramatic influence on the behavior for certain critical values of the parameters p, q, R, L .

Before stating the results, we need to introduce a bit more notation and terminology. Let $\mathbb{P}_{\omega, k}$ denote the probability measure for the random walk started at $X_0 = k$ in the initial environment ω , and let $\mathbb{P}_{\omega} = \mathbb{P}_{\omega, 0}$. Also, let \mathbb{E}_{ω} and $\mathbb{E}_{\omega, k}$ denote, respectively, expectations with respect to the measures \mathbb{P}_{ω} and $\mathbb{P}_{\omega, k}$. Finally, for $x \in \mathbb{Z}$, let N_x be the total number of visits to site x :

$$N_x = |\{n \geq 0 : X_n = x\}|. \quad (1.1)$$

We say that the random walk path (X_n) is ¹:

- *recurrent* if $N_0 = \infty$.
- *right transient*, or *transient to $+\infty$* , if $\lim_{n \rightarrow \infty} X_n = +\infty$, and *left transient*, or *transient to $-\infty$* , if $\lim_{n \rightarrow \infty} X_n = -\infty$.
- *ballistic* if $\liminf_{n \rightarrow \infty} |X_n|/n > 0$.

Our first theorem gives the cutoff point for left/right transience.

Theorem 1. *Define $\alpha = \alpha(p, q, R, L) \in (0, 1)$ by*

$$\alpha = \frac{p \cdot [(1-q)q^R(1-(1-p)^L)] + q \cdot [p(1-p)^L(1-q^R)]}{[(1-q)q^R(1-(1-p)^L)] + [p(1-p)^L(1-q^R)]}. \quad (1.2)$$

- *If $\alpha > 1/2$ then the random walk (X_n) is \mathbb{P}_{ω} a.s. right transient, for any initial environment ω .*
- *If $\alpha < 1/2$ then the random walk (X_n) is \mathbb{P}_{ω} a.s. left transient, for any initial environment ω .*

We will call the vector (p, q, R, L) the *parameter quadruple* for the random walk (X_n) . In light of Theorem 1, we say that the parameter quadruple (p, q, R, L) is *critical* if $\alpha(p, q, R, L) = 1/2$, and *noncritical* otherwise. Our next theorem shows that in the noncritical case, the random walk is not just transient, but in fact ballistic.

Theorem 2. *If $\alpha(p, q, R, L) \neq 1/2$, then there exists a $\beta = \beta(p, q, R, L) > 0$ such that, for any initial environment ω ,*

$$\liminf_{n \rightarrow \infty} \frac{|X_n|}{n} \geq \beta, \quad \mathbb{P}_{\omega} \text{ a.s.} \quad (1.3)$$

Moreover, if $\alpha > 1/2$ ($\alpha < 1/2$) and the initial environment $\omega(x)$ is constant for $x \geq m$ ($x \leq -m$) then $\mathbb{E}_{\omega}(N_x)$ is also constant for $x \geq m$ ($x \leq -m$), and denoting this common value by γ ,

$$\lim_{n \rightarrow \infty} \frac{|X_n|}{n} = \frac{1}{\gamma}, \quad \mathbb{P}_{\omega} \text{ a.s.} \quad (1.4)$$

Here, m can be any nonnegative integer.

The following proposition characterizes some properties of the fundamental function α . We choose to analyze α as a function of p for fixed R, L, q ; of course, a similar analysis also works to analyze α as a function of q for fixed R, L, p .

¹Note that these definitions do not have any a.s. qualifications, and are simply statements about the (random) path $(X_n) = (X_0, X_1, \dots)$. Thus, the random walk (X_n) has some probability of being right transient, some probability of being left transient, and some probability of being recurrent. Typically one says that a random walk (X_n) is recurrent/right transient/left transient if, according to our definitions, it is a.s. recurrent/right transient/left transient. However, for our model there are some situations (see Theorem 8) where there is positive probability both for transience to $+\infty$ and for transience to $-\infty$, so for consistency we will speak of all of these properties probabilistically.

Proposition 1. Let R, L, q be fixed and consider α as a function of p , $\alpha(p) \equiv \alpha(p, q, R, L)$.

(i) If $q = 1/2$, then

$$\alpha(1/2) = 1/2, \quad \alpha(p) < 1/2 \text{ for } p < 1/2, \quad \alpha(p) > 1/2 \text{ for } p > 1/2.$$

(ii) If $q < 1/2$, then there exists a unique critical point $p_0 = p_0(q, R, L) \in (1/2, 1)$ such that

$$\alpha(p_0) = 1/2, \quad \alpha(p) < 1/2 \text{ for } p < p_0, \quad \alpha(p) > 1/2 \text{ for } p > p_0. \quad (1.5)$$

(iii) For $q < 1/2$ the critical point p_0 from (ii) satisfies

$$\begin{aligned} q + p_0(q, R, L) &< 1, \quad \text{if } R < L; \\ q + p_0(q, R, L) &> 1, \quad \text{if } R > L. \end{aligned} \quad (1.6)$$

Also, for any fixed R and L , $p_0(q, R, L)$ is a decreasing function of q , for $q \in (0, 1/2)$.

(iv) If $q < 1/2$ and $L = 1$, then

$$p_0 = \frac{1 - 2q + q^{R+1}}{1 - 2q + q^R}. \quad (1.7)$$

If $q > 1/2$ and $L = 1$, then (1.5) still holds with $p_0 = \frac{1-2q+q^{R+1}}{1-2q+q^R}$ as long as $1 - 2q + q^{R+1} > 0$. However, if $1 - 2q + q^{R+1} \leq 0$, then $\alpha(p) > 1/2$, for all $p \in (0, 1)$.

(v) If $q < 1/2$ and $L = R$, then $p_0 = 1 - q$. If $q > 1/2$ and $L = R$, then $1 - q$ is still a critical point (i.e. $\alpha(1 - q) = 1/2$), but it is not always unique.

(vi) For any R, L, q , $\lim_{p \rightarrow 1} \alpha(p) = 1$. In particular, $\alpha > 1/2$ for all sufficiently large p .

Remark 1. Part (v) shows that $\alpha(p)$ is not always a monotonic function of p , and, in fact, often it is not. Consequently, increasing p (with fixed q, R, L) may sometimes change the process from the right transient regime to the left transient regime. However, this phenomena can only occur when $q > 1/2$, by part (ii), in which case the process has negative feedback at all critical points. Illustrative plots are given in Figure 1.

Remark 2. As noted before the proposition, we could have considered α as a function of q for fixed p, R, L . We note, in particular, that in the case that $p > 1/2$, there exists a unique critical point $q_0 = q_0(p, R, L) \in (0, 1/2)$, and when in addition, $R = 1$, one has

$$q_0 = \frac{p(1-p)^L}{2p-1+(1-p)^L}. \quad (1.8)$$

Moreover, if $R = 1$ and $p \leq 1/2$, then there is still a unique critical point q_0 given by (1.8) as long as $2p - 1 + (1 - p)^L > 0$. However, if $2p - 1 + (1 - p)^L \leq 0$, then $\alpha < 1/2$, for all $q \in (0, 1)$.

In general for cookie-type random walks, it is very difficult to obtain an explicit formula for the speed in the ballistic regime. However, the additional level of interaction between the random walker and the environment in the site-based feedback case makes a calculation of the speed possible in some situations. Before moving on to the critical case, we present two results that give an exact characterization of the limiting speed with R or L equal to 1, in certain initial environments. We assume that $\alpha > 1/2$, but analogous results for $\alpha < 1/2$ are easily inferred by symmetry considerations. Specifically, if $\alpha(p, q, R, L) < 1/2$ then $\alpha(1 - q, 1 - p, L, R) > 1/2$, and the speed to $-\infty$ with parameters p, q, R, L in an initial environment ω is the same as the speed to $+\infty$ with parameters $1 - q, 1 - p, L, R$ in an initial environment ω' defined by $\omega'(x) = \omega(-x)^*$, $x \in \mathbb{Z}$, where $(q, i)^* = (1 - q, i)$ and $(p, i)^* = (1 - p, i)$.

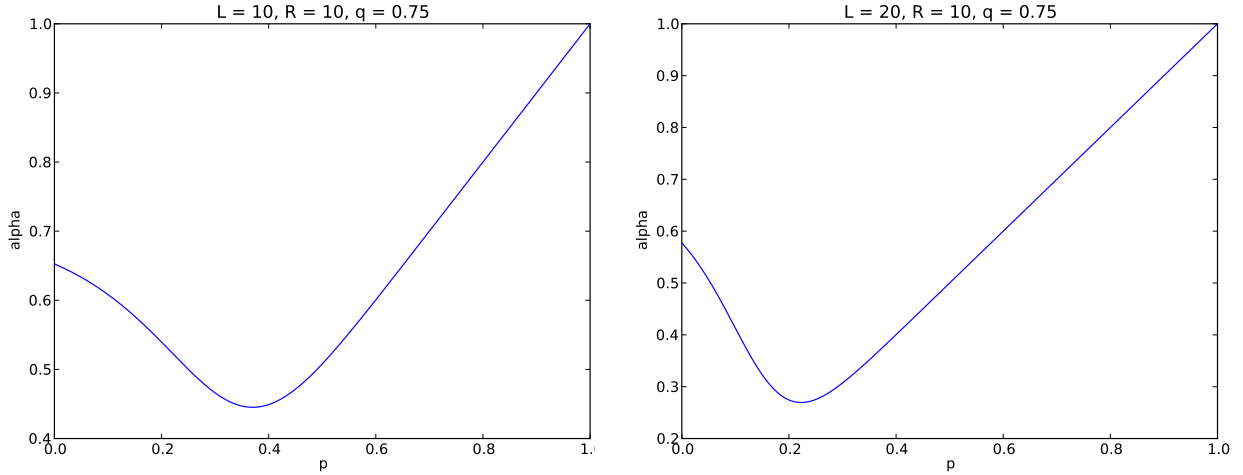


Figure 1: Plots of $\alpha(p)$ with $L = 10, R = 10, q = 0.75$ (left) and $L = 20, R = 10, q = 0.75$ (right). In both cases, as p increases from 0 to 1 the parameter quadruple (p, q, R, L) passes from right transient ($\alpha > 1/2$), to left transient ($\alpha < 1/2$), and back to right transient.

Theorem 3. *Let $L = 1$ and $\alpha > 1/2$. If $\omega(x) = (q, 0)$ in a neighborhood of $+\infty$, then*

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = \frac{1 - t_*}{1 + t_*}, \mathbb{P}_\omega \text{ a.s.}, \quad (1.9)$$

where t_* is the unique root of the polynomial

$$P(t) = (1 - q) + (pq - p - 1)t + (p + q)t^2 - pqt^3 - (p - q)q^R(t^R - t^{R+1}) \quad (1.10)$$

in the interval $(1 - q, 1)$.

Theorem 4. *Let $R = 1$ and $\alpha > 1/2$. Assume that the limiting right density of (p, i) sites $d_i \equiv \lim_{n \rightarrow \infty} \frac{1}{n} |\{0 \leq x \leq n - 1 : \omega(x) = (p, i)\}|$ exists, for each $0 \leq i \leq L - 1$, and let $d_L = 1 - \sum_{i=0}^{L-1} d_i$ denote the limiting right density of $(q, 0)$ sites. Then,*

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = \frac{1}{\sum_{i=0}^L a_i d_i}, \mathbb{P}_\omega \text{ a.s.}, \quad (1.11)$$

where

$$a_0 = \frac{1 + (p/q - 1)(1 - p)^L}{(2p - 1) - (p/q - 1)(1 - p)^L}, \quad (1.12)$$

and, for $1 \leq i \leq L$,

$$a_i = \frac{1 + (p/q - 1)(1 - p)^{L-i}}{p} + \left(\frac{(1 - p) + (p/q - 1)(1 - p)^{L-i}}{p} \right) a_0. \quad (1.13)$$

Remark. In the case $R = L = 1$ with $\alpha > 1/2$ (i.e. with $p + q > 1$), the speed $s = \lim_{n \rightarrow \infty} \frac{X_n}{n}$ from (1.11) reduces to

$$s = \frac{p + q - 1}{1 + (1 - 2d)(p - q)},$$

where $d = d_0$ is the limiting right density of $(p, 0)$ sites in the initial environment ω . Interestingly, this formula for s is invariant under the interchange $p \leftrightarrow q$, $d \leftrightarrow 1 - d$. That is, the speed of the walk is the same with positive or negative reinforcement, as long as the pair of bias parameters and the limiting right density of sites initially in the configuration with the more favorable bias for jumping right remain the same.

We now turn to the critical case, $\alpha = 1/2$. Here, there are two possibilities: positive feedback with $q < 1/2 < p$ or negative feedback with $p < 1/2 < q$. In the case of positive feedback the situation is somewhat simpler, but in both cases the analysis is more delicate than before, and the transience/recurrence of the random walk often depends heavily on the initial environment.

Our first result shows that there always exist initial environments for which the random walk is recurrent.

Theorem 5. *If $\alpha(p, q, R, L) = 1/2$, then there exist initial environments ω for which the random walk (X_n) is a.s. recurrent. In particular, in the positive feedback case, $q < p$, the random walk (X_n) is \mathbb{P}_ω a.s. recurrent for any initial environment ω with $\omega(x) = (q, 0)$ for x in a neighborhood of $+\infty$ and $\omega(x) = (p, 0)$ for x in a neighborhood of $-\infty$.*

The next two theorems, and concomitant corollary and proposition, concern the situation that either R or L is 1. In this case, we can give an essentially complete description of when the random walk is recurrent, right transient, or left transient. However, for technical reasons, we will need to assume in many instances that the initial environment ω is constant either in a neighborhood of $+\infty$, a neighborhood of $-\infty$, or both. Our first result indicates that, when R or L is equal to 1, only one of the two directions is possible for transience.

Theorem 6. *Assume $\alpha = 1/2$.*

- *If $R = 1$ and $q < p$, then the random walk (X_n) is \mathbb{P}_ω a.s. not transient to $+\infty$, for any initial environment ω .*
- *If $R = 1$ and $p < q$, then the random walk (X_n) is \mathbb{P}_ω a.s. not transient to $+\infty$, for any environment ω which is constant in a neighborhood of $+\infty$.*
- *If $L = 1$ and $q < p$, then the random walk (X_n) is \mathbb{P}_ω a.s. not transient to $-\infty$, for any initial environment ω .*
- *If $L = 1$ and $p < q$, then the random walk (X_n) is \mathbb{P}_ω a.s. not transient to $-\infty$, for any environment ω which is constant in a neighborhood of $-\infty$.*

The following corollary is an immediate consequence of this theorem and part (ii) of Lemma 1 in section 2.2.

Corollary 1. *Let $R = L = 1$ and $p = 1 - q$; so $\alpha = 1/2$.*

- *If $q < p$ then the random walk (X_n) is \mathbb{P}_ω a.s. recurrent, for any initial environment ω .*
- *If $p < q$ then the random walk (X_n) is \mathbb{P}_ω a.s. recurrent, for any initial environment ω which is constant in neighborhoods of $+\infty$ and $-\infty$.*

Our next theorem gives specific conditions to determine if the random walk is recurrent or transient to $+\infty$ in the case $L = 1$ and $R > 1$ (which, by Theorem 6, are the only possibilities). By symmetry considerations, if $R = 1$, instead of $L = 1$, then the result obtained for $L = 1$ will hold with the roles of q, p, R and $\pm\infty$ replaced by $1 - p, 1 - q, L$ and $\mp\infty$ respectively.

Theorem 7. *Assume that $L = 1$, $R \geq 2$, and $\alpha = 1/2$. Thus, by (1.7), $p = p_0 = \frac{1-2q+q^{R+1}}{1-2q+q^R}$. In the case of negative feedback, $p < q$, assume also that the initial environment ω is constant in a neighborhood of $-\infty$.*

- (i) If $\omega(x) = (q, 0)$ in a neighborhood of $+\infty$, then the random walk (X_n) is \mathbb{P}_ω a.s. recurrent.
- (ii) If $\omega(x) = (q, i)$ in a neighborhood of $+\infty$, $1 \leq i \leq R-1$, then the random walk (X_n) is

$$\begin{aligned} &\mathbb{P}_\omega \text{ a.s. recurrent if } P_{R,i}(q) \geq 0, \text{ and} \\ &\mathbb{P}_\omega \text{ a.s. transient to } +\infty \text{ if } P_{R,i}(q) < 0, \end{aligned}$$

where

$$\begin{aligned} P_{R,i}(q) = &(2R-1)q^{R+2} - (3R+1)q^{R+1} + (R+1)q^R \\ &- 2q^{R+2-i} + 3q^{R+1-i} - q^{R-i} + (1-2q)^2. \end{aligned} \quad (1.14)$$

- (iii) If $\omega(x) = (p, 0)$ in a neighborhood of $+\infty$, then the random walk (X_n) is

$$\begin{aligned} &\mathbb{P}_\omega \text{ a.s. recurrent if } P_{R,R}(q) \geq 0, \text{ and} \\ &\mathbb{P}_\omega \text{ a.s. transient to } +\infty \text{ if } P_{R,R}(q) < 0, \end{aligned}$$

where

$$P_{R,R}(q) = (2R-1)q^{R+2} - (3R+1)q^{R+1} + (R+1)q^R + 2q^2 - q. \quad (1.15)$$

Moreover, for each $R \geq 2$ there exists a unique root $q_*(R) \in (0, 1/2)$ of the polynomial $P_{R,R}(q)$, $P_{R,R}(q) > 0$ for $q > q_*(R)$, $P_{R,R}(q) < 0$ for $q < q_*(R)$, and $\lim_{R \rightarrow \infty} q_*(R) = 1/2$.

Remark 1. In the case of positive feedback, $q < p$, one can also determine between right transience and recurrence for some environments that are not constant in a neighborhood of $+\infty$, using the comparison lemma given in section 2.3. In particular, $(p, 0)$ is the most favorable environment for right transience in the positive feedback case, so if the random walk is not right transient with the $(p, 0)$ environment in a neighborhood of $+\infty$, then it is not right transient for any initial environment.

Remark 2. $P_{R,R}(q)$ is the same polynomial one obtains by substituting $i = R$ into the definition of $P_{R,i}(q)$ in (1.14).

Remark 3. For any $1 \leq i \leq R$, $P_{R,i}(q)$ has a double root at 1. $P_{R,R}(q)$ also has a single root at 0 and factors as $P_{R,R}(q) = q(1-q)^2 \tilde{P}_{R,R}(q)$ where

$$\tilde{P}_{R,R}(q) = -1 + \sum_{j=1}^{R-3} jq^{j+1} + (2R-1)q^{R-1}. \quad (1.16)$$

Here, the sum is defined to be 0, for $R = 2, 3$.

Remark 4. Using (1.16) one finds that $q_*(2) = 1/3$ and $q_*(3) = 1/\sqrt{5} \approx 0.447$. Using a combination of analytical techniques and computer generated plots one also finds the following behavior for $P_{R,i}$, $1 \leq i \leq R-1$. For $R = 2, 3, 4$, $P_{R,i}(q) \geq 0$ for all $1 \leq i \leq R-1$ and $q \in (0, 1)$. For $R = 5, 6$, $P_{R,i}(q) \geq 0$ for all $1 \leq i \leq R-2$ and $q \in (0, 1)$. However, $P_{5,4}(q) < 0$ if (and only if) $q \in (a, b)$, where $a \approx .410$ and $b \approx .473$, and $P_{6,5}(q) < 0$ if (and only if) $q \in (a, b)$, where $a \approx .391$ and $b \approx .490$. For $i = 5, 6$ there are ranges of q for which $P_{7,i}(q) < 0$.

The next proposition characterizes asymptotic properties of the function $P_{R,i}(q)$ in the limit of large R , for two different cases of $i = i_R$. In the first case, $R - i_R$ grows to infinity, so the process must jump right many consecutive times from a given site to switch it to the p -mode, starting in the (q, i_R) initial environment. In the second case, $i_R = R - k$, for a fixed k , so the process need jump right only k consecutive times from any site to switch it to the p -mode, starting in the (q, i_R) initial environment. The proof of both cases is straightforward, and is left to the reader.

Proposition 2.

(i) If $(R - i_R) \rightarrow \infty$ as $R \rightarrow \infty$ then, for any fixed $q \in (0, 1)$, $P_{R, i_R}(q) > 0$ for all sufficiently large R .

(ii) Let $i_R = R - k$, and define

$$Q_k(q) = (1 - 2q)^2 - q^k + 3q^{k+1} - 2q^{k+2}. \quad (1.17)$$

If $Q_k(q) > 0$ then $P_{R, i_R}(q) > 0$ for sufficiently large R , and if $Q_k(q) < 0$ then $P_{R, i_R}(q) < 0$ for sufficiently large R .

Remark. The polynomial $Q_k(q)$ factors as $(2q - 1)(2q - 1 + q^k - q^{k+1})$. Using this representation it is not hard to verify the following facts: $Q_k(q) > 0$ for $q > 1/2$, and there exists an $a_k \in (\frac{1}{2}(3 - \sqrt{5}), 1/2) \approx (.382, 1/2)$ such that $Q_k(q) > 0$ for $q \in (0, a_k)$ and $Q_k(q) < 0$ for $q \in (a_k, 1/2)$. Furthermore, a_k is increasing in k and $\lim_{k \rightarrow \infty} a_k = 1/2$.

Corollary 1 showed that if $R = L = 1$ then in the critical case $p = 1 - q$ the random walk is always recurrent (assuming constant initial environment in neighborhoods of $\pm\infty$ in the negative feedback regime). When $R = L = 2$ the behavior in the critical case is much more complicated; in particular, for appropriate initial environments, there is a positive probability for both transience to $+\infty$ and transience to $-\infty$.

Theorem 8. Let $R = L = 2$ and assume that $p = 1 - q$; so $\alpha = \frac{1}{2}$. In the negative feedback case, $q > \frac{1}{2}$, assume also that the initial environment ω is constant in a neighborhood of $+\infty$ and in a neighborhood of $-\infty$. Let

$$q_1^* = \frac{\sqrt{13} - 3}{2} \approx 0.303$$

and let

$$q_2^* \approx .682 \text{ be the unique root in } (0, 1) \text{ of } q^3 + q - 1.$$

(i) Let $q < q_1^*$.

a. If $\omega(x) = (p, 0)$ in a neighborhood of $+\infty$ and $\omega(x) \neq (q, 0)$ for any x in a neighborhood of $-\infty$, then the random walk (X_n) is \mathbb{P}_ω a.s. transient to $+\infty$.

b. If $\omega(x) = (q, 0)$ in a neighborhood of $-\infty$ and $\omega(x) \neq (p, 0)$ for any x in a neighborhood of $+\infty$, then the random walk (X_n) is \mathbb{P}_ω a.s. transient to $-\infty$.

c. If $\omega(x) \neq (p, 0)$ for any x in a neighborhood of $+\infty$ and $\omega(x) \neq (q, 0)$ for any x in a neighborhood of $-\infty$, then the random walk (X_n) is \mathbb{P}_ω a.s. recurrent.

d. If $\omega(x) = (p, 0)$ in a neighborhood of $+\infty$ and $\omega(x) = (q, 0)$ in a neighborhood of $-\infty$, then $\mathbb{P}_\omega(X_n \rightarrow +\infty) = 1 - \mathbb{P}_\omega(X_n \rightarrow -\infty) \in (0, 1)$.

(ii) Let $q > q_2^*$. Then (a)-(d) of (i) hold with the roles of $(p, 0)$ and $(q, 0)$ reversed.

(iii) Let $q \in [q_1^*, q_2^*]$. Then the random walk (X_n) is \mathbb{P}_ω a.s. recurrent.

Remark. The proof of Theorem 8 relies on the eigen-decomposition of a 2×2 matrix. In principal, one could also apply similar methods to characterize the transience/recurrence properties for general $R, L \geq 2$. Specifically, to determine if there is positive probability of right transience one needs to diagonalize an $L \times L$ matrix, and to determine if there is positive probability of left transience one needs to diagonalize an $R \times R$ matrix. For $R = L = 3$ this is possible analytically, but in general it is not. Nevertheless, it could be done numerically for reasonably sized R and L . In the case $R \neq L$, one would also need to determine numerically the critical value/s of p such that $\alpha(p, q, R, L) = 1/2$, as the entries of these matrices depend on both p and q .

The remainder of the paper is organized as follows. In section 2 we introduce some important constructions that will be central to our proofs and establish a number of simple lemmas. In section 3 we prove Theorems 1–4 concerning the behavior of the random walk (X_n) in the noncritical case. In section 4 we prove Theorems 5–8 concerning the behavior of the random walk in the critical case. Finally, in section 5 we prove Proposition 1 which characterizes properties of the important function α .

2 Preliminaries

In this section we introduce a basic framework for proving the theorems stated above and establish a number of useful lemmas. Section 2.1 gives constructions of the single site Markov chains $(Y_n^x)_{n \in \mathbb{N}}$ and the right jumps Markov chain $(Z_x)_{x \geq 0}$, which will be the primary tools used in the proofs of Theorems 1, 2, 5, 7 and 8. Section 2.2 gives three simple lemmas that will be used in a number of places. The first two concern conditions for transience, and the other relates hitting times to speed. Finally, section 2.3 gives an important lemma comparing the possibility of transience in different environments.

2.1 Auxilliary Markov Chains

2.1.1 The Single Site Markov Chains $(Y_n^x)_{n \in \mathbb{N}}$

Let M be the stochastic transition matrix on the set of single site configurations Λ , with nonzero entries defined as follows:

$$\begin{aligned} M_{(p,i)(p,0)} &= p, M_{(p,i)(p,i+1)} = 1 - p, \quad \text{for } 0 \leq i \leq L - 2. \\ M_{(p,L-1)(p,0)} &= p, M_{(p,L-1)(q,0)} = 1 - p. \\ M_{(q,i)(q,0)} &= 1 - q, M_{(q,i)(q,i+1)} = q, \quad \text{for } 0 \leq i \leq R - 2. \\ M_{(q,R-1)(q,0)} &= 1 - q, M_{(q,R-1)(p,0)} = q. \end{aligned}$$

For $x \in \mathbb{Z}$, let $(Y_n^x)_{n \in \mathbb{N}}$ be the Markov chain with state space Λ , transition matrix M , and initial state $\omega(x)$. We refer to the chain $(Y_n^x)_{n \in \mathbb{N}}$ as the *single site Markov chain* at x . It is the Markovian sequence of configurations at site x that would occur if x were to be visited infinitely often. That is,

$$\mathbb{P}(C_{n+1}^x = \lambda' | C_n^x = \lambda, N_x \geq n + 1) = M_{\lambda\lambda'}, \quad \lambda, \lambda' \in \Lambda$$

where N_x is the total number of visits to site x , as above, and C_n^x is the configuration at site x immediately after the n -th visit.

The *extended single site chain* at x , $(\widehat{Y}_n^x)_{n \in \mathbb{N}} = (Y_n^x, J_n^x)_{n \in \mathbb{N}}$, is the Markov chain whose states are pairs (λ, j) , where $\lambda \in \Lambda$ denotes the current configuration at site x and $j \in \{1, -1\}$ represents the next jump from x (1 for right, -1 for left). The state space of this chain is $\widehat{\Lambda} = \Lambda \times \{1, -1\}$ and the transition matrix \widehat{M} is defined by:

$$\begin{aligned} \widehat{M}_{((p,i),1)((p,0),1)} &= p, \widehat{M}_{((p,i),1)((p,0),-1)} = 1 - p, \quad \text{for } 0 \leq i \leq L - 1. \\ \widehat{M}_{((p,i),-1)((p,i+1),1)} &= p, \widehat{M}_{((p,i),-1)((p,i+1),-1)} = 1 - p, \quad \text{for } 0 \leq i \leq L - 2. \\ \widehat{M}_{((p,L-1),-1)((q,0),1)} &= q, \widehat{M}_{((p,L-1),-1)((q,0),-1)} = 1 - q. \\ \widehat{M}_{((q,i),-1)((q,0),1)} &= q, \widehat{M}_{((q,i),-1)((q,0),-1)} = 1 - q, \quad \text{for } 0 \leq i \leq R - 1. \\ \widehat{M}_{((q,i),1)((q,i+1),1)} &= q, \widehat{M}_{((q,i),1)((q,i+1),-1)} = 1 - q, \quad \text{for } 0 \leq i \leq R - 2. \\ \widehat{M}_{((q,R-1),1)((p,0),1)} &= p, \widehat{M}_{((q,R-1),1)((p,0),-1)} = 1 - p. \end{aligned}$$

The initial state \widehat{Y}_1^x for the chain has the following distribution:

- If $\omega(x) = (p, i)$, for some $0 \leq i \leq L - 1$, then

$$\mathbb{P}\left(\widehat{Y}_1^x = ((p, i), 1)\right) = p, \quad \mathbb{P}\left(\widehat{Y}_1^x = ((p, i), -1)\right) = 1 - p. \quad (2.1)$$

- If $\omega(x) = (q, i)$, for some $0 \leq i \leq R - 1$, then

$$\mathbb{P}\left(\widehat{Y}_1^x = ((q, i), 1)\right) = q, \quad \mathbb{P}\left(\widehat{Y}_1^x = ((q, i), -1)\right) = 1 - q. \quad (2.2)$$

By construction, the sequence of site configurations (Y_n^x) obtained by projection from this extended Markov chain state sequence (\widehat{Y}_n^x) with transition matrix \widehat{M} and initial state distributed according to (2.1) and (2.2) has the same law as above, when defined directly by the transition matrix M with initial state $\omega(x)$.

Coupling to the Random Walk (X_n)

For a given initial position x_0 and initial environment $\omega = \{\omega(x)\}_{x \in \mathbb{Z}}$ one can construct the random walk $(X_n)_{n \geq 0}$ according to the following two step procedure, similar to that given in [5] for cookie random walks.

1. Run the extended single site Markov chains $(\widehat{Y}_n^x)_{n \in \mathbb{N}}$ at each site x independently.
2. Walk deterministically from the initial point x_0 according to the corresponding “jump pattern” $\{J_n^x\}_{n \in \mathbb{N}, x \in \mathbb{Z}}$. That is, upon the the k_{th} visit to site x , the walk jumps right if $J_k^x = 1$ and left if $J_k^x = -1$. Formally, we have:
 - $X_0 = x_0$.
 - For $n \geq 0$, $X_{n+1} = X_n + J_{K_n}^{X_n}$ where $K_n = |\{0 \leq m \leq n : X_m = X_n\}|$.

By definition of the extended single site chains, the random walk $(X_n)_{n \geq 0}$ constructed by this two step procedure will have the correct law, and in the sequel we always assume our random walk (X_n) to be defined in this fashion. We also denote by \mathbb{P}_ω the probability measure for the extended single site chains, run independently at each site x , with initial environment $\omega = \{\omega(x)\}_{x \in \mathbb{Z}}$. This is a slight abuse of notation since the probability measure $\mathbb{P}_\omega \equiv \mathbb{P}_{\omega, 0}$ introduced in section 1 also specifies the initial position of the random walk as $X_0 = 0$. However, things should be clear from the context.

Stationary Distribution

Since Λ is finite and M is an irreducible transition matrix, there exists a unique stationary probability distribution π on Λ satisfying $\pi = \pi M$. Solving the linear system $\{\pi = \pi M, \sum_{\lambda \in \Lambda} \pi_\lambda = 1\}$, one obtains the following explicit form for π (see Appendix A.1):

$$\begin{aligned} \pi_{(p,i)} &= \frac{p(1-q)q^R}{(1-q)q^R(1-(1-p)^L) + p(1-p)^L(1-q^R)} \cdot (1-p)^i, \quad 0 \leq i \leq L-1. \\ \pi_{(q,i)} &= \frac{p(1-q)(1-p)^L}{(1-q)q^R(1-(1-p)^L) + p(1-p)^L(1-q^R)} \cdot q^i, \quad 0 \leq i \leq R-1. \end{aligned} \quad (2.3)$$

In particular,

$$\begin{aligned} \pi_p &\equiv \sum_{i=0}^{L-1} \pi_{(p,i)} = \frac{(1-q)q^R(1-(1-p)^L)}{(1-q)q^R(1-(1-p)^L) + p(1-p)^L(1-q^R)}, \quad \text{and} \\ \pi_q &\equiv \sum_{i=0}^{R-1} \pi_{(q,i)} = \frac{p(1-p)^L(1-q^R)}{(1-q)q^R(1-(1-p)^L) + p(1-p)^L(1-q^R)}. \end{aligned} \quad (2.4)$$

So, defining $\phi : \widehat{\Lambda} \rightarrow \{0, 1\}$ by $\phi(\lambda, j) = \mathbb{1}\{j = 1\}$, we have

$$\mathbb{E}_{\widehat{\pi}}(\phi) = p \cdot \pi_p + q \cdot \pi_q = \alpha, \quad (2.5)$$

where $\widehat{\pi}$ is the stationary distribution for the transition matrix \widehat{M} , and $\alpha \in (0, 1)$ is as in Theorem 1. It follows, by the ergodic theorem for finite-state Markov chains, that the limiting fraction of right jumps in the sequence $(J_n^x)_{n \in \mathbb{N}}$ is equal to α a.s., for each site x .

2.1.2 The Right Jumps Markov Chain $(Z_x)_{x \geq 0}$

The *right jumps Markov chain* $(Z_x)_{x \geq 0}$ is defined as follows:

- $Z_0 = 1$.
- For $x \geq 1$,

$$Z_x = \Theta_x - Z_{x-1} \quad \text{where} \quad \Theta_x = \inf \left\{ n \geq 0 : \sum_{m=1}^n \mathbb{1}\{J_m^x = -1\} = Z_{x-1} \right\}, \quad (2.6)$$

with the convention that $\sum_{m=1}^0 \mathbb{1}\{J_m^x = -1\} = 0$. That is, Θ_x is the first time that there are Z_{x-1} left jumps in the sequence $(J_n^x)_{n \in \mathbb{N}}$, and $Z_x = \Theta_x - Z_{x-1}$ is the total number of right jumps in the sequence $(J_n^x)_{n \in \mathbb{N}}$ before there are Z_{x-1} left jumps.

For an initial environment ω , we denote the probability measure for the right jumps chain (Z_x) also by \mathbb{P}_ω . This is simply the projection of the measure \mathbb{P}_ω for the extended single site Markov chains, of which the right jumps chain is a deterministic function.

Relation to the Random Walk (X_n)

We denote by T_x the first hitting time of site x ,

$$T_x = \inf \{ n \geq 0 : X_n = x \}, \quad x \in \mathbb{Z}.$$

Also, we say that a jump pattern $\{J_n^x\}_{n \in \mathbb{N}, x \in \mathbb{Z}}$ is *non-degenerate* if

$$|\{n : J_{n+1}^x \neq J_n^x\}| = \infty, \quad \text{for each } x \in \mathbb{Z}.$$

Clearly, for any initial environment ω , the corresponding jump pattern $\{J_n^x\}_{n \in \mathbb{N}, x \in \mathbb{Z}}$ is non-degenerate \mathbb{P}_ω a.s. The following important proposition relating transience/recurrence of the random walk (X_n) to survival of the Markov chain (Z_x) is shown in [5]².

Proposition 3. *If $X_0 = 1$ and $\{J_n^x\}_{n \in \mathbb{N}, x \in \mathbb{Z}}$ is non-degenerate, then*

$$T_0 = \infty \quad \text{if and only if } Z_x > 0, \quad \text{for all } x > 0. \quad (2.7)$$

Moreover, if $T_0 < \infty$ then, for each $x \in \mathbb{N}$, Z_x is equal to the number of right jumps of the process (X_n) from site x before hitting 0.

²The terminology there is slightly different. The jump pattern is referred to as an *arrow environment* and denoted by a . After the arrow environment is chosen (according to some random rule which differs depending on the model) the walker follows the directional arrows deterministically on its walk.

2.2 Basic Lemmas

For $n \geq 0$ we denote by \mathcal{A}_n^+ the event that the random walk steps right at time n and never returns to its time- n location, and by \mathcal{A}_n^- the event that the random walk steps left at time n and never returns:

$$\mathcal{A}_n^+ = \{X_m > X_n, \forall m > n\} \quad \text{and} \quad \mathcal{A}_n^- = \{X_m < X_n, \forall m > n\}. \quad (2.8)$$

The following simple facts will be needed in several instances below. A proof is provided in Appendix B.

Lemma 1. *For any initial environment ω :*

- (i) $\mathbb{P}_\omega(\mathcal{A}_0^+) > 0$ if and only if $\mathbb{P}_\omega(X_n \rightarrow \infty) > 0$, and $\mathbb{P}_\omega(\mathcal{A}_0^-) > 0$ if and only if $\mathbb{P}_\omega(X_n \rightarrow -\infty) > 0$.
- (ii) $\mathbb{P}_\omega(X_n \rightarrow \infty) = \mathbb{P}_\omega(\liminf_{n \rightarrow \infty} X_n > -\infty)$, and $\mathbb{P}_\omega(X_n \rightarrow -\infty) = \mathbb{P}_\omega(\limsup_{n \rightarrow \infty} X_n < \infty)$.
- (iii) $\mathbb{P}_\omega(X_n \rightarrow \infty) = 1$ if $\mathbb{P}_\omega(X_n \rightarrow \infty) > 0$ and $\mathbb{P}_\omega(X_n \rightarrow -\infty) = 0$.
 $\mathbb{P}_\omega(X_n \rightarrow -\infty) = 1$ if $\mathbb{P}_\omega(X_n \rightarrow -\infty) > 0$ and $\mathbb{P}_\omega(X_n \rightarrow \infty) = 0$.

Combining Proposition 3 and part (i) of Lemma 1 gives the following useful lemma.

Lemma 2. *For any initial environment ω ,*

$$\mathbb{P}_\omega(\mathcal{A}_0^+) = \mathbb{P}_\omega(X_1 = 1) \cdot \mathbb{P}_\omega(Z_x > 0, \forall x > 0). \quad (2.9)$$

Consequently, $\mathbb{P}_\omega(X_n \rightarrow \infty) > 0$ if and only if $\mathbb{P}_\omega(Z_x > 0, \forall x > 0) > 0$.

Proof. Fix any initial environment ω , and let ω' denote the environment at time 1 induced by jumping right from $X_0 = 0$ starting in ω :

$$\{\omega_0 = \omega, X_0 = 0, X_1 = 1\} \implies \omega_1 = \omega'.$$

Since $\omega(x) = \omega'(x)$, for all $x > 0$, the distribution of the random variables $(J_n^x)_{n,x>0}$, is the same in the two environments ω and ω' . Thus,

$$\mathbb{P}_\omega(Z_x > 0, \forall x > 0) = \mathbb{P}_{\omega'}(Z_x > 0, \forall x > 0).$$

So,

$$\begin{aligned} \mathbb{P}_\omega(\mathcal{A}_0^+) &\equiv \mathbb{P}_{\omega,0}(X_n > 0, \forall n > 0) \\ &= \mathbb{P}_{\omega,0}(X_1 = 1) \cdot \mathbb{P}_{\omega,0}(X_n > 0, \forall n > 1 | X_1 = 1) \\ &= \mathbb{P}_{\omega,0}(X_1 = 1) \cdot \mathbb{P}_{\omega',1}(X_n > 0, \forall n > 0) \\ &\stackrel{(*)}{=} \mathbb{P}_{\omega,0}(X_1 = 1) \cdot \mathbb{P}_{\omega'}(Z_x > 0, \forall x > 0) \\ &= \mathbb{P}_{\omega,0}(X_1 = 1) \cdot \mathbb{P}_\omega(Z_x > 0, \forall x > 0). \end{aligned}$$

This proves (2.9), and the ‘‘consequently’’ part of the proposition follows immediately from part (i) of Lemma 1. Step (*) follows from Proposition 3³. \square

For the proofs of Theorems 2 and 4 we will need the following lemma relating hitting times to speed. The same result is shown in [6, Lemma 2.1.17], for the case $C < \infty$ without the a priori assumption that $X_n \rightarrow \infty$. It is easy to see that with this assumption the claim also holds in the case $C = \infty$.

³In the proof we have used the explicit notation $\mathbb{P}_{\omega,0}$, rather than simply \mathbb{P}_ω , for the random walk variables X_n , $n \geq 0$, to emphasize that the initial position $X_0 = 0$ plays a role in their distribution. By contrast, $\mathbb{P}_\omega, \mathbb{P}_{\omega'}$ are used for the distribution of the right jumps Markov chain $(Z_x)_{x \geq 0}$, where the initial position of the random walk plays no role.

Lemma 3. *If $\lim_{n \rightarrow \infty} X_n = \infty$ and $\lim_{x \rightarrow \infty} T_x/x = C \in (0, \infty]$, then*

$$\lim_{n \rightarrow \infty} X_n/n = 1/C.$$

We note that, although stated in [6, Lemma 2.1.17] in the context of random walks in random environment, the proof is entirely non-probabilistic and holds for *any* nearest neighbor walk trajectory (X_0, X_1, \dots) such that $X_n \rightarrow \infty$ and $\lim_{x \rightarrow \infty} T_x/x = C$.

2.3 Comparison of Environments

Let \prec be the ordering on the set of single site configurations Λ defined by

$$(q, 0) \prec \dots \prec (q, R-1) \prec (p, L-1) \prec \dots \prec (p, 0).$$

We write $\lambda \preceq \tilde{\lambda}$ if $\lambda \prec \tilde{\lambda}$ or $\lambda = \tilde{\lambda}$, and $\omega \preceq \tilde{\omega}$ if $\omega(x) \preceq \tilde{\omega}(x)$, for all $x \in \mathbb{Z}$. In this case, we also say that the environment $\tilde{\omega}$ *dominates* the environment ω . The following lemma relating the possibility of right transience in different environments will be important for the analysis of transience and recurrence in the critical case $\alpha = 1/2$.

Lemma 4. *If $q < p$ and $\omega \preceq \tilde{\omega}$, then $\mathbb{P}_\omega(\mathcal{A}_0^+) \leq \mathbb{P}_{\tilde{\omega}}(\mathcal{A}_0^+)$. In particular, by Lemma 1, if $q < p$, $\omega \preceq \tilde{\omega}$, and $\mathbb{P}_{\tilde{\omega}}(X_n \rightarrow \infty) = 0$, then $\mathbb{P}_\omega(X_n \rightarrow \infty) = 0$.*

For the proof it will be convenient to introduce the following definitions.

- The threshold function $f : \Lambda \times [0, 1] \rightarrow \{1, -1\}$ is defined by

$$\begin{aligned} f(\lambda, u) &= \mathbf{1}\{u \leq p\} - \mathbf{1}\{u > p\}, & \text{if } \lambda \in \Lambda_p \\ f(\lambda, u) &= \mathbf{1}\{u \leq q\} - \mathbf{1}\{u > q\}, & \text{if } \lambda \in \Lambda_q. \end{aligned}$$

- The transition function $g : \Lambda \times \{1, -1\} \rightarrow \Lambda$ is defined by

$$g(\lambda, j) = \lambda' \iff \{Y_n^x = \lambda, J_n^x = j\} \text{ implies } Y_{n+1}^x = \lambda'.$$

That is, $g(\lambda, j)$ is the (deterministic) next configuration at site x if the walk jumps in direction j from site x when x is in configuration λ .

Proof of Lemma 4. For $x \in \mathbb{Z}$, let $(Y_n^x, J_n^x)_{n \in \mathbb{N}}$ and $(\tilde{Y}_n^x, \tilde{J}_n^x)_{n \in \mathbb{N}}$ denote, respectively, the state sequences of the extended single site Markov chains at x for the environments ω and $\tilde{\omega}$. Also, let $(U_n^x)_{x \in \mathbb{Z}, n \in \mathbb{N}}$ be i.i.d. uniform $([0, 1])$ random variables. For each x , we will use the i.i.d. sequence $(U_n^x)_{n \in \mathbb{N}}$ to couple the state sequences $(Y_n^x, J_n^x)_{n \in \mathbb{N}}$ and $(\tilde{Y}_n^x, \tilde{J}_n^x)_{n \in \mathbb{N}}$ in such a way that $J_n^x \leq \tilde{J}_n^x$, for all n . By independence, this coupling at each individual site x passes to a coupling of the entire joint processes $(Y_n^x, J_n^x)_{x \in \mathbb{Z}, n \in \mathbb{N}}$ and $(\tilde{Y}_n^x, \tilde{J}_n^x)_{x \in \mathbb{Z}, n \in \mathbb{N}}$, with the correct law. This final larger coupling will be used to show that $\mathbb{P}_\omega(\mathcal{A}_0^+) \leq \mathbb{P}_{\tilde{\omega}}(\mathcal{A}_0^+)$.

Step 1: The Coupling

For a fixed site x , we construct the sequences $(Y_n^x, J_n^x)_{n \in \mathbb{N}}$ and $(\tilde{Y}_n^x, \tilde{J}_n^x)_{n \in \mathbb{N}}$ inductively from the i.i.d. random variables $(U_n^x)_{n \in \mathbb{N}}$ as follows.

- $Y_1^x = \omega(x)$ and $\tilde{Y}_1^x = \tilde{\omega}(x)$.
- For $n \geq 1$,

$$J_n^x = f(Y_n^x, U_n^x), Y_{n+1}^x = g(Y_n^x, J_n^x) \quad \text{and} \quad \tilde{J}_n^x = f(\tilde{Y}_n^x, U_n^x), \tilde{Y}_{n+1}^x = g(\tilde{Y}_n^x, \tilde{J}_n^x).$$

Clearly, the sequences $(Y_n^x, J_n^x)_{n \in \mathbb{N}}$ and $(\tilde{Y}_n^x, \tilde{J}_n^x)_{n \in \mathbb{N}}$ each have the appropriate marginal laws under this coupling. Moreover, by considering the various possible cases for $Y_n^x, \tilde{Y}_n^x \in \Lambda$ and possible ranges for $U_n^x \in [0, 1]$ one finds that, since $q < p$, whatever the value of U_n^x is:

$$Y_n^x \preceq \tilde{Y}_n^x \implies J_n^x \leq \tilde{J}_n^x \text{ and } Y_{n+1}^x \preceq \tilde{Y}_{n+1}^x.$$

Since $Y_1^x = \omega(x) \preceq \tilde{\omega}(x) = \tilde{Y}_1^x$ it follows, by induction, that

$$J_n^x \leq \tilde{J}_n^x, \text{ for all } n. \quad (2.10)$$

Step 2: Relation to the probability of \mathcal{A}_0^+

Let $(Z_x)_{x \geq 0}$ and $(\tilde{Z}_x)_{x \geq 0}$ denote, respectively, the right jumps Markov chains constructed from the jump patterns $(J_n^x)_{x \in \mathbb{Z}, n \in \mathbb{N}}$ and $(\tilde{J}_n^x)_{x \in \mathbb{Z}, n \in \mathbb{N}}$ according to (2.6). Also, for $x, k \geq 0$ define $\Theta_{x,k}$ and $\tilde{\Theta}_{x,k}$ by

$$\Theta_{x,k} = \inf \left\{ n : \sum_{m=1}^n \mathbb{1}\{J_m^x = -1\} = k \right\}, \quad \tilde{\Theta}_{x,k} = \inf \left\{ n : \sum_{m=1}^n \mathbb{1}\{\tilde{J}_m^x = -1\} = k \right\}.$$

If $Z_{x-1} \leq \tilde{Z}_{x-1}$, then applying the definition (2.6) gives

$$\begin{aligned} Z_x &= \sum_{m=1}^{\Theta_{x,Z_{x-1}}} \mathbb{1}\{J_m^x = 1\} \stackrel{(a)}{\leq} \sum_{m=1}^{\tilde{\Theta}_{x,Z_{x-1}}} \mathbb{1}\{J_m^x = 1\} \\ &\leq \sum_{m=1}^{\tilde{\Theta}_{x,\tilde{Z}_{x-1}}} \mathbb{1}\{J_m^x = 1\} \stackrel{(b)}{\leq} \sum_{m=1}^{\tilde{\Theta}_{x,\tilde{Z}_{x-1}}} \mathbb{1}\{\tilde{J}_m^x = 1\} = \tilde{Z}_x. \end{aligned}$$

Here, (a) follows from (2.10), which implies $\Theta_{x,k} \leq \tilde{\Theta}_{x,k}$ for any k , and (b) follows directly from (2.10). Since $Z_0 = \tilde{Z}_0 = 1$, it follows, by induction, that

$$Z_x \leq \tilde{Z}_x, \text{ for all } x \in \mathbb{Z}. \quad (2.11)$$

Now, since (2.10) and (2.11) both hold with probability 1, under our coupling, it follows from Lemma 2 that

$$\begin{aligned} \mathbb{P}_\omega(\mathcal{A}_0^+) &= \mathbb{P}_\omega(X_1 = 1) \cdot \mathbb{P}_\omega(Z_x > 0, \forall x > 0) \\ &= \mathbb{P}_\omega(J_1^0 = 1) \cdot \mathbb{P}_\omega(Z_x > 0, \forall x > 0) \\ &\leq \mathbb{P}_{\tilde{\omega}}(J_1^0 = 1) \cdot \mathbb{P}_{\tilde{\omega}}(Z_x > 0, \forall x > 0) \\ &= \mathbb{P}_{\tilde{\omega}}(X_1 = 1) \cdot \mathbb{P}_{\tilde{\omega}}(Z_x > 0, \forall x > 0) \\ &= \mathbb{P}_{\tilde{\omega}}(\mathcal{A}_0^+). \end{aligned}$$

Here, we have dropped the tildes on all random variables corresponding to the initial environment $\tilde{\omega}$, since the probability measure $\mathbb{P}_{\tilde{\omega}}$ is now explicit. \square

3 The Noncritical Case

Here we analyze the behavior of the random walk (X_n) for $\alpha \neq 1/2$, proving Theorems 1–4. We begin in section 3.1 with a key lemma for the survival probability of the right jumps Markov chain (Z_x) , from which we derive a number of useful corollaries. Using these results, Theorem 1 on the cutoff for

right/left transience and Theorem 2 on ballisticity of the random walk are then proved in sections 3.2 and 3.3. Theorems 3 and 4 on the exact speed of the random walk in certain special cases are proved afterward in sections 3.4 and 3.5.

Throughout we use the following notation:

- $T_x^{(i)}$, $x \in \mathbb{Z}$ and $i \in \mathbb{N}$, is the i -th hitting time of site x .

$$T_x^{(1)} = T_x \quad \text{and} \quad T_x^{(i+1)} = \inf\{n > T_x^{(i)} : X_n = x\}, \quad (3.1)$$

with the convention $T_x^{(j)} = \infty$, for all $j > i$, if $T_x^{(i)} = \infty$.

- N_x is the total number of visits to site x , as in (1.1), and N_x^y is the number of visits to site x up to time T_y .

$$N_x = |\{n \geq 0 : X_n = x\}|, \quad x \in \mathbb{Z}.$$

$$N_x^y = |\{0 \leq n \leq T_y : X_n = x\}|, \quad x, y \in \mathbb{Z}.$$

- R_x is the total number of right jumps from site x , and L_x is the total number of left jumps from site x .

$$R_x = |\{n \geq 0 : X_n = x, X_{n+1} = x + 1\}|, \quad x \in \mathbb{Z}.$$

$$L_x = |\{n \geq 0 : X_n = x, X_{n+1} = x - 1\}|, \quad x \in \mathbb{Z}.$$

- B_x is the farthest distance the random walk ever steps backward from site x after hitting x for the first time.

$$B_x = \sup\{k \geq 0 : \exists n \geq T_x \text{ with } X_n = x - k\}, \quad x \in \mathbb{Z}.$$

In the case $T_x = \infty$, $B_x \equiv 0$.

- \mathcal{A}_n^+ , given by (2.8), is the event that the random walk steps to the right at time n and never returns to its time- n location.
- \mathcal{B}_ϵ , $0 < \epsilon < 1$, is the event that $B_x \leq \epsilon x$, for all sufficiently large x .

$$\mathcal{B}_\epsilon = \{\exists N \in \mathbb{N} \text{ s.t. } B_x \leq \epsilon x, \forall x \geq N\}. \quad (3.2)$$

3.1 Survival of Right Jumps Markov Chain (Z_x)

Lemma 5. *If $\alpha = \alpha(p, q, R, L) > 1/2$, then there exists some $\beta = \beta(p, q, R, L) > 0$ such that, for any initial environment ω ,*

$$\mathbb{P}_\omega(Z_x > 0, \forall x > 0) \geq \beta.$$

Proof. Fix p, q, R, L such that $\alpha > 1/2$ and any initial environment ω . Define $0 < \epsilon < 1/4$ by the relation $\alpha = 1/2 + 2\epsilon$, and for $\hat{\lambda} = (\lambda, j) \in \hat{\Lambda}$, let $\phi(\hat{\lambda}) = \mathbf{1}\{j = 1\}$.

By (2.5) we have $\alpha = \mathbb{E}_{\hat{\pi}}(\phi)$, where $\hat{\pi}$ is the stationary distribution for the extended single site transition matrix \hat{M} . So, by standard large deviation bounds for finite-state Markov chains, there exist some $0 < a < 1$ and $n_0 \in \mathbb{N}$ such that for any initial state $\hat{\lambda} \in \hat{\Lambda}$ the Markov chain (\hat{Y}_n) with transition matrix \hat{M} satisfies

$$\mathbb{P}_{\hat{\lambda}}\left(\frac{1}{n} \sum_{m=1}^n \phi(\hat{Y}_m) \leq 1/2 + \epsilon\right) = \mathbb{P}_{\hat{\lambda}}\left(\frac{1}{n} \sum_{m=1}^n \phi(\hat{Y}_m) \leq \mathbb{E}_{\hat{\pi}}(\phi) - \epsilon\right) \leq a^n, \quad n \geq n_0.$$

Using this estimate we obtain the following important inequality:

$$\begin{aligned}
& \mathbb{P}_\omega(Z_x \leq n(1/2 + \epsilon)/(1/2 - \epsilon) \mid Z_{x-1} = n) \\
&= \mathbb{P}_\omega(\Theta_x \leq n/(1/2 - \epsilon) \mid Z_{x-1} = n) \\
&= \mathbb{P}_\omega\left(\exists n \leq m \leq n/(1/2 - \epsilon) : \sum_{i=1}^m (1 - \phi(\widehat{Y}_i^x)) = n\right) \\
&= \mathbb{P}_\omega\left(\exists n \leq m \leq n/(1/2 - \epsilon) : \frac{1}{m} \sum_{i=1}^m \phi(\widehat{Y}_i^x) = \frac{m-n}{m}\right) \\
&\leq \mathbb{P}_\omega\left(\exists n \leq m \leq n/(1/2 - \epsilon) : \frac{1}{m} \sum_{i=1}^m \phi(\widehat{Y}_i^x) \leq 1/2 + \epsilon\right) \\
&\leq \mathbb{P}_\omega\left(\exists m \geq n : \frac{1}{m} \sum_{i=1}^m \phi(\widehat{Y}_i^x) \leq 1/2 + \epsilon\right) \\
&\leq \sum_{m=n}^{\infty} a^m = \frac{a^n}{1-a}, \text{ for all } n \geq n_0 \text{ and } x \in \mathbb{N}. \tag{3.3}
\end{aligned}$$

Now, define $b > 1$ by $b = \frac{1/2+\epsilon}{1/2-\epsilon}$, and take $n_1 \geq n_0$ sufficiently large that $\frac{a^{n_1}}{1-a} < 1$. Thus, $\frac{a^{\lceil n_1 b^{x-1} \rceil}}{1-a} < 1, \forall x \in \mathbb{N}$. Applying the inequality (3.3) gives,

$$\begin{aligned}
& \mathbb{P}_\omega(Z_x > 0, \forall x > 0) \\
&\geq \mathbb{P}_\omega(Z_x \geq n_1 b^x, \forall x > 0) \\
&= \mathbb{P}_\omega(Z_1 \geq n_1 b) \cdot \prod_{x=2}^{\infty} \mathbb{P}_\omega(Z_x \geq n_1 b^x \mid Z_1 \geq n_1 b, \dots, Z_{x-1} \geq n_1 b^{x-1}) \\
&\geq \mathbb{P}_\omega(Z_1 \geq n_1 b) \cdot \prod_{x=2}^{\infty} \mathbb{P}_\omega(Z_x \geq n_1 b^x \mid Z_{x-1} = \lceil n_1 b^{x-1} \rceil) \\
&\geq (\min\{p, q\})^{\lceil n_1 b \rceil} \cdot \prod_{x=2}^{\infty} \left(1 - \frac{a^{\lceil n_1 b^{x-1} \rceil}}{1-a}\right) \equiv \beta.
\end{aligned}$$

Note that $\sum_{x=2}^{\infty} \frac{a^{\lceil n_1 b^{x-1} \rceil}}{1-a} < \infty$, so $\prod_{x=2}^{\infty} \left(1 - \frac{a^{\lceil n_1 b^{x-1} \rceil}}{1-a}\right) > 0$. \square

Corollary 2. *If $\alpha = \alpha(p, q, R, L) > 1/2$ then there exists some $\beta = \beta(p, q, R, L) > 0$, such that for any initial environment ω and random walk path (x_0, x_1, \dots, x_n) ,*

$$\mathbb{P}_{\omega, x_0}(\mathcal{A}_n^+ \mid X_0 = x_0, \dots, X_n = x_n) \geq \beta. \tag{3.4}$$

Proof. Since the claimed bound is uniform in the initial environment ω , it suffices to consider the case $x_0 = n = 0$. By Lemma 5, there exists some $\beta' > 0$ such that $\mathbb{P}_\omega(Z_x > 0, \forall x > 0) \geq \beta'$, for any initial environment ω . Thus, by Lemma 2,

$$\mathbb{P}_\omega(\mathcal{A}_0^+) \geq \min\{p, q\} \cdot \beta' \equiv \beta$$

for any initial environment ω . \square

Corollary 3. *If $\alpha > 1/2$ then, for any initial environment ω and site $x \geq 0$,*

$$\mathbb{P}_\omega(N_x \geq k) \leq (1 - \beta)^{k-1}, \text{ for all } k \geq 1$$

where $\beta > 0$ is the constant in Corollary 2.

Proof. Let $A_x^{(i)}$ be the set of all random walk paths (x_0, x_1, \dots, x_n) , of any length n , which end in an i -th hitting time of site x . That is, $\{X_0 = x_0, X_1 = x_1, \dots, X_n = x_n\} \implies T_x^{(i)} = n$. For brevity we denote (X_0, \dots, X_n) as X_0^n and (x_0, \dots, x_n) as x_0^n . By Corollary 2, for any $i \geq 1$, we have

$$\begin{aligned}
& \mathbb{P}_\omega(T_x^{(i+1)} < \infty | T_x^{(i)} < \infty) \\
&= \sum_{x_0^n \in A_x^{(i)}} \mathbb{P}_\omega(X_0^n = x_0^n | T_x^{(i)} < \infty) \cdot \mathbb{P}_\omega(T_x^{(i+1)} < \infty | T_x^{(i)} < \infty, X_0^n = x_0^n) \\
&= \sum_{x_0^n \in A_x^{(i)}} \mathbb{P}_\omega(X_0^n = x_0^n | T_x^{(i)} < \infty) \cdot \mathbb{P}_\omega(T_x^{(i+1)} < \infty | X_0^n = x_0^n) \\
&\leq \sum_{x_0^n \in A_x^{(i)}} \mathbb{P}_\omega(X_0^n = x_0^n | T_x^{(i)} < \infty) \cdot \mathbb{P}_\omega((\mathcal{A}_n^+)^c | X_0^n = x_0^n) \\
&\leq (1 - \beta).
\end{aligned}$$

Hence, for each $k \geq 1$,

$$\mathbb{P}_\omega(N_x \geq k) = \mathbb{P}_\omega(T_x^{(1)} < \infty) \cdot \prod_{i=1}^{k-1} \mathbb{P}_\omega(T_x^{(i+1)} < \infty | T_x^{(i)} < \infty) \leq (1 - \beta)^{k-1}.$$

□

Corollary 4. *If $\alpha > 1/2$ then, for any initial environment ω and site $x \geq 0$,*

$$\mathbb{P}_\omega(B_x \geq k) \leq (1 - \beta)^k, \quad \text{for all } k \geq 1 \tag{3.5}$$

where $\beta > 0$ is the constant in Corollary 2. In particular, by the Borel-Cantelli lemma,

$$\mathbb{P}_\omega(\mathcal{B}_\epsilon) = 1, \quad \text{for each } 0 < \epsilon < 1.$$

Proof. The proof is similar to that of Corollary 3. For $x \in \mathbb{Z}$, let $\tau_x^{(0)}$ be the first hitting time of site x , and let $\tau_x^{(i)}$, $i \in \mathbb{N}$, be the first time greater than $\tau_x^{(i-1)}$ at which the walk steps backward from its position $x - (i - 1)$ at time $\tau_x^{(i-1)}$. That is, $\tau_x^{(0)} = T_x$, and for $i \geq 1$,

$$\begin{aligned}
\tau_x^{(i)} &= \inf\{n > \tau_x^{(i-1)} : X_n < X_{\tau_x^{(i-1)}}\} \\
&= \inf\{n > T_x : X_n = x - i\}
\end{aligned}$$

with the convention $\tau_x^{(j)} = \infty$, for all $j > i$, if $\tau_x^{(i)} = \infty$. Also, let $A_x^{(i)}$ be the set of all random walk paths (x_0, x_1, \dots, x_n) , of any length n , which end in an i -th “back step time” from site x . That is, $\{X_0 = x_0, X_1 = x_1, \dots, X_n = x_n\} \implies \tau_x^{(i)} = n$. As above, we denote (X_0, \dots, X_n) as X_0^n and (x_0, \dots, x_n) as x_0^n . By Corollary 2, for any $i \geq 0$, we have

$$\begin{aligned}
& \mathbb{P}_\omega(\tau_x^{(i+1)} < \infty | \tau_x^{(i)} < \infty) \\
&= \sum_{x_0^n \in A_x^{(i)}} \mathbb{P}_\omega(X_0^n = x_0^n | \tau_x^{(i)} < \infty) \cdot \mathbb{P}_\omega(\tau_x^{(i+1)} < \infty | \tau_x^{(i)} < \infty, X_0^n = x_0^n) \\
&= \sum_{x_0^n \in A_x^{(i)}} \mathbb{P}_\omega(X_0^n = x_0^n | \tau_x^{(i)} < \infty) \cdot \mathbb{P}_\omega(\tau_x^{(i+1)} < \infty | X_0^n = x_0^n) \\
&\leq \sum_{x_0^n \in A_x^{(i)}} \mathbb{P}_\omega(X_0^n = x_0^n | \tau_x^{(i)} < \infty) \cdot \mathbb{P}_\omega((\mathcal{A}_n^+)^c | X_0^n = x_0^n) \\
&\leq (1 - \beta).
\end{aligned}$$

So, for each $k \geq 1$,

$$\mathbb{P}_\omega(B_x \geq k) = \mathbb{P}_\omega(\tau_x^{(0)} < \infty) \cdot \prod_{i=0}^{k-1} \mathbb{P}_\omega(\tau_x^{(i+1)} < \infty | \tau_x^{(i)} < \infty) \leq (1 - \beta)^k.$$

□

3.2 Proof of Theorem 1

Proof of Theorem 1. If $\alpha > 1/2$ then Corollary 4 implies that B_0 is \mathbb{P}_ω a.s. finite, for any initial environment ω . Thus, by part (ii) of Lemma 1, for $\alpha > 1/2$ we must have $X_n \rightarrow \infty$, \mathbb{P}_ω a.s., for any initial environment ω . It follows by symmetry that, for $\alpha < 1/2$ and any initial environment ω , $X_n \rightarrow -\infty$, \mathbb{P}_ω a.s. □

3.3 Proof of Theorem 2

For the proof of Theorem 2 we will assume that $\alpha > 1/2$, the case $\alpha < 1/2$ follows by symmetry considerations. The primary ingredients for the proof are Corollaries 3 and 4, above, and Lemmas 6 and 7, given below. Lemma 6 is a simple consequence of Theorem 1. Lemma 7 shows that, when $\alpha > 1/2$, the sequence (N_x) obeys a strong law of large numbers. The proof of this fact is somewhat lengthy and is deferred to Appendix C.

Lemma 6. *Assume that $\alpha > 1/2$ and $X_0 = 0$.*

(i) *For all $x \geq 0$, the random variables N_x , L_x , and R_x are each independent of the environment to the left of site x when site x is first reached:*

$$N_x, L_x, R_x \perp \{\omega_{T_x}(y), y < x\}.$$

(ii) *N_x^y and N_y are independent, for all $0 \leq x < y$.*

(iii) *If, for some $y \geq 0$, $\omega(x)$ is constant for $x \geq y$, then N_x and N_y have the same distribution for all $x \geq y$. Similarly, if $\omega(x) = \omega(y)$ for all $x \geq y$, then R_x and R_y have the same distribution for all $x \geq y$, and L_x and L_y have the same distribution for all $x \geq y$.*

Proof. Since $\alpha > 1/2$ and $X_0 = 0$, Theorem 1 shows that T_x is a.s. finite, for each $x \geq 0$, and that regardless of the environment to the left of site x at time T_x , the walk returns to site x with probability 1 each time it steps left from x . This implies (i). Now, (ii) and (iii) follow easily since (i) shows that the distribution of N_x , L_x , and R_x are each entirely determined by the values of $\omega_{T_x}(y), y \geq x$, which are the same as the original values $\omega(y), y \geq x$. □

Lemma 7. *If $\alpha > 1/2$ then, for any initial environment ω ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x=1}^n (N_x - \mathbb{E}_\omega(N_x)) = 0, \mathbb{P}_\omega \text{ a.s.}$$

Proof of Theorem 2, Equation (1.3), with $\alpha > 1/2$. Let $\alpha > 1/2$, and fix any initial environment ω . Also, let $\beta > 0$ be the constant defined in Corollary 2. We will show that:

$$(i) \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{y=1}^x N_y \leq 1/\beta, \mathbb{P}_\omega \text{ a.s.}$$

$$(ii) \limsup_{x \rightarrow \infty} T_x/x \leq \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{y=1}^x N_y, \mathbb{P}_\omega \text{ a.s.}$$

$$(iii) \liminf_{n \rightarrow \infty} X_n/n \geq (\limsup_{x \rightarrow \infty} T_x/x)^{-1}, \mathbb{P}_\omega \text{ a.s.}$$

The result (1.3) follows directly from these three facts.

Proof of (i): This is immediate from Lemma 7 and Corollary 3.

Proof of (ii): Since $\alpha > 1/2$, $X_n \rightarrow \infty$ \mathbb{P}_ω a.s. So, $\sum_{x \leq 0} N_x$ is \mathbb{P}_ω a.s. finite. Thus, \mathbb{P}_ω a.s. we have

$$\limsup_{x \rightarrow \infty} \frac{T_x}{x} = \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{y=-\infty}^x N_y \leq \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{y=-\infty}^x N_y = \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{y=1}^x N_y.$$

Proof of (iii): For $0 < \epsilon < 1$, let $\mathcal{B}'_\epsilon = \mathcal{B}_\epsilon \cap \{T_x < \infty, \forall x > 0\}$, where \mathcal{B}_ϵ is defined by (3.2). On the event \mathcal{B}'_ϵ , for all sufficiently large x and $T_x \leq n < T_{x+1}$, we have

$$\frac{X_n}{n} \geq \frac{x - \epsilon x}{T_{x+1}} = (1 - \epsilon) \frac{x + 1}{T_{x+1}} - \frac{1 - \epsilon}{T_{x+1}}.$$

So,

$$\liminf_{n \rightarrow \infty} \frac{X_n}{n} \geq \liminf_{x \rightarrow \infty} \left((1 - \epsilon) \frac{x + 1}{T_{x+1}} - \frac{1 - \epsilon}{T_{x+1}} \right) = (1 - \epsilon) \cdot \left(\limsup_{x \rightarrow \infty} T_x/x \right)^{-1}.$$

The result follows since $\mathbb{P}_\omega(\mathcal{B}'_\epsilon) = 1$, for each $\epsilon > 0$, due to Corollary 4 and the fact that the random walk (X_n) is a.s. right transient with $\alpha > 1/2$. \square

Proof of Theorem 2, Equation (1.4), with $\alpha > 1/2$. By assumption $\omega(x)$ is constant for $x \geq m$, so Lemma 6 implies N_x and N_m are equal in law, for all $x \geq m$, under \mathbb{P}_ω . Thus,

$$\mathbb{E}_\omega(N_x) = \mathbb{E}_\omega(N_m) \equiv \gamma, \text{ for all } x \geq m. \quad (3.6)$$

To show that $\lim_{n \rightarrow \infty} X_n/n = 1/\gamma$, \mathbb{P}_ω a.s., note first that (3.6) and Lemma 7 imply that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{y=1}^x N_y = \gamma, \mathbb{P}_\omega \text{ a.s.} \quad (3.7)$$

So, by point (ii) above, we have

$$\limsup_{x \rightarrow \infty} T_x/x \leq \gamma, \mathbb{P}_\omega \text{ a.s.} \quad (3.8)$$

On the other hand, on the event $\mathcal{B}'_\epsilon = \mathcal{B}_\epsilon \cap \{T_x < \infty, \forall x > 0\}$, we have

$$\liminf_{x \rightarrow \infty} \frac{T_x}{x} = \liminf_{x \rightarrow \infty} \frac{1}{x} \sum_{y=-\infty}^x N_y \geq \liminf_{x \rightarrow \infty} \frac{1}{x} \sum_{y=1}^{\lfloor (1-\epsilon)x \rfloor} N_y = \liminf_{x \rightarrow \infty} \frac{1}{x} \sum_{y=1}^{\lfloor (1-\epsilon)x \rfloor} N_y. \quad (3.9)$$

Since $\mathbb{P}_\omega(\mathcal{B}'_\epsilon) = 1$, for each $\epsilon > 0$, and the RHS of (3.9) is equal to $(1 - \epsilon)\gamma$, \mathbb{P}_ω a.s., by (3.7), this implies

$$\liminf_{x \rightarrow \infty} T_x/x \geq \gamma, \mathbb{P}_\omega \text{ a.s.} \quad (3.10)$$

Together, (3.8) and (3.10) imply $\lim_{x \rightarrow \infty} T_x/x = \gamma$, \mathbb{P}_ω a.s., so the result follows from Lemma 3. \square

3.4 Proof of Theorem 3

The proof of Theorem 3 is based on the speed formula given in Theorem 2, and uses the assumptions on L and ω to obtain a more explicit expression for γ .

Proof of Theorem 3. We will prove the theorem under the assumption $\omega(x) = (q, 0)$, for all $x \geq 0$. The case $\omega(x) = (q, 0)$ in a neighborhood of $+\infty$ follows immediately from this. The main observation is that since $L = 1$ and the random walk starts from $X_0 = 0$ in an environment ω satisfying $\omega(x) = (q, 0)$, for all $x \geq 0$, we have

$$\omega_n(x) = (q, 0), \text{ for each } n \geq 0 \text{ and } x > X_n.$$

That is, the environment to the right of the current position of the random walk always consists entirely of sites in the $(q, 0)$ configuration. Consequently, when the walk jumps right the environment both at its current position and to its right consists entirely of sites in the $(q, 0)$ configuration:

$$\{X_{n-1} = x - 1 \text{ and } X_n = x\} \implies \omega_n(y) = (q, 0), \text{ for all } y \geq x. \quad (3.11)$$

Using this fact we will show that:

- (i) $\gamma \equiv \mathbb{E}_\omega(N_0) = \frac{1+\eta}{1-\eta}$, where $\eta \equiv \mathbb{P}_\omega(T_{-1} < \infty)$.
- (ii) η satisfies $P(\eta) = 0$, where P is as in (1.10).

Also, using direct calculus arguments we will show that:

- (iii) The polynomial $P(t)$ has a unique real root in the interval $(0,1)$.

Clearly, $\eta > \mathbb{P}_\omega(T_{-1} = 1) = 1 - q$, and by Corollary 2, we know $\eta < 1$. Thus, the theorem follows from points (i)-(iii) and Theorem 2.

Proof of (i): Since $\alpha > 1/2$ the random walk returns to site 0 with probability 1 every time it steps left from 0, and by (3.11), applied in the case $x = 0$, we know that at each time n when the random walk returns to site 0 after stepping left on its last visit, we have $\omega_n(x) = \omega_0(x) = (q, 0)$, for all $x \geq 0$. Therefore, since $\mathbb{P}_{\omega'}(T_{-1} < \infty)$ does not depend on the values of $\omega'(x)$, $x < 0$, it follows that L_0 is a geometric random variable with distribution

$$\mathbb{P}_\omega(L_0 = k) = \eta^k(1 - \eta), \quad k \geq 0.$$

Hence, by Lemma 6,

$$\mathbb{E}_\omega(N_0) = \mathbb{E}_\omega(R_0 + L_0) \stackrel{(*)}{=} [\mathbb{E}_\omega(L_1) + 1] + \mathbb{E}_\omega(L_0) = 2\mathbb{E}_\omega(L_0) + 1 = \frac{1 + \eta}{1 - \eta}.$$

Step (*) follows from the fact that $R_0 = L_1 + 1$ a.s., since the random walk is a.s. transient to $+\infty$.

Proof of (ii): For $i \geq 0$, let A_i be the event that the random walk steps right from site 0 and eventually returns i times without ever jumping left from 0, and let A'_i be the event that the random walk steps right from site 0 and eventually returns i times without stepping left from 0, but then does step left on its next visit:

$$\begin{aligned} A_i &= \{N_0 \geq i + 1, T_{-1} > T_0^{(i+1)}\}, \\ A'_i &= \{N_0 \geq i + 1, T_{-1} = T_0^{(i+1)} + 1\}. \end{aligned}$$

Clearly, $\mathbb{P}_\omega(A_0) = 1$. We claim also that:

$$\mathbb{P}_\omega(A'_i|A_i) = \begin{cases} (1-q), & \text{for } 0 \leq i \leq R-1 \\ (1-p), & \text{for } i \geq R \end{cases} \quad (3.12)$$

and

$$\mathbb{P}_\omega(A_{i+1}|A_i) = \begin{cases} q\eta, & \text{for } 0 \leq i \leq R-1 \\ p\eta, & \text{for } i \geq R. \end{cases} \quad (3.13)$$

To see (3.12), note that after jumping right from site 0 and returning i times in a row, site 0 will be in configuration (q, i) , for $0 \leq i \leq R-1$, and in configuration $(p, 0)$ for $i \geq R$. Thus, for $0 \leq i \leq R-1$, we have

$$\mathbb{P}_\omega(A'_i|A_i) = \mathbb{P}_\omega\left(X_{T_0^{(i+1)}+1} = -1 \mid \omega_{T_0^{(i+1)}}(0) = (q, i)\right) = (1-q)$$

and, for $i \geq R$, we have

$$\mathbb{P}_\omega(A'_i|A_i) = \mathbb{P}_\omega\left(X_{T_0^{(i+1)}+1} = -1 \mid \omega_{T_0^{(i+1)}}(0) = (p, 0)\right) = (1-p).$$

Now, (3.13) follows from (3.12) and the following calculation which is valid for all $i \geq 0$:

$$\begin{aligned} \mathbb{P}_\omega(A_{i+1}|A_i) &= \mathbb{P}_\omega(X_{T_0^{(i+1)}+1} = 1|A_i) \cdot \mathbb{P}_\omega(T_0^{(i+2)} < \infty | A_i, X_{T_0^{(i+1)}+1} = 1) \\ &= \mathbb{P}_\omega((A'_i)^c|A_i) \cdot \eta. \end{aligned}$$

The second equality above follows from (3.11), which implies that on the event $\{X_{T_0^{(i+1)}+1} = 1\}$, all sites $x \geq 1$ are in the $(q, 0)$ configuration at time $T_0^{(i+1)} + 1$.

Now, from (3.12) and (3.13), along with the fact $\mathbb{P}_\omega(A_0) = 1$, we conclude that

$$\mathbb{P}_\omega(A'_i) = \left(\prod_{j=1}^i \mathbb{P}_\omega(A_j|A_{j-1}) \right) \cdot \mathbb{P}_\omega(A'_i|A_i) = \begin{cases} (q\eta)^i(1-q), & \text{for } 0 \leq i \leq R-1 \\ (q\eta)^R(p\eta)^{i-R}(1-p), & \text{for } i \geq R. \end{cases}$$

So,

$$\eta = \mathbb{P}_\omega(T_{-1} < \infty) = \sum_{i=0}^{\infty} \mathbb{P}_\omega(A'_i) = (1-q) \frac{1-(q\eta)^R}{1-q\eta} + (1-p) \frac{(q\eta)^R}{1-p\eta}.$$

For $0 < \eta < 1$, this condition is equivalent to $P(\eta) = 0$.

Proof of (iii): From point (ii) above we know that $\eta \in (0, 1)$ is a root of the polynomial $P(t)$. We now show that there cannot be any other roots in $(0, 1)$. Observe that $t = 1$ is a root of P and that P factors as

$$P(t) = (1-t) \left(1 - q + (pq - p - q)t + pqt^2 - (p-q)q^R t^R \right) \equiv (1-t)Q(t).$$

Thus, we need to show that the only root of Q in $(0, 1)$ is η . Observe that $\frac{1}{q} > 1$ is a root of Q . For $R \geq 3$, we have

$$Q''(t) = 2pq - (p-q)q^R R(R-1)t^{R-2}.$$

So, if $R \geq 3$ and $q \geq p$ then Q is convex and can have at most two real roots. Also, if $R \in \{1, 2\}$ then Q is quadratic and, thus, has at most two real roots. In either case, this completes the proof.

Now assume that $R \geq 3$ and $q < p$. In this case, Q'' has one real root. Thus, Q' can have no more than two real roots. Let t^+ denote the largest root of Q in $(0, 1)$. We will show below that $Q(1) < 0$. Using this along with the facts that $Q(t^+) = Q(\frac{1}{q}) = 0$ and $Q(t) < 0$ for sufficiently large t , it follows that Q' has two roots in (t^+, ∞) . If there were another root $t^- \in (0, 1)$ of Q , then Q' would have to have a root in (t^-, t^+) , but this is impossible since Q' cannot have more than two real roots. It remains to show that $Q(1) < 0$.

We have

$$Q(1) = 1 - 2q + q^{R+1} - p(1 - 2q + q^R). \quad (3.14)$$

Since we are assuming that $\alpha > \frac{1}{2}$, it follows from Proposition 1 that $p > \frac{1-2q+q^{R+1}}{1-2q+q^R} \equiv p_0$, if $1 - 2q + q^{R+1} > 0$. On the other hand, if $1 - 2q + q^{R+1} \leq 0$, then $p \in (0, 1)$ is unrestricted. In the former case, it follows from (3.14) that for any q , $Q(1) < 1 - 2q + q^{R+1} - p_0(1 - 2q + q^R) = 0$. In the latter case:

- If $1 - 2q + q^R = 0$, then (3.14) implies $Q(1) = 1 - 2q + q^{R+1} < 0$.
- If $1 - 2q + q^R > 0$, then (3.14) implies $Q(1) < 1 - 2q + q^{R+1} \leq 0$, for all $p \in (0, 1)$.
- If $1 - 2q + q^R < 0$, then (3.14) implies $Q(1) < 1 - 2q + q^{R+1} - (1 - 2q + q^R) < 0$, for all $p \in (0, 1)$.

□

3.5 Proof of Theorem 4

Unlike the proof of Theorem 3 for the speed with $L = 1$, the proof of Theorem 4 for the speed with $R = 1$ does not rely on the implicit characterization of the speed given by Theorem 2 in terms of γ . Instead, the proof is based on a direct method for estimating the hitting times T_x for large x .

Proof of Theorem 4. For $0 \leq i \leq L-1$, we define a_i to be the expected hitting time of site 1, starting from site 0, in an initial environment with all sites $x < 0$ in the $(p, 0)$ configuration and site 0 in the (p, i) configuration. Also, we define a_L to be the expected hitting time of site 1, starting from site 0, in an initial environment with all sites $x < 0$ in the $(p, 0)$ configuration and site 0 in the $(q, 0)$ configuration.

$$a_i = \mathbb{E}_{\omega^{(i)}}(T_1), \quad 0 \leq i \leq L,$$

where the environments $\omega^{(i)}$ satisfy:

$$\begin{aligned} \omega^{(i)}(x) &= (p, 0), \quad x < 0 \quad \text{and} \quad 0 \leq i \leq L. \\ \omega^{(i)}(0) &= (p, i), \quad 0 \leq i \leq L-1. \\ \omega^{(L)}(0) &= (q, 0). \end{aligned}$$

The proof proceeds in two steps. First we set up a linear system of equations for the a_i 's, which can be solved to obtain the desired speed formula in the case that the initial environment ω satisfies $\omega(x) = (p, 0)$, for all $x < 0$. Then, using this result, we show that the same speed formula holds in the general case.

Case (1): $\omega(x) = (p, 0)$, for all $x < 0$.

Since $\alpha > 1/2$, T_x is a.s. finite for each $x > 0$, and we define Δ_x , $x \geq 0$, by

$$\Delta_x = T_{x+1} - T_x.$$

The key observation is that because $R = 1$ and the random walk starts at $X_0 = 0$ in an environment ω satisfying $\omega(x) = (p, 0)$, for all $x < 0$, we have

$$\omega_n(x) = (p, 0), \quad \text{for each } n \geq 0 \text{ and } x < X_n.$$

That is, the environment to the left of the current position of the random walk always consists entirely of sites in the $(p, 0)$ configuration. Applying this fact at the random time T_x it follows that, for each $x > 0$, Δ_x is independent of $\Delta_0, \dots, \Delta_{x-1}$ and has distribution:

$$\begin{aligned} \mathbb{P}_\omega(\Delta_x = k) &= \mathbb{P}_{\omega^{(i)}}(T_1 = k), \quad \text{if } \omega(x) = (p, i), \quad 0 \leq i \leq L-1. \\ \mathbb{P}_\omega(\Delta_x = k) &= \mathbb{P}_{\omega^{(L)}}(T_1 = k), \quad \text{if } \omega(x) = (q, 0). \end{aligned}$$

Thus, defining

$$\begin{aligned} A_i^x &= \{0 \leq y \leq x-1 : \omega(y) = (p, i)\}, \quad 0 \leq i \leq L-1 \\ A_L^x &= \{0 \leq y \leq x-1 : \omega(y) = (q, 0)\} \end{aligned}$$

and applying the strong law of large numbers for the i.i.d. random variables $\{\Delta_y : \omega(y) = (p, i)\}$ and $\{\Delta_y : \omega(y) = (q, 0)\}$ we have that \mathbb{P}_ω a.s.

$$\lim_{x \rightarrow \infty} T_x/x = \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{i=0}^L \sum_{y \in A_i^x} \Delta_y = \lim_{x \rightarrow \infty} \sum_{i=0}^L \frac{|A_i^x|}{x} \left(\frac{1}{|A_i^x|} \sum_{y \in A_i^x} \Delta_y \right) = \sum_{i=0}^L d_i a_i. \quad (3.15)$$

So, by Lemma 3,

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = \frac{1}{\sum_{i=0}^L d_i a_i}, \quad \mathbb{P}_\omega \text{ a.s.} \quad (3.16)$$

Now, by conditioning on the first step of the walk it is easy to see that the following relations between the a_i 's hold:

$$\begin{aligned} a_i &= p \cdot 1 + (1-p) \cdot (1 + a_0 + a_{i+1}), \quad 0 \leq i \leq L-1. \\ a_L &= q \cdot 1 + (1-q) \cdot (1 + a_0 + a_L). \end{aligned} \quad (3.17)$$

One possible solution to the system (3.17) is $a_0 = a_1 = \dots = a_L = \infty$. However, by (3.16), this implies $X_n/n \rightarrow 0$, \mathbb{P}_ω a.s., which contradicts Theorem 2. Also, if $a_j = \infty$, for any j , then to satisfy (3.17) we must have $a_i = \infty$, for all i , which, as just shown, cannot happen. Over the real numbers the system (3.17) has a unique solution given by (1.12) and (1.13). This is shown in Appendix A.2.

Case (2): General Case

Fix any initial environment ω such that the limiting right densities d_i exist, and let $s = 1/(\sum_{i=0}^L d_i a_i)$. Also, for an arbitrary environment ω' , let τ denote the last hitting time of site 0 (which is a.s. finite by Theorem 1).

We observe that:

1. For any ω' ,

$$\mathbb{P}_{\omega'}(\tau = 0) = \mathbb{P}_{\omega''}(\tau = 0) > 0,$$

by Corollary 2, where ω'' is the environment defined by

$$\omega''(x) = \omega'(x), \quad x \geq 0 \quad \text{and} \quad \omega''(x) = (p, 0), \quad x < 0. \quad (3.18)$$

⁴Of course, in order to apply the strong law to conclude that $\lim_{x \rightarrow \infty} \frac{1}{|A_i^x|} \sum_{y \in A_i^x} \Delta_y = a_i$, we need $|A_i^x| \rightarrow \infty$. However, if $|A_i^x| \not\rightarrow \infty$, for some i , then $d_i = 0$. So, $\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{y \in A_i^x} \Delta_y = 0 = d_i a_i$, and (3.15) still holds.

2. For any environment ω' with $\mathbb{P}_\omega(\omega_\tau = \omega') > 0$, we have

$$\mathbb{P}_\omega(X_n/n \rightarrow s | \omega_\tau = \omega') = \mathbb{P}_{\omega'}(X_n/n \rightarrow s | \tau = 0) = \mathbb{P}_{\omega''}(X_n/n \rightarrow s | \tau = 0),$$

where ω'' is defined by (3.18).

3. For any environment ω' with $\mathbb{P}_\omega(\omega_\tau = \omega') > 0$, $\omega'(x) = \omega(x)$, for all but finitely many x . So, the limiting right densities d'_i of states in each configuration for the environment ω' are the same as the limiting right densities d_i for the initial environment ω .

It follows from these three observations and the result for Case (1) that, for any environment ω' with $\mathbb{P}_\omega(\omega_\tau = \omega') > 0$,

$$\mathbb{P}_\omega(X_n/n \rightarrow s | \omega_\tau = \omega') = \mathbb{P}_{\omega''}(X_n/n \rightarrow s | \tau = 0) = \mathbb{P}_{\omega''}(X_n/n \rightarrow s) = 1.$$

Hence, $X_n/n \rightarrow s$, \mathbb{P}_ω a.s. □

4 The Critical Case

Here we analyze the transience/recurrence properties of the random walk (X_n) in the critical case $\alpha = 1/2$, proving Theorems 5–8. We begin in section 4.1 with an important lemma for transience/recurrence of Markov chains on \mathbb{N}_0 . Then, in section 4.2 we establish a framework relating the right jumps Markov chain (Z_x) to the setup of this lemma. Using this framework, Theorem 5 is proved in section 4.3, Theorem 7 in section 4.4 and Theorem 8 in section 4.5. Theorem 6 is proved in section 4.6, using other methods.

4.1 Transience and Recurrence for Markov Chains on \mathbb{N}_0

Let $(Z_x)_{x \geq 0}$ be a time-homogenous Markov chain on the state space $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ with step distribution $U(n)$. That is,

$$\mathbb{P}(Z_{x+1} = m | Z_x = n) = \mathbb{P}(U(n) = m), \quad n, m \geq 0.$$

We will say that the chain (Z_x) is *irreducible and aperiodic with the exception of state 0* if 0 is an absorbing state, but it is possible to redefine the transition probabilities from 0 to make the chain irreducible and aperiodic. Also, we will say that the chain (Z_x) is *irreducible and aperiodic with the possible exception of state 0* if it is either irreducible and aperiodic or irreducible and aperiodic with the exception of state 0. Finally, we will say that the step distribution $U(n)$ is *well concentrated* if

$$\mu \equiv \lim_{n \rightarrow \infty} \mathbb{E}(U(n))/n \tag{4.1}$$

exists and there exist constants $C, c > 0$ and $N \in \mathbb{N}$ such that:

$$\mathbb{P}(|U(n) - \mu n| > \epsilon n) \leq C e^{-c\epsilon^2 n}, \quad \text{for } 0 < \epsilon \leq 1 \text{ and } n \geq N. \tag{4.2}$$

$$\mathbb{P}(|U(n) - \mu n| > \epsilon n) \leq C e^{-c\epsilon n}, \quad \text{for } \epsilon \geq 1 \text{ and } n \geq N. \tag{4.3}$$

In this case, we define also the quantities $\rho(n)$, $\nu(n)$, and $\theta(n)$ by

$$\rho(n) = \mathbb{E}(U(n) - \mu n), \quad \nu(n) = \mathbb{E}((U(n) - \mu n)^2)/n, \quad \theta(n) = 2\rho(n)/\nu(n). \tag{4.4}$$

The following lemma is essentially Theorem 1.3 from [3]⁵.

Lemma 8. *Let $(Z_x)_{x \geq 0}$ be a time-homogenous Markov chain on state space \mathbb{N}_0 , which is irreducible and aperiodic with the possible exception of state 0 and has well concentrated step distribution $U(n)$. Also, denote by \mathbb{P}_k the probability measure for the chain (Z_x) started from $Z_0 = k$. Then the following hold for any initial state $k \geq 1$.*

(i) *If $\mu < 1$, then $\mathbb{P}_k(Z_x > 0, \forall x \geq 0) = 0$.*

(ii) *If $\mu > 1$, then $\mathbb{P}_k(Z_x > 0, \forall x \geq 0) > 0$.*

(iii) *If $\mu = 1$, $\liminf_{n \rightarrow \infty} \nu(n) > 0$, and $\theta(n) < 1 + \frac{1}{\ln(n)} - \frac{a(n)}{n^{1/2}}$ for sufficiently large n , for some function $a(n) \rightarrow \infty$, then $\mathbb{P}_k(Z_x > 0, \forall x \geq 0) = 0$.*

(iv) *If $\mu = 1$, $\liminf_{n \rightarrow \infty} \nu(n) > 0$, and $\theta(n) > 1 + \frac{2}{\ln(n)} + \frac{a(n)}{n^{1/2}}$ for sufficiently large n , for some function $a(n) \rightarrow \infty$, then $\mathbb{P}_k(Z_x > 0, \forall x \geq 0) > 0$.*

4.2 Step Distribution of the Right Jumps Markov chain

By definition (2.6) for the right jumps Markov chain $(Z_x)_{x \geq 0}$,

$$\mathbb{P}(Z_{x+1} = m | Z_x = n) = \mathbb{P}(U(n, x+1) = m), \quad n, m, x \geq 0,$$

where $U(n, x)$ is the (random) number of right jumps in the sequence $(J_k^x)_{k \in \mathbb{N}}$ before the time of the n -th left jump:

$$U(n, x) = \inf \left\{ \ell \geq 0 : \sum_{k=1}^{\ell} \mathbb{1}\{J_k^x = -1\} = n \right\} - n. \quad (4.6)$$

If the initial environment $\omega(x)$ is constant for all $x \geq 0$, then the distribution of the jump sequence $(J_k^x)_{k \in \mathbb{N}}$ is the same for all $x \geq 0$, so the distribution of $U(n, x)$ is also the same for all $x \geq 0$. In this case, the right jumps chain $(Z_x)_{x \geq 0}$ is time-homogeneous (where x is the time variable) with step distribution $U(n) = U(n, x)$. It is also irreducible and aperiodic with the exception of state 0. For example, redefining the transition probabilities from state 0 as $\mathbb{P}(Z_{x+1} = 1 | Z_x = 0) = 1$ would make the chain irreducible and aperiodic.

For the remainder of this section we assume $\omega(x) = \omega(0)$, for all $x \geq 0$. For our analysis of the step distribution $U(n)$ we fix an arbitrary site $x \geq 0$ and decompose $U(n) = U(n, x)$ as

$$U(n) = \sum_{j=1}^n \Gamma_j, \quad (4.7)$$

where Γ_j is the number of right jumps in the sequence $(J_k^x)_{k \in \mathbb{N}}$ between the $(j-1)$ -th and j -th left jumps. That is, $\Gamma_j = k_j - k_{j-1} - 1$, where $k_0 = 0$ and, for $j \geq 1$, $k_j = \inf\{k > k_{j-1} : J_k^x = -1\}$.

⁵There are three small differences. First, in Theorem 1.3 of [3] the chain (Z_x) is required to be truly irreducible and aperiodic, without the possible exception of state 0. Second, instead of (4.2) and (4.3) the following somewhat stronger concentration condition for $U(n)$ is assumed: There exist $c > 0$ and $N \in \mathbb{N}$ such that

$$\mathbb{P}(|U(n) - \mu n| > \epsilon n) \leq 2e^{-c\epsilon^2 n}, \quad \text{for all } \epsilon > 0 \text{ and } n \geq N. \quad (4.5)$$

Finally, there is no assumption that $\liminf_{n \rightarrow \infty} \nu(n) > 0$ for cases (iii) and (iv).

Allowing the possible exception of state 0 in the irreducible and aperiodic hypothesis clearly has no effect, since the probability of ever hitting state 0, starting from a state $k \geq 1$, depends only on the transition probabilities from the nonzero states. Also, the concentration condition (4.5) is used in [3] only to bound the error terms in certain Taylor series expansions, and these estimates remain valid if (4.2) and (4.3) hold instead, so there is no issue with using the weaker concentration condition. However, the proof of cases (iii) and (iv) given in [3] actually works as stated only if $\liminf_{n \rightarrow \infty} \nu(n) > 0$, so we require this condition also in our statement.

We think of the $(\Gamma_j)_{j=1}^n$ as the values obtained in n “sessions,” and denote by $\omega^j = \omega^j(x)$ the configuration at site x at the beginning of the j -th session. Thus, $\omega^1 = \omega(x)$ and, for $j \geq 2$, $\omega^j = Y_{k_{j-1}+1}^x$ is the configuration at site x immediately after the $(j-1)$ -th left jump in the sequence $(J_k^x)_{k \in \mathbb{N}}$. It is straightforward to see that conditioned on ω^j , Γ_j is independent of $\Gamma_1, \dots, \Gamma_{j-1}$ and has the following distribution:

$$\begin{aligned} \Gamma_j &\sim S_i, \text{ if } \omega^j = (q, i), \text{ for some } 0 \leq i \leq R-1, \text{ and} \\ \Gamma_j &\sim S_R, \text{ if } \omega^j = (p, i), \text{ for some } 0 \leq i \leq L-1, \end{aligned} \quad (4.8)$$

where S_0, \dots, S_R are random variables with law

$$\mathbb{P}(S_i = k) = \begin{cases} q^k(1-q), & 0 \leq k \leq R-i-1; \\ q^{R-i}p^{k-(R-i)}(1-p), & k \geq R-i. \end{cases} \quad (4.9)$$

In particular, S_R is a standard geometric random variable with parameter $1-p$.

Now, the configuration ω^{j+1} at the beginning of the next session is determined entirely by ω^j and Γ_j . More precisely, ω^{j+1} is the (deterministic) configuration obtained by jumping right Γ_j times from site x , starting in configuration ω^j , and then jumping left once. Thus, assuming that $L \geq 2$:

$$\begin{aligned} \text{If } \omega^j = (q, i), 0 \leq i \leq R-1, \text{ then } \omega^{j+1} &= \begin{cases} (p, 1), & \text{if } \Gamma_j \geq R-i; \\ (q, 0), & \text{if } \Gamma_j < R-i. \end{cases} \\ \text{If } \omega^j = (p, i), 0 \leq i \leq L-2, \text{ then } \omega^{j+1} &= \begin{cases} (p, 1), & \text{if } \Gamma_j \geq 1; \\ (p, i+1), & \text{if } \Gamma_j = 0. \end{cases} \\ \text{If } \omega^j = (p, L-1), \text{ then } \omega^{j+1} &= \begin{cases} (p, 1), & \text{if } \Gamma_j \geq 1; \\ (q, 0), & \text{if } \Gamma_j = 0. \end{cases} \end{aligned} \quad (4.10)$$

If $L = 1$ then the configuration ω^j at the beginning of each of the right jumps sessions after the first is always $(q, 0)$, since the configuration at site x immediately after a left jump is $(q, 0)$.

From (4.8)-(4.10) it follows that, for any $L \geq 2$, the sequence of configurations $(\omega^j)_{j=1}^n$ is a Markov chain with (initial state $\omega(x)$ and) transition matrix \widehat{A} given by

$$\begin{aligned} \widehat{A}_{(q,i),(q,0)} &= 1 - q^{R-i}, \quad \widehat{A}_{(q,i),(p,1)} = q^{R-i} \quad \text{for } 0 \leq i \leq R-1; \\ \widehat{A}_{(p,i),(p,1)} &= p, \quad \widehat{A}_{(p,i),(p,i+1)} = 1 - p \quad \text{for } 0 \leq i \leq L-2; \\ \widehat{A}_{(p,L-1),(p,1)} &= p, \quad \widehat{A}_{(p,L-1),(q,0)} = 1 - p. \end{aligned} \quad (4.11)$$

In the case $L = 1$, $(\omega^j)_{j=1}^n$ is still a Markov chain, but it is degenerate. The transition matrix \widehat{A} has $\widehat{A}_{\lambda,(q,0)} = 1$, for all $\lambda \in \Lambda$.

In either case, the transition matrix \widehat{A} is indecomposable, and the L states $\Lambda_0 \equiv \{(q, 0), (p, 1), \dots, (p, L-1)\}$ constitute a closed, irreducible set of states. We denote by A the corresponding transition matrix obtained from \widehat{A} by restricting to these L states, and by ψ the unique invariant measure for A (for $L = 1$, $A = \psi = 1$). Also, we denote by $e_{(p,i)}$ the unit L -vector with a 1 in the position of state (p, i) , and by $e_{(q,0)}$ the unit L -vector with a 1 in the position of $(q, 0)$. Finally, we let E denote the L -vector with components, $E_{(q,0)} = \mathbb{E}(S_0)$ and $E_{(p,i)} = \mathbb{E}(S_R)$, for $i \in \{1, \dots, L-1\}$.

Basic Lemmas

The following lemmas characterize some key properties of the step distribution $U(n) = \sum_{j=1}^n \Gamma_j$ in the critical case, $\alpha = 1/2$. Proofs are deferred to Appendix D, but in all three cases use the underlying Markov chain $(\omega^j)_{j=1}^n$.

Lemma 9. *If $\alpha = 1/2$, then*

$$\mu \equiv \lim_{n \rightarrow \infty} \frac{\mathbb{E}(U(n))}{n} = \langle \psi, E \rangle = 1, \quad (4.12)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product of two real vectors.

Lemma 10. *If $\alpha = 1/2$, then the step distribution $U(n)$ is well concentrated.*

Lemma 11. *If $\alpha = 1/2$, then $\liminf_{n \rightarrow \infty} \nu(n) > 0$, where $\nu(n)$ is given by (4.4).*

Remark. The assumption $\alpha = 1/2$ is actually not necessary for the conclusions of Lemmas 10 and 11 to hold, but it simplifies the writing of the proofs slightly and is the only case where we will apply them.

Setup for the Proofs of Theorems 5, 7, and 8

For the proofs of Theorems 5, 7, and 8 below we will adopt the framework given here for analyzing the step distribution $U(n)$ of the right jumps Markov chain, without further mention, whenever the initial environment ω is constant over all $x \geq 0$. In this case, since $\alpha = 1/2$ for all three of these theorems, we know $\mu = 1$ by Lemma 9. Thus, (4.4) becomes

$$\rho(n) = \mathbb{E}(U(n) - n), \quad \nu(n) = \mathbb{E}((U(n) - n)^2)/n, \quad \theta(n) = 2\rho(n)/\nu(n) \quad (4.13)$$

and it follows from Lemmas 2, 8, 10, and 11 that

$$\begin{aligned} \mathbb{P}_\omega(X_n \rightarrow \infty) &= 0, \quad \text{if } \theta(n) \leq 1 + O\left(\frac{1}{n}\right) \quad \text{and} \\ \mathbb{P}_\omega(X_n \rightarrow \infty) &> 0, \quad \text{if } \lim_{n \rightarrow \infty} \theta(n) > 1. \end{aligned} \quad (4.14)$$

In all cases one of these two possibilities for $\theta(n)$ will occur.

4.3 Proof of Theorem 5

Proof of Theorem 5. If the initial environment ω satisfies $\omega(x) = \lambda \in \Lambda_0$ for all $x \geq 0$, then the Markov chain representation of section 4.2 and Lemma 9 give

$$\begin{aligned} \mathbb{E}(\Gamma_j) &= \sum_{\lambda'} \mathbb{P}(\omega^j = \lambda' | \omega^1 = \lambda) \cdot \mathbb{E}(\Gamma_j | \omega^j = \lambda') \\ &= \langle e_\lambda A^{j-1}, E \rangle = \langle \psi, E \rangle + \langle (e_\lambda A^{j-1} - \psi), E \rangle = 1 + O(a^j), \end{aligned}$$

for some $0 < a < 1$, which depends on the matrix A . Thus, in this case, for all $n \geq j$ we have from (4.13)

$$\rho(n) = \sum_{i=1}^n \mathbb{E}(\Gamma_i) - n = \left(\sum_{i=1}^j \mathbb{E}(\Gamma_i) - j \right) + O(a^j) = \rho(j) + O(a^j).$$

Also, by Lemma 11, we know that there exists some $\epsilon > 0$, which can be chosen uniformly over $\lambda \in \Lambda_0$, such that $\liminf_{n \rightarrow \infty} \nu(n) \geq \epsilon$ if $\omega(x) = \lambda$, for $x \geq 0$. Combining these observations we see that there exists some $n_0 \in \mathbb{N}$ satisfying

$$\nu^{(\lambda)}(n) \geq \epsilon/2 \quad \text{and} \quad \rho^{(\lambda)}(n) - \rho^{(\lambda)}(n_0) \leq \epsilon/4, \quad \text{for all } \lambda \in \Lambda_0 \text{ and } n \geq n_0, \quad (4.15)$$

where $\rho^{(\lambda)}(n)$ and $\nu^{(\lambda)}(n)$ are the quantities $\rho(n)$ and $\nu(n)$ with initial environment $\omega(x) = \lambda$, $x \geq 0$.

Now, for any fixed j , Lemma 9 implies

$$\begin{aligned} j &= \sum_{i=1}^j \langle \psi, E \rangle = \sum_{i=1}^j \langle \psi A^{j-1}, E \rangle = \sum_{i=1}^j \sum_{\lambda \in \Lambda_0} \psi_\lambda \langle e_\lambda A^{j-1}, E \rangle \\ &= \sum_{i=1}^j \sum_{\lambda \in \Lambda_0} \psi_\lambda \cdot \mathbb{E}(\Gamma_j^{(\lambda)}) = \sum_{\lambda \in \Lambda_0} \psi_\lambda \cdot \mathbb{E}(U^{(\lambda)}(j)), \end{aligned}$$

where $\Gamma_j^{(\lambda)}$ and $U^{(\lambda)}(j)$ are the random variables Γ_j and $U(j)$, with initial environment $\omega(x) = \lambda$, $x \geq 0$. Thus, for any fixed j ,

$$\sum_{\lambda \in \Lambda_0} \psi_\lambda \cdot [\mathbb{E}(U^{(\lambda)}(j)) - j] = \sum_{\lambda \in \Lambda_0} \psi_\lambda \cdot \rho^{(\lambda)}(j) = 0,$$

so there exists some $\lambda_j \in \Lambda_0$ such that

$$\rho^{(\lambda_j)}(j) \leq 0. \quad (4.16)$$

Define $\lambda^* = \lambda_{n_0}$. Then, by (4.15) and (4.16), $\rho^{(\lambda^*)}(n) \leq \rho^{(\lambda^*)}(n_0) + \epsilon/4 = \rho^{(\lambda_{n_0})}(n_0) + \epsilon/4 \leq \epsilon/4$, for $n \geq n_0$. So, by (4.15),

$$\theta^{(\lambda^*)}(n) = \frac{2\rho^{(\lambda^*)}(n)}{\nu^{(\lambda^*)}(n)} \leq \frac{2 \cdot (\epsilon/4)}{\epsilon/2} = 1, \text{ for } n \geq n_0.$$

It follows, from (4.14), that $\mathbb{P}_\omega(X_n \rightarrow \infty) = 0$ for any initial environment ω with $\omega(x) = \lambda^*$, for all $x \geq 0$. Thus, also, $\mathbb{P}_\omega(X_n \rightarrow \infty) = 0$ for any initial environment ω which is equal to λ^* in a neighborhood of $+\infty$. An analogous argument shows that there exists some λ_* such that $\mathbb{P}_\omega(X_n \rightarrow -\infty) = 0$, for any initial environment ω which is equal to λ_* in a neighborhood of $-\infty$. So, by part (ii) of Lemma 1, the random walk (X_n) is \mathbb{P}_ω a.s. recurrent for any initial environment ω which is equal to λ^* in a neighborhood of $+\infty$ and equal to λ_* in a neighborhood of $-\infty$. By Lemma 4, and symmetry considerations, we may take $\lambda^* = (q, 0)$ and $\lambda_* = (p, 0)$ in the case of positive feedback, $q < p$. \square

4.4 Proof of Theorem 7

For notational convenience in the proof of Theorem 7 we define

$$\lambda_0 = (q, 0), \dots, \lambda_{R-1} = (q, R-1), \lambda_R = (p, 0).$$

As discussed above, in the case $L = 1$ the transition matrix \widehat{A} is degenerate with $\widehat{A}_{\lambda, (q, 0)} = 1$, for all $\lambda \in \Lambda$. Thus, in this case, $\omega^j = (q, 0)$, for all $j \geq 2$ (independent of the values of the Γ_j 's). Also, with $L = 1$, ψ is simply the length-1 vector 1 and E is simply the length-1 vector $\mathbb{E}(S_0)$. The following facts are immediate from this.

1. If $L = 1$ and $\omega(x) = \lambda_i$, $x \geq 0$, then

$$\Gamma_1, \Gamma_2, \dots \text{ are independent with } \Gamma_1 \sim S_i \text{ and } \Gamma_j \sim S_0, \quad j \geq 2. \quad (4.17)$$

2. If $L = 1$ and $\alpha = 1/2$ then, by Lemma 9,

$$\mathbb{E}(S_0) = \langle \psi, E \rangle = 1. \quad (4.18)$$

Using these facts we now prove Theorem 7.

Proof of Theorem 7. By assumption $\alpha = 1/2$ and $L = 1$, and the initial environment is a constant in a neighborhood of $-\infty$ in the case of negative feedback, $p < q$. Thus, by Theorem 6, the probability of the random walk (X_n) being transient to $-\infty$ is equal to 0⁶. So, by Lemma 1, the probability of being transient to $+\infty$ is either 0 or 1, and if it is 0, the process is recurrent. Moreover, without loss of generality, clearly we may assume that the initial environment ω is constant for all $x \geq 0$ (rather than only in a neighborhood of $+\infty$). Thus, it suffices to show the following to establish the transience/recurrence claims in the theorem:

- If $\omega(x) = \lambda_0$ for all $x \geq 0$, then $\mathbb{P}_\omega(X_n \rightarrow \infty) = 0$.
 - If $\omega(x) = \lambda_i$ for all $x \geq 0$, $1 \leq i \leq R$, then $\mathbb{P}_\omega(X_n \rightarrow \infty) = 0$ if and only if $P_{R,i}(q) \geq 0$.
- (4.19)

For the remainder of the proof we assume that

$$\omega(x) = \lambda_i, \quad x \geq 0, \quad (4.20)$$

for some $0 \leq i \leq R$. To determine if there is positive probability of transience to $+\infty$ we will calculate $\theta(n) = 2\rho(n)/\nu(n)$ and apply (4.14).

We begin with $\rho(n)$. Since $L = 1$ and $\alpha = 1/2$, $\mathbb{E}(S_0) = 1$, by (4.18). Thus, by (4.13) and (4.17),

$$\rho(n) = \mathbb{E}(S_i) + (n-1)\mathbb{E}(S_0) - n = \mathbb{E}(S_i) - 1. \quad (4.21)$$

A direct computation yields

$$\begin{aligned} \mathbb{E}(S_i) &= \sum_{k=0}^{R-i-1} k \cdot q^k (1-q) + \sum_{k=R-i}^{\infty} k \cdot q^{R-i} p^{k-(R-i)} (1-p) \\ &= \frac{1}{1-q} \left[-(1-q)q^{R-i}(R-i) + (1-q^{R-i})q \right] + \\ &\quad \frac{1}{1-p} \left(\frac{q}{p} \right)^{R-i} \left[(1-p)p^{R-i}(R-i) + p^{R-i+1} \right]. \end{aligned} \quad (4.22)$$

Since $L = 1$ and $\alpha = 1/2$, Proposition 1 implies

$$p = p_0 = \frac{1-2q+q^{R+1}}{1-2q+q^R}, \quad 1-p = 1-p_0 = \frac{q^R - q^{R+1}}{1-2q+q^R}. \quad (4.23)$$

Substituting for p above we obtain, after some lengthy simplifications,

$$\mathbb{E}(S_i) = \frac{1-2q+q^{i+1}}{q^i(1-q)}.$$

Thus, by (4.21),

$$\rho(n) = \frac{1-2q-q^i+2q^{i+1}}{q^i(1-q)}, \quad \text{for all } n. \quad (4.24)$$

We now turn to the calculation of $\nu(n)$. From (4.13) we recall that $\nu(n) = \mathbb{E}[(U(n) - n)^2]/n$. Using the independence of the random variables $\Gamma_1, \dots, \Gamma_n$ we have

$$\begin{aligned} \mathbb{E}[(U(n) - n)^2] &= \mathbb{E} \left[\left(\sum_{j=1}^n \Gamma_j - (n-1 + \mathbb{E}(S_i)) - (1 - \mathbb{E}(S_i)) \right)^2 \right] = \\ &= \text{Var}(S_i) + (n-1)\text{Var}(S_0) + (1 - \mathbb{E}(S_i))^2 = n[\mathbb{E}(S_0^2) - 1] + O(1). \end{aligned} \quad (4.25)$$

⁶Theorem 6 is not proved till later in section 4.6, but the proof is independent of the proof of this theorem.

A tedious computation gives

$$\begin{aligned}\mathbb{E}(S_0^2) &= \sum_{k=0}^{R-1} k^2 \cdot q^k (1-q) + \sum_{k=R}^{\infty} k^2 \cdot q^R p^{k-R} (1-p) \\ &= \frac{q + q^2 - q^R (R^2 - (2R^2 - 2R - 1)q + (R-1)^2 q^2)}{(1-q)^2} + \\ &\quad \frac{q^R (R^2 - (2R^2 - 2R - 1)p + (R-1)^2 p^2)}{(1-p)^2}.\end{aligned}\tag{4.26}$$

Substituting for p from (4.23) and doing a lot of algebra, one eventually finds that

$$\mathbb{E}(S_0^2) = \frac{1}{q^R(1-q)^2} \left[2 - 8q + 8q^2 + (2R+1)q^R + (2-6R)q^{R+1} + (4R-5)q^{R+2} \right].\tag{4.27}$$

Finally, from (4.25) and (4.27), we have

$$\begin{aligned}\nu(n) &= \frac{\mathbb{E}[(U(n) - n)^2]}{n} = \mathbb{E}(S_0^2) - 1 + O\left(\frac{1}{n}\right) \\ &= \frac{1}{q^R(1-q)^2} \left[2 - 8q + 8q^2 + 2Rq^R + (4-6R)q^{R+1} + (4R-6)q^{R+2} \right] + O\left(\frac{1}{n}\right).\end{aligned}\tag{4.28}$$

Now, combining (4.24) and (4.28) shows that $\theta(n) = \theta + O\left(\frac{1}{n}\right)$ where

$$\begin{aligned}\theta &= \frac{q^{R-i}(1-q)(1-2q-q^i+2q^{i+1})}{1-4q+4q^2+Rq^R+(2-3R)q^{R+1}+(2R-3)q^{R+2}} \\ &= \frac{q^{R-i} - 3q^{R-i+1} + 2q^{R-i+2} - q^R + 3q^{R+1} - 2q^{R+2}}{1-4q+4q^2+Rq^R+(2-3R)q^{R+1}+(2R-3)q^{R+2}}.\end{aligned}\tag{4.29}$$

For $1 \leq i \leq R$, $\theta \leq 1$ is equivalent to $P_{R,i}(q) \geq 0$, and in the case $i = 0$, θ is 0. Thus, (4.19) follows from (4.14).

It remains only to show the claims concerning the polynomial $P_{R,R}(q)$. For these we will use the factored representation $P_{R,R}(q) = q(1-q)^2 \tilde{P}_{R,R}(q)$, where $\tilde{P}_{R,R}(q) = -1 + \sum_{j=1}^{R-3} j q^{j+1} + (2R-1)q^{R-1}$, as in (1.16). Since, $P_{R,R}(q)$ and $\tilde{P}_{R,R}(q)$ have the same sign for all $q \in (0, 1)$, it suffices to prove the claims for the polynomial $\tilde{P}_{R,R}(q)$.

Now, clearly, $\tilde{P}_{R,R}$ is increasing, and $\tilde{P}_{R,R}(0) = -1$. For $R \geq 4$, one can rewrite $\tilde{P}_{R,R}$ as $\tilde{P}_{R,R}(q) = -1 + \left(\frac{q}{1-q}\right)^2 [1 - (R-2)q^{R-3} + (R-3)q^{R-2}] + (2R-1)q^{R-1}$. Using this, we find that $\tilde{P}_{R,R}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{R-1}$, for all $R \geq 2$. Consequently, $\tilde{P}_{R,R}$ has a unique root $q_*(R) \in (0, \frac{1}{2})$, with $\tilde{P}_{R,R}(q) < 0$ for $q < q_*(R)$ and $\tilde{P}_{R,R}(q) > 0$ for $q > q_*(R)$. Furthermore,

$$\begin{aligned}\tilde{P}_{R+1,R+1}(q) - \tilde{P}_{R,R}(q) &= (2R+1)q^R - (R+1)q^{R-1} = \\ &= q^{R-1} [(2R+1)q - (R+1)] \leq -\frac{1}{2}q^{R-1} < 0, \text{ for } q \in [0, \frac{1}{2}].\end{aligned}\tag{4.30}$$

So, $q_*(R)$ is increasing in R . Also, we have $\tilde{P}_{\infty,\infty}(q) \equiv \lim_{R \rightarrow \infty} \tilde{P}_{R,R}(q) = \frac{2q-1}{(1-q)^2}$. Since the root of $\tilde{P}_{\infty,\infty}$ is at $q = \frac{1}{2}$, it follows that $\lim_{R \rightarrow \infty} q_*(R) = \frac{1}{2}$. \square

4.5 Proof of Theorem 8

For the proof of Theorem 8 we will need the following lemma.

Lemma 12. *If the initial environment ω is constant in a neighborhood of $+\infty$ ($-\infty$) and $\mathbb{P}_\omega(X_n \rightarrow +\infty) > 0$ ($\mathbb{P}_\omega(X_n \rightarrow -\infty) > 0$), then*

$$\mathbb{P}_\omega(X_n \rightarrow +\infty) + \mathbb{P}_\omega(X_n \rightarrow -\infty) = 1.$$

Proof. We will prove the claim for the case of constant initial environment in a neighborhood of $+\infty$; the other claim follows from symmetry considerations. Thus, we assume the environment ω satisfies $\mathbb{P}_\omega(X_n \rightarrow \infty) > 0$ and $\omega(x) = \lambda$, $x \geq N$, for some $\lambda \in \Lambda$ and $N \in \mathbb{N}$. Also, we let ω' be any environment with $\omega'(x) = \lambda$, for all $x \geq 0$. Since we assume $\mathbb{P}_\omega(X_n \rightarrow \infty) > 0$ it follows from part (i) of Lemma 1 that $\mathbb{P}_{\omega'}(X_n \rightarrow \infty), \mathbb{P}_{\omega'}(\mathcal{A}_0^+) > 0$, and we define $\delta = \mathbb{P}_{\omega'}(\mathcal{A}_0^+)$.

If the random walk (X_n) is run starting in ω , then at every time n that it first hits a site $x \geq N$ the environment at time n is equal to λ at all sites $y \geq x$. Thus,

$$\begin{aligned} \mathbb{P}_\omega(\mathcal{A}_n^+ | X_0 = x_0, \dots, X_n = x_n) &= \mathbb{P}_{\omega'}(\mathcal{A}_0^+) = \delta, \\ \text{for any path } (x_0, \dots, x_n) \text{ satisfying } &x_0 = 0, x_n \geq N, x_m < x_n \text{ for } m < n. \end{aligned} \quad (4.31)$$

We define stopping times τ_i, τ'_i and stopping points z_i as follows:

- $\tau_0 = T_N, z_0 = N$.
- For $i \geq 1$,

$$\begin{aligned} \tau'_i &= \inf\{n > \tau_{i-1} : X_n = z_{i-1}\}, \\ z_i &= \sup\{X_n : \tau_{i-1} \leq n < \tau'_i\}, \\ \tau_i &= \inf\{n > \tau'_i : X_n = z_i + 1\}. \end{aligned}$$

Of course, not all these stopping times are necessarily finite. We think of the times as an ordered list $\tau_0, \tau'_1, \tau_1, \tau'_2, \tau_2, \dots$ and if any element in this list is equal to ∞ then all elements to the right of it are defined to be ∞ as well. By construction, the list is an increasing sequence of positive integers up till the point of the first ∞ , and it follows from (4.31) that, for any $i \geq 1$,

$$\mathbb{P}_\omega(\tau'_i < \infty | \tau_{i-1} < \infty) \leq 1 - \delta.$$

Thus, for any $i \geq 1$,

$$\mathbb{P}_\omega(\tau_i < \infty) = \mathbb{P}_\omega(\tau_0 < \infty) \cdot \prod_{j=1}^i \left(\mathbb{P}_\omega(\tau'_j < \infty | \tau_{j-1} < \infty) \cdot \mathbb{P}_\omega(\tau_j < \infty | \tau'_j < \infty) \right) \leq (1 - \delta)^i.$$

So, $\mathbb{P}_\omega(\tau_i < \infty, \text{ for all } i > 0) = 0$. However, by part (ii) of Lemma 1, we also have $\mathbb{P}_\omega(X_n \not\rightarrow +\infty \text{ and } X_n \not\rightarrow -\infty) \leq \mathbb{P}_\omega(\tau_i < \infty, \text{ for all } i > 0)$. Thus, $\mathbb{P}_\omega(X_n \rightarrow +\infty) + \mathbb{P}_\omega(X_n \rightarrow -\infty) = 1$. \square

Proof of Theorem 8. We will show that if $\omega(x) = (p, 0)$ for $x \geq 0$, then $\theta(n)$ from (4.13) satisfies $\theta(n) \leq 1 + O(\frac{1}{n})$, if $q \geq q_1^*$, and $\lim_{n \rightarrow \infty} \theta(n) > 1$, if $q < q_1^*$. We will also show that if $\omega(x) = (p, 1)$ for $x \geq 0$ or $\omega(x) = (q, 1)$ for $x \geq 0$, then $\theta(n)$ satisfies $\theta(n) \leq 1 + O(\frac{1}{n})$ for all $q \in (0, 1)$. Finally, we will show that if $\omega(x) = (q, 0)$ for $x \geq 0$, then $\theta(n)$ satisfies $\theta(n) \leq 1 + O(\frac{1}{n})$, if $q \leq q_2^*$, and $\lim_{n \rightarrow \infty} \theta(n) > 1$, if $q > q_2^*$. It follows, by (4.14), that:

- If $\omega(x) = (p, 0)$ for $x \geq 0$, then $\mathbb{P}_\omega(X_n \rightarrow \infty) > 0$ if and only if $q < q_1^*$.
- If $\omega(x) = (q, 0)$ for $x \geq 0$, then $\mathbb{P}_\omega(X_n \rightarrow \infty) > 0$ if and only if $q > q_2^*$.
- If $\omega(x) = (p, 1)$ for $x \geq 0$ or $\omega(x) = (q, 1)$ for $x \geq 0$, then $\mathbb{P}_\omega(X_n \rightarrow \infty) = 0$. (4.32)

Clearly, (4.32) is still valid if the hypothesis “ $x \geq 0$ ” is changed to “ x in a neighborhood of $+\infty$ ”. Thus, this will prove the theorem, in light of Lemmas 1 and 12, and the symmetry that holds because $R = L$ and $p = 1 - q$, along with Lemma 4 in the case $q < p$, where the theorem sometimes allows for nonconstant environments in a neighborhood of $+\infty$ or $-\infty$.

By definition, $\theta(n) = 2\rho(n)/\nu(n)$. The calculation of the two components $\rho(n)$ and $\nu(n)$ will be done separately, but we begin first with some general setup that will be used in both cases. Throughout we will use implicitly the following basic fact many times, which is immediate from the construction of the joint process (ω^j, Γ_j) :

$$\text{Conditioned on } \omega^i, (\omega^j, \Gamma_j)_{j=1}^{i-1} \text{ and } (\Gamma_j)_{j=i}^{\infty} \text{ are independent.}$$

Setup

Since $L = 2$, the transition matrix A defined in section 4.2 corresponds to the recurrent states $(p, 1)$ and $(q, 0)$. Ordering the states in the order they appear here, and using the fact that $R = 2$ and the assumption $p = 1 - q$, it follows from (4.11) that

$$A = \begin{pmatrix} p & 1-p \\ q^2 & 1-q^2 \end{pmatrix} = \begin{pmatrix} 1-q & q \\ q^2 & 1-q^2 \end{pmatrix}.$$

This matrix A has eigenvalues

$$\eta_1 = 1, \quad \eta_2 = 1 - q - q^2 \tag{4.33}$$

with corresponding left eigenvectors

$$w_1 = (q, 1), \quad w_2 = (-1, 1). \tag{4.34}$$

For future reference we observe that $|\eta_2| < 1$, for any $q \in (0, 1)$, and that the unit vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ decompose as

$$e_1 = c_1 w_1 + c_2 w_2, \quad e_2 = d_1 w_1 + d_2 w_2 \tag{4.35}$$

where

$$c_1 = \frac{1}{q+1}, \quad c_2 = -\frac{1}{q+1}, \quad d_1 = \frac{1}{q+1}, \quad d_2 = \frac{q}{q+1}. \tag{4.36}$$

With $R = 2$ and $p = 1 - q$ the distribution of the random variables S_0, S_1 and S_2 given in (4.9) becomes:

$$\begin{aligned} \mathbb{P}(S_0 = k) &= q^k(1-q), \quad k = 0, 1 \quad \text{and} \quad \mathbb{P}(S_0 = k) = q^3(1-q)^{k-2}, \quad k \geq 2. \\ \mathbb{P}(S_1 = 0) &= 1-q \quad \text{and} \quad \mathbb{P}(S_1 = k) = q^2(1-q)^{k-1}, \quad k \geq 1. \\ \mathbb{P}(S_2 = k) &= q(1-q)^k, \quad k \geq 0. \end{aligned} \tag{4.37}$$

Direct calculations yield

$$\begin{aligned} \mathbb{E}(S_0) &= 2q, \quad \mathbb{E}(S_1) = 1, \quad \mathbb{E}(S_2) = (1-q)/q, \\ \mathbb{E}(S_0^2) &= 2(1+q), \quad \mathbb{E}(S_2^2) = (1-q)(2-q)/q^2, \\ \mathbb{E}(S_0|S_0 \geq 2) &= (1+q)/q, \quad \mathbb{E}(S_2|S_2 \geq 1) = 1/q. \end{aligned} \tag{4.38}$$

By (4.8) we have $\mathbb{E}(\Gamma_i|\omega^i = (p, 1)) = \mathbb{E}(S_2)$ and $\mathbb{E}(\Gamma_i|\omega^i = (q, 0)) = \mathbb{E}(S_0)$, and with our chosen state ordering $(p, 1), (q, 0)$ the expectation vector E from section 4.2 becomes $E = (\mathbb{E}(S_2), \mathbb{E}(S_0))$.

Thus, since $\omega^{i+\ell}$ is distributed as $e_1 A^\ell$ when $\omega^i = (p, 1)$ and as $e_2 A^\ell$ when $\omega^i = (q, 0)$, we have

$$\begin{aligned} & \mathbb{E}\left(\Gamma_{i+\ell} \mid \omega^i = (p, 1)\right) \\ &= \mathbb{P}\left(\omega^{i+\ell} = (p, 1) \mid \omega^i = (p, 1)\right) \cdot \mathbb{E}(S_2) + \mathbb{P}\left(\omega^{i+\ell} = (q, 0) \mid \omega^i = (p, 1)\right) \cdot \mathbb{E}(S_0) \\ &= \langle e_1 A^\ell, E \rangle = c_1 \langle w_1, E \rangle \eta_1^\ell + c_2 \langle w_2, E \rangle \eta_2^\ell = 1 + \left(\frac{1-2q}{q}\right) \eta_2^\ell, \end{aligned} \quad (4.39)$$

and

$$\begin{aligned} & \mathbb{E}\left(\Gamma_{i+\ell} \mid \omega^i = (q, 0)\right) \\ &= \mathbb{P}\left(\omega^{i+\ell} = (p, 1) \mid \omega^i = (q, 0)\right) \cdot \mathbb{E}(S_2) + \mathbb{P}\left(\omega^{i+\ell} = (q, 0) \mid \omega^i = (q, 0)\right) \cdot \mathbb{E}(S_0) \\ &= \langle e_2 A^\ell, E \rangle = d_1 \langle w_1, E \rangle \eta_1^\ell + d_2 \langle w_2, E \rangle \eta_2^\ell = 1 + (2q-1) \eta_2^\ell, \end{aligned} \quad (4.40)$$

for any $\ell \geq 0$. Similarly, denoting $E' = (\mathbb{E}(S_2^2), \mathbb{E}(S_0^2))$, we have

$$\begin{aligned} & \mathbb{E}\left(\Gamma_{i+\ell}^2 \mid \omega^i = (p, 1)\right) \\ &= \langle e_1 A^\ell, E' \rangle = c_1 \langle w_1, E' \rangle \eta_1^\ell + c_2 \langle w_2, E' \rangle \eta_2^\ell = \frac{2-q+3q^2}{q(1+q)} + O(|\eta_2|^\ell). \end{aligned} \quad (4.41)$$

Calculation of $\rho(n)$

We will write $\rho^{(p,0)}(n)$ for the quantity $\rho(n)$ when the initial environment is $\omega(x) = (p, 0)$, $x \geq 0$, and similarly we denote by $\rho^{(q,0)}(n)$, $\rho^{(p,1)}(n)$, and $\rho^{(q,1)}(n)$ the quantity $\rho(n)$ with initial environments $(q, 0)$, $(p, 1)$, and $(q, 1)$ for $x \geq 0$. In all cases, we have, from (4.13), $\rho(n) = \mathbb{E}(U(n)) - n = \sum_{j=1}^n \mathbb{E}(\Gamma_j) - n$, where the expectation is (implicitly) the expectation conditioned on $\omega^1 = (p, 0)$, $(q, 0)$, $(p, 1)$, or $(q, 1)$.

Using (4.39) with $i = 1$, along with (4.33), gives

$$\begin{aligned} \rho^{(p,1)}(n) &= \sum_{j=1}^n \mathbb{E}\left(\Gamma_j \mid \omega^1 = (p, 1)\right) - n = \sum_{j=1}^n \left[1 + \left(\frac{1-2q}{q}\right) \eta_2^{j-1}\right] - n \\ &= \left(\frac{1-2q}{q}\right) \frac{1}{1-\eta_2} + O(|\eta_2|^n) = \frac{1-2q}{q^2(1+q)} + O(|\eta_2|^n). \end{aligned} \quad (4.42)$$

Similarly, using (4.40) with $i = 1$, along with (4.33), gives

$$\begin{aligned} \rho^{(q,0)}(n) &= \sum_{j=1}^n \mathbb{E}\left(\Gamma_j \mid \omega^1 = (q, 0)\right) - n = \sum_{j=1}^n \left[1 + (2q-1) \eta_2^{j-1}\right] - n \\ &= (2q-1) \frac{1}{1-\eta_2} + O(|\eta_2|^n) = \frac{2q-1}{q(1+q)} + O(|\eta_2|^n). \end{aligned} \quad (4.43)$$

Now, if $\omega^1 = (p, 0)$ then Γ_1 is distributed according to S_2 , and $\omega^2 = (p, 1)$ with probability 1, by (4.10). Thus, by (4.38) and (4.42),

$$\begin{aligned} \rho^{(p,0)}(n) &= \left[\mathbb{E}\left(\Gamma_1 \mid \omega^1 = (p, 0)\right) - 1\right] + \left[\sum_{j=2}^n \mathbb{E}\left(\Gamma_j \mid \omega^2 = (p, 1)\right) - (n-1)\right] \\ &= [\mathbb{E}(S_2) - 1] + \rho^{(p,1)}(n-1) = \frac{(1-2q)(1+q+q^2)}{q^2(1+q)} + O(|\eta_2|^n). \end{aligned} \quad (4.44)$$

Finally, when $\omega^1 = (q, 1)$ it follows from (4.10) that $\omega^2 = (p, 1)$ if $\Gamma_1 \geq 1$ and $\omega^2 = (q, 0)$ if $\Gamma_1 = 0$. Since Γ_1 is distributed according to S_1 when $\omega^1 = (q, 1)$, we have

$$\begin{aligned}\rho^{(q,1)}(n) &= \mathbb{E}\left(\Gamma_1 \mid \omega^1 = (q, 1)\right) + \sum_{j=2}^n \mathbb{E}\left(\Gamma_j \mid \omega^1 = (q, 1)\right) - n \\ &= \mathbb{E}(S_1) + \mathbb{P}(S_1 = 0) \sum_{j=2}^n \mathbb{E}\left(\Gamma_j \mid \omega^2 = (q, 0)\right) + \mathbb{P}(S_1 \geq 1) \sum_{j=2}^n \mathbb{E}\left(\Gamma_j \mid \omega^2 = (p, 1)\right) - n \\ &= \mathbb{E}(S_1) - 1 + (1 - q)\rho^{(q,0)}(n - 1) + q\rho^{(p,1)}(n - 1) = \frac{1 - 2q}{1 + q} + O(|\eta_2|^n),\end{aligned}\tag{4.45}$$

by (4.38), (4.42), and (4.43).

Calculation of $\nu(n)$

From (4.13) we have,

$$\nu(n) = \frac{1}{n} \mathbb{E}[(U(n) - n)^2] = \frac{1}{n} \mathbb{E}\left[\left(\sum_{i=1}^n \Gamma_i - n\right)^2\right].\tag{4.46}$$

We will show below that with $\omega(x) = (p, 1)$ for $x \geq 0$, i.e. with $\omega^1 = (p, 1)$,

$$\nu(n) = \frac{2(1 - q)(1 - q + q^2)}{q^2(q + 1)} + O\left(\frac{1}{n}\right).\tag{4.47}$$

Similar calculations show that (4.47) also holds with $\omega^1 = (p, 0)$, $(q, 0)$ and $(q, 1)$. The latter are omitted for the sake of brevity. However, this asymptotic equivalence of $\nu(n)$ for different values of ω^1 is to be expected from (4.46), since the distribution of ω^j converges exponentially fast to the stationary distribution of the matrix A , for any value of the initial state ω^1 . Thus, one can couple the joint processes $(\omega^j, \Gamma_j)_{j=1}^\infty$ starting from two different values of ω^1 in such a way that the probability that the tails $(\omega^j, \Gamma_j)_{j=n}^\infty$ are not the same in the two processes decays exponentially in n .

We now proceed to the calculation of $\nu(n)$ with $\omega^1 = (p, 1)$; thus, henceforth, all expectations are conditioned on $\omega^1 = (p, 1)$ if not otherwise stated. From (4.46) we have

$$\nu(n) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}(\Gamma_i^2) + \frac{2}{n} \sum_{1 \leq i < j \leq n} \mathbb{E}(\Gamma_i \Gamma_j) - 2 \sum_{i=1}^n \mathbb{E}(\Gamma_i) + n.\tag{4.48}$$

By (4.41), the first term on the right hand side of (4.48) is

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}(\Gamma_i^2) = \frac{1}{n} \sum_{i=1}^n \left[\frac{2 - q + 3q^2}{q(1 + q)} + O(|\eta_2|^{i-1}) \right] = \frac{2 - q + 3q^2}{q(1 + q)} + O\left(\frac{1}{n}\right).\tag{4.49}$$

Also, by (4.42), the third term on the right hand side of (4.48) is

$$-2 \sum_{i=1}^n \mathbb{E}(\Gamma_i) = -2(n + \rho^{(p,1)}(n)) = -2n + \frac{2(2q - 1)}{q^2(1 + q)} + O(|\eta_2|^n).\tag{4.50}$$

To compute the second term on the right hand side of (4.48) we observe that, by (4.8) and (4.10), the following hold.

- (a) If $\omega^i = (p, 1)$ then:
 Γ_i is distributed as S_2 , and $\omega^{i+1} = (p, 1)$ if $\Gamma_i \geq 1$.

(b) If $\omega^i = (q, 0)$ then:

Γ_i is distributed as S_0 , and $\omega^{i+1} = (p, 1)$ if $\Gamma_i \geq 2$ and $(q, 0)$ otherwise.

From (a) and (4.37)-(4.39) we have

$$\begin{aligned}\mathbb{E}\left(\Gamma_i \Gamma_j \middle| \omega^i = (p, 1)\right) &= \mathbb{P}(S_2 \geq 1) \cdot \mathbb{E}(S_2 | S_2 \geq 1) \cdot \mathbb{E}\left(\Gamma_j \middle| \omega^{i+1} = (p, 1)\right) \\ &= \frac{1-q}{q} + \left[\frac{(1-q)(1-2q)}{q^2}\right] \eta_2^{j-(i+1)},\end{aligned}\quad (4.51)$$

and from (b) and (4.37)-(4.40) we have

$$\begin{aligned}\mathbb{E}\left(\Gamma_i \Gamma_j \middle| \omega^i = (q, 0)\right) &= \mathbb{P}(S_0 \geq 2) \cdot \mathbb{E}(S_0 | S_0 \geq 2) \cdot \mathbb{E}\left(\Gamma_j \middle| \omega^{i+1} = (p, 1)\right) \\ &\quad + \mathbb{P}(S_0 = 1) \cdot 1 \cdot \mathbb{E}\left(\Gamma_j \middle| \omega^{i+1} = (q, 0)\right) \\ &= 2q + (1-2q)(1+q^2) \eta_2^{j-(i+1)}.\end{aligned}\quad (4.52)$$

Since ω^i is distributed as $e_1 A^{i-1}$, under our assumption $\omega^1 = (p, 1)$, (4.51) and (4.52) give

$$\begin{aligned}\mathbb{E}(\Gamma_i \Gamma_j) &= \mathbb{P}\left(\omega^i = (p, 1)\right) \cdot \mathbb{E}\left(\Gamma_i \Gamma_j \middle| \omega^i = (p, 1)\right) + \mathbb{P}\left(\omega^i = (q, 0)\right) \cdot \mathbb{E}\left(\Gamma_i \Gamma_j \middle| \omega^i = (q, 0)\right) \\ &= \left\langle e_1 A^{i-1}, \left(\frac{1-q}{q} + \left[\frac{(1-q)(1-2q)}{q^2}\right] \eta_2^{j-(i+1)}, 2q + (1-2q)(1+q^2) \eta_2^{j-(i+1)}\right) \right\rangle.\end{aligned}$$

Using the decomposition (4.35)-(4.36) this simplifies to

$$\mathbb{E}(\Gamma_i \Gamma_j) = 1 + \left[\frac{(1-2q)(1-q+q^2)}{q}\right] \eta_2^{j-(i+1)} + \left[\frac{1-2q}{q}\right] \eta_2^{i-1} + C \eta_2^{j-2} \quad (4.53)$$

after a bit of algebra, where C is an unimportant constant which depends only on q . So, using (4.33), we find

$$\begin{aligned}&\frac{2}{n} \sum_{1 \leq i < j \leq n} \mathbb{E}(\Gamma_i \Gamma_j) \\ &= \frac{2}{n} \left(\frac{n(n-1)}{2} + \left[\frac{(1-2q)(1-q+q^2)}{q}\right] \frac{n}{1-\eta_2} + \left[\frac{1-2q}{q}\right] \frac{n}{1-\eta_2} + O(1) \right) \\ &= n - 1 + \frac{2(1-2q)(2-q+q^2)}{q^2(1+q)} + O\left(\frac{1}{n}\right).\end{aligned}\quad (4.54)$$

Combining (4.48)-(4.50) and (4.54) and simplifying one arrives at (4.47).

Calculation and Analysis of $\theta(n)$

Recall that, in general, $\theta(n) = 2\rho(n)/\nu(n)$. We denote by $\theta^{(p,0)}(n)$ the quantity $\theta(n)$ with $\omega^1 = (p, 0)$ and define $\theta^{(p,0)} = \lim_{n \rightarrow \infty} \theta^{(p,0)}(n)$. Also, we define the analogous quantities for $(q, 0)$, $(p, 1)$, and $(q, 1)$. From (4.42)-(4.45) and (4.47) we have

$$\theta^{(p,0)}(n) = \theta^{(p,0)} + O\left(\frac{1}{n}\right) \quad \text{where} \quad \theta^{(p,0)} = \frac{(1-2q)(1+q+q^2)}{(1-q)(1-q+q^2)}, \quad (4.55)$$

$$\theta^{(p,1)}(n) = \theta^{(p,1)} + O\left(\frac{1}{n}\right) \quad \text{where} \quad \theta^{(p,1)} = \frac{1-2q}{(1-q)(1-q+q^2)}, \quad (4.56)$$

$$\theta^{(q,0)}(n) = \theta^{(q,0)} + O\left(\frac{1}{n}\right) \quad \text{where} \quad \theta^{(q,0)} = \frac{(2q-1)q}{(1-q)(1-q+q^2)}, \quad (4.57)$$

$$\theta^{(q,1)}(n) = \theta^{(q,1)} + O\left(\frac{1}{n}\right) \quad \text{where} \quad \theta^{(q,1)} = \frac{q^2(1-2q)}{(1-q)(1-q+q^2)}. \quad (4.58)$$

From (4.55) it follows that $\theta^{(p,0)} > 1$ is equivalent to $1 - 3q - q^2 > 0$. Consequently, $\theta^{(p,0)} > 1$ if and only if $q \in (0, q_1^*)$, where q_1^* is as in the statement of the theorem. From (4.56) it follows that $\theta^{(p,1)} > 1$ is equivalent to $q > 2$; thus, we always have $\theta^{(p,1)} \leq 1$. From (4.58) it follows that $\theta^{(q,1)} > 1$ is equivalent to $q^3 + q^2 - 2q + 1 < 0$. Since $q^3 + q^2 - 2q + 1 = q^3 + (1 - q)^2$, we always have $\theta^{(q,1)} \leq 1$. Finally, from (4.57) it follows that $\theta^{(q,0)} > 1$ is equivalent to $q^3 + q - 1 > 0$. Thus, $\theta^{(q,0)} > 1$ if and only if $q > q_2^*$, where q_2^* is as in the statement of the theorem. This establishes all claims made in the first paragraph of the proof. \square

4.6 Proof of Theorem 6

Thus far the proofs of transience/recurrence for the random walk (X_n) have centered around an analysis of the right jumps Markov chain (Z_x) . For the proof of Theorem 6, we will need to construct another auxiliary process called the left jumps Markov chain.

Consider the random walk $(X_n)_{n \geq 0}$ started from $X_0 = 0$ and restricted to $\mathbb{N} \cup \{0, -1\}$ by the following modification of its transition mechanism: when the walker is at a site $x \geq 0$, it behaves as before, but at the site -1 it jumps right with probability one. Denote the modified random walk by $(\tilde{X}_n)_{n \geq 0}$. Note that the modified random walk can be defined in terms of the extended single site Markov chains, $(\hat{Y}_n^x)_{n \in \mathbb{N}} = (Y_n^x, J_n^x)_{n \in \mathbb{N}}$, $x \geq 0$, along with an appropriately defined deterministic single site mechanism at $x = -1$. Fix $N \in \mathbb{Z}^+$ and let $\tilde{T}_N = \inf\{n \geq 0 : \tilde{X}_n = N\}$ denote the first time the modified random walk hits N . Note that \tilde{T}_N is almost surely finite. We define a process $(\tilde{W}_x^{(N)})_{x=0}^N$ by setting $\tilde{W}_x^{(N)}$ equal to the number of times the modified walk (\tilde{X}_n) jumps left from site x before time \tilde{T}_N . That is,

$$\tilde{W}_x^{(N)} = |\{n \leq \tilde{T}_N - 1 : \tilde{X}_n = x, \tilde{X}_{n+1} = x - 1\}|.$$

We will refer to this process $(\tilde{W}_x^{(N)})_{x=0}^N$ as the *left jumps N -chain*. It can also be defined directly in terms of the jump sequences $(J_k^x)_{k \in \mathbb{N}}$, $0 \leq x \leq N$:

$$\tilde{W}_N^{(N)} \equiv 0, \quad \tilde{W}_x^{(N)} = \Theta_x^{(N)} - \tilde{W}_{x+1}^{(N)} - 1, \quad x \in \{N-1, N-2, \dots, 0\}, \quad (4.59)$$

where

$$\Theta_x^{(N)} = \inf \left\{ n \geq 1 : \sum_{k=1}^n \mathbb{1}\{J_k^x = 1\} = \tilde{W}_{x+1}^{(N)} + 1 \right\}. \quad (4.60)$$

That is, $\tilde{W}_x^{(N)}$ is the number of left jumps in the jump sequence $(J_k^x)_{k \in \mathbb{N}}$ before the $(\tilde{W}_{x+1}^{(N)} + 1)$ -th right jump. In particular, $\tilde{W}_x^{(N)}$ is independent of $\tilde{W}_{x+2}^{(N)}, \dots, \tilde{W}_N^{(N)}$ conditioned on $\tilde{W}_{x+1}^{(N)}$, so the sequence $(\tilde{W}_N^{(N)}, \dots, \tilde{W}_0^{(N)})$ is Markovian. The distribution of the jump sequence $(J_k^x)_{k \in \mathbb{N}}$ is the same for all $x \geq 0$, if the initial environment ω is constant for all $x \geq 0$. So, in this case, the transition probabilities

$$\begin{aligned} \mathbb{P}(\tilde{W}_x^{(N)} = \ell | \tilde{W}_{x+1}^{(N)} = m) \\ = \mathbb{P} \left(\inf \left\{ n \geq 1 : \sum_{k=1}^n \mathbb{1}\{J_k^x = 1\} = m + 1 \right\} - (m + 1) = \ell \right) \end{aligned}$$

are independent of N and $x \in \{0, \dots, N-2\}$, and we may define a single time-homogeneous Markov chain $(W_n)_{n=0}^\infty$ such that $(\tilde{W}_N^{(N)}, \tilde{W}_{N-1}^{(N)}, \dots, \tilde{W}_0^{(N)})$ has the same distribution as (W_0, W_1, \dots, W_N) , for all N .

We call $(W_n)_{n=0}^\infty$ the *left jumps Markov chain*. The following proposition characterizes the transience or recurrence of the original random walk (X_n) in terms of the positive recurrence or non-positive recurrence of the left jumps Markov chain.

Proposition 4. *If $X_0 = 0$ and the initial environment $\omega(x)$ is constant for $x \geq 0$, then the random walk (X_n) has positive probability of being transient to $+\infty$ if and only if the left jumps Markov chain (W_n) is positive recurrent.*

Proof. Arguments exactly like the proof of part (iii) of Lemma 1 show that the modified random walk (\tilde{X}_n) either has probability 1 of being transient to $+\infty$ or probability 1 of being recurrent, and clearly the former occurs if and only if the original random walk (X_n) has a positive probability of being transient to $+\infty$. Thus, it suffices to show that the left jumps Markov chain (W_n) is positive recurrent if and only if the modified random walk (\tilde{X}_n) is transient to $+\infty$.

Now, by construction of the left jumps Markov chain (W_n) , we know W_N and $\tilde{W}_0^{(N)}$ have the same distribution for each $N > 0$, where $\tilde{W}_0^{(N)}$ is the number of jumps of the modified random walk (\tilde{X}_n) from 0 to -1 before it first reaches N . Thus, the distribution of W_N is stochastically increasing, and it converges to a limiting finite distribution if and only if the modified random walk (\tilde{X}_n) is transient. On the other hand, since $(W_n)_{n \geq 0}$ is a (time-homogeneous) irreducible, aperiodic, Markov chain, the distribution of W_N converges to a finite limiting distribution if and only if this chain is positive recurrent. \square

We now use Proposition 4 to prove Theorem 6.

Proof of Theorem 6. By symmetry it suffices to treat the case $R = 1$. In the statement of the theorem, it is assumed that the initial environment is constant in a neighborhood of $+\infty$ in the negative feedback case. For the proof, we will make this assumption even in the positive feedback case. This causes no problem because in the positive feedback case if we can prove that the probability of transience to $+\infty$ is 0 for any constant environment then, by Lemma 4, it is also true for any non-constant initial environment. Without loss of generality, we may assume also that the initial environment is constant for all $x \geq 0$.

In this case, by Proposition 4, it suffices to show the left jumps Markov chain (W_n) is not positive recurrent. By construction of the left jumps chain we have

$$\mathbb{P}(W_n = \cdot | W_{n-1} = m) = \mathbb{P}(\tilde{W}_x^{(N)} = \cdot | \tilde{W}_{x+1}^{(N)} = m),$$

where the right hand side is independent of N and $x \in \{0, \dots, N-2\}$ (due to the assumption on the initial environment). Now, if we condition on $\tilde{W}_{x+1}^{(N)} = m$, it follows from (4.59) and (4.60) that $\tilde{W}_x^{(N)}$ is equal to the number of left jumps in the jump sequence $(J_k^x)_{k \in \mathbb{N}}$ before the time of the $(m+1)$ -th right jump.

Similarly to the analysis of the right jumps chain, we decompose $\tilde{W}_x^{(N)}$ as

$$\tilde{W}_x^{(N)} = \sum_{j=1}^{m+1} V_j,$$

where V_j is the number of left jumps in the sequence $(J_k^x)_{k \in \mathbb{N}}$ between the $(j-1)$ -th and j -th right jumps. Since $R = 1$, the configuration at site x is always $(p, 0)$ immediately after a right jump from site x . So, the starting configuration for each of the “left jump sessions” after the first one is $(p, 0)$, independent of the number of left jumps in all previous sessions. It follows that the random variables V_1, \dots, V_{m+1} are independent and V_2, \dots, V_{m+1} are i.i.d. with common distribution V , which is the distribution of the number of left jumps from site x before the first right jump, starting in the $(p, 0)$ configuration:

$$\mathbb{P}(V = k) = \begin{cases} (1-p)^k p, & 0 \leq k \leq L-1; \\ (1-p)^L (1-q)^{k-L} q, & k \geq L. \end{cases}$$

(This is analogous to the situation $L = 1$ for the right jumps Markov chain, where $U(m) = \sum_{j=1}^m \Gamma_j$ with $\Gamma_1, \dots, \Gamma_m$ independent and $\Gamma_2, \dots, \Gamma_m$ i.i.d.)

We now show that since $\alpha = \frac{1}{2}$, $\mathbb{E}(V) = 1$. After a somewhat messy calculation and some algebraic simplification, one finds that

$$\mathbb{E}(V) = \frac{(1-p)q + (1-p)^L(p-q)}{pq}.$$

From this it follows that $\mathbb{E}(V) = 1$ if and only if $q = \frac{p(1-p)^L}{2p-1+(1-p)^L}$. Since $R = 1$ and $\alpha = 1/2$, we know that $q = q_0 = \frac{p(1-p)^L}{2p-1+(1-p)^L}$ by Remark 2 after Proposition 1. So, we conclude that $\mathbb{E}(V) = 1$.

We have now shown that

$$\mathbb{P}(W_n = \cdot | W_{n-1} = m) = \mathbb{P}\left(\sum_{j=1}^{m+1} V_j = \cdot\right),$$

where V_1, \dots, V_{m+1} are independent and V_2, \dots, V_{m+1} are i.i.d. with mean 1. So, the Markov chain (W_n) has the transition probabilities of a critical branching process with immigration. The immigration term V_1 depends on the initial environment, but is always nonnegative and not identically zero with finite mean. Also, clearly $\mathbb{E}(V^2) < \infty$, so the branching terms have finite variance. It thus follows from [7] that $\frac{W_n}{n}$ converges in law to a certain nonzero limiting distribution, which implies the Markov chain $(W_n)_{n \geq 0}$ cannot be positive recurrent. \square

5 Analysis of α

In this section we prove Proposition 1, which characterizes some properties of the important quantity

$$\alpha = p \cdot \pi_p + q \cdot \pi_q \tag{5.1}$$

that determines the direction of transience for our random walk (away from borderline critical case). We recall from (2.4) that

$$\begin{aligned} \pi_p &= \frac{(1-q)q^R(1-(1-p)^L)}{(1-q)q^R(1-(1-p)^L) + p(1-p)^L(1-q^R)}, \\ \pi_q &= \frac{p(1-p)^L(1-q^R)}{(1-q)q^R(1-(1-p)^L) + p(1-p)^L(1-q^R)}. \end{aligned}$$

The various pieces of the proposition will be proved separately, but we begin first with two useful observations.

(I) For any fixed q, R, L the quantity

$$\frac{\pi_p}{\pi_q} = \frac{(1-q)q^R}{1-q^R} \cdot \frac{1-(1-p)^L}{p(1-p)^L} \tag{5.2}$$

satisfies $\lim_{p \rightarrow 1} \left(\frac{\pi_p}{\pi_q}\right) = \infty$. Since $\pi_p + \pi_q = 1$, this implies $\lim_{p \rightarrow 1} \pi_p = 1$.

(II) For any fixed q, R, L the quantity $\frac{\pi_p}{\pi_q}$ satisfies

$$\frac{d}{dp} \left(\frac{\pi_p}{\pi_q}\right) = \frac{(1-q)q^R}{1-q^R} \cdot \frac{p(L+1) + (1-p)^{L+1} - 1}{p^2(1-p)^{L+1}} > 0, \quad \forall p \in (0, 1).$$

Since $\pi_p + \pi_q = 1$, this implies $\frac{d}{dp}(\pi_p) > 0, \forall p \in (0, 1)$. So,

$$\begin{aligned} \frac{d}{dp}(\alpha) &= \frac{d}{dp}(p \cdot \pi_p + q \cdot \pi_q) = \frac{d}{dp}(p \cdot \pi_p + q \cdot (1 - \pi_p)) \\ &= \pi_p + (p - q) \cdot \frac{d}{dp}(\pi_p) > 0, \quad \text{for all } p \geq q. \end{aligned} \tag{5.3}$$

Proof of (vi): By (I), $\lim_{p \rightarrow 1} \alpha = \lim_{p \rightarrow 1} (p \cdot \pi_p + q \cdot \pi_q) = 1 \cdot 1 + q \cdot 0 = 1$.

Proof of (i): This is immediate from (5.1) since $\pi_p + \pi_q = 1$ and $\pi_p, \pi_q > 0$, for any p, q .

Proof of (ii): If $q < 1/2$, then $\alpha < 1/2$ for all $p \leq 1/2$, by (5.1). But, by (II) and (vi), we also know that $\alpha(p)$ is monotonically increasing on the interval $[1/2, 1) \subset [q, 1)$, with $\lim_{p \rightarrow 1} \alpha(p) = 1$. Thus, the claim follows by continuity of $\alpha(p)$.

Proof of (iii): Plugging $p = 1 - q$ into (1.2) and simplifying one finds that

$$\begin{aligned} \alpha(1 - q) < 1/2 &\iff q^R(1/2 - q) - q^L(1/2 - q) < 0, \text{ and} \\ \alpha(1 - q) > 1/2 &\iff q^R(1/2 - q) - q^L(1/2 - q) > 0. \end{aligned}$$

Thus, for $q < 1/2$ and $R > L$, $\alpha(1 - q) < 1/2$, which implies $p_0 > 1 - q$. While, for $q < 1/2$ and $R < L$, $\alpha(1 - q) > 1/2$, which implies $p_0 < 1 - q$. This proves (1.6).

Now, by (II) and symmetry considerations, for any fixed R, L, p we know that $d/dq(\alpha) > 0$ for $q \leq p$. Thus, for any $0 < q < q' < 1/2$, we have

$$\alpha(p_0(q, R, L), q', R, L) > \alpha(p_0(q, R, L), q, R, L) = 1/2,$$

which implies $p_0(q', R, L) < p_0(q, R, L)$. So, p_0 is a decreasing function of q , for $q \in (0, 1/2)$.

Proof of (iv): Plugging $L = 1$ into (1.2) and simplifying one finds that

$$\alpha = 1/2 \iff p(1 - 2q + q^R) = 1 - 2q + q^{R+1}$$

and, similarly,

$$\begin{aligned} \alpha < 1/2 &\iff p(1 - 2q + q^R) < 1 - 2q + q^{R+1}, \\ \alpha > 1/2 &\iff p(1 - 2q + q^R) > 1 - 2q + q^{R+1}. \end{aligned}$$

(iv) follows by considering separately the two cases $1 - 2q + q^{R+1} > 0$ and $1 - 2q + q^{R+1} \leq 0$.

Proof of (v): If $L = R$, then plugging in $p = 1 - q$ into (1.2) gives $\alpha = 1/2$. So, by (ii), if $q < 1/2$ then $p_0 = 1 - q$ is the unique critical point. On the other hand, for any $q > 1/2$, if $L = R$ is sufficiently large then there exists another critical point $p'_0 > 1 - q$. This follows from (vi), continuity of α , and the following claim.

Claim: For any fixed $q > 1/2$, if $L = R$ is sufficiently large then $\frac{d}{dp}(\alpha)|_{p=1-q} < 0$. Thus, there exists some $\epsilon > 0$ such that $\alpha(1 - q + \epsilon) < 1/2$.

Proof: Computing $\frac{d}{dp}(\alpha)$ directly from (1.2) and then substituting $L = R$ and $p = 1 - q$, one finds, after some lengthy simplifications, that the condition $\frac{d}{dp}(\alpha)|_{p=1-q} < 0$ is equivalent to the condition

$$R(1 - q)(1 - 2q) + q(1 - q^R) < 0.$$

For fixed $q > 1/2$, this condition is satisfied for all sufficiently large R . □

A Solution of Linear Systems

A.1 Stationary Distribution of Single Site Markov Chains

Here we solve the linear system $\{\pi = \pi M, \sum_{\lambda} \pi_{\lambda} = 1\}$ for the stationary distribution π of the single site Markov chain transition matrix M . In expanded form this system becomes

$$\pi_{(p,i)} = (1-p) \cdot \pi_{(p,i-1)}, \quad 1 \leq i \leq L-1 \quad (\text{A.1})$$

$$\pi_{(p,0)} = p \cdot \pi_p + q \cdot \pi_{(q,R-1)} \quad (\text{A.2})$$

$$\pi_{(q,i)} = q \cdot \pi_{(q,i-1)}, \quad 1 \leq i \leq R-1 \quad (\text{A.3})$$

$$\pi_{(q,0)} = (1-q) \cdot \pi_q + (1-p) \cdot \pi_{(p,L-1)} \quad (\text{A.4})$$

$$\pi_p + \pi_q = 1, \quad (\text{A.5})$$

where $\pi_p = \sum_{i=0}^{L-1} \pi_{(p,i)}$ and $\pi_q = \sum_{i=0}^{R-1} \pi_{(q,i)}$. Applying (A.1) and (A.3) repeatedly gives

$$\pi_{(p,i)} = (1-p)^i \cdot \pi_{(p,0)}, \quad 0 \leq i \leq L-1; \quad (\text{A.6})$$

$$\pi_{(q,i)} = q^i \cdot \pi_{(q,0)}, \quad 0 \leq i \leq R-1. \quad (\text{A.7})$$

Hence,

$$\pi_p = \sum_{i=0}^{L-1} (1-p)^i \cdot \pi_{(p,0)} = \frac{1-(1-p)^L}{p} \cdot \pi_{(p,0)}, \quad (\text{A.8})$$

$$\pi_q = \sum_{i=0}^{R-1} q^i \cdot \pi_{(q,0)} = \frac{1-q^R}{1-q} \cdot \pi_{(q,0)}. \quad (\text{A.9})$$

Plugging (A.7) and (A.8) into (A.2) gives

$$\pi_{(p,0)} = p \cdot \left(\frac{1-(1-p)^L}{p} \cdot \pi_{(p,0)} \right) + q \cdot (q^{R-1} \cdot \pi_{(q,0)}),$$

which implies

$$\pi_{(p,0)} = \pi_{(q,0)} \cdot \frac{q^R}{(1-p)^L}. \quad (\text{A.10})$$

But, by (A.5), (A.8), and (A.9), we also have

$$\frac{1-(1-p)^L}{p} \cdot \pi_{(p,0)} + \frac{1-q^R}{1-q} \cdot \pi_{(q,0)} = 1$$

or, equivalently,

$$\pi_{(p,0)} = \left(1 - \pi_{(q,0)} \frac{1-q^R}{1-q} \right) \cdot \frac{p}{1-(1-p)^L}. \quad (\text{A.11})$$

Equating the right hand sides of (A.10) and (A.11) and solving for $\pi_{(q,0)}$ gives

$$\pi_{(q,0)} = \frac{p(1-q)(1-p)^L}{(1-q)q^R(1-(1-p)^L) + p(1-p)^L(1-q^R)}.$$

Substituting this value of $\pi_{(q,0)}$ into (A.10) gives an explicit expression for $\pi_{(p,0)}$, and the values of $\pi_{(q,i)}$, $1 \leq i \leq R-1$, and $\pi_{(p,i)}$, $1 \leq i \leq L-1$, are then easily found by substituting the expressions for $\pi_{(p,0)}$ and $\pi_{(q,0)}$ in (A.6) and (A.7), giving (2.3).

A.2 Expected Hitting Times with $R = 1$

Here we solve the linear system (3.17) for the expected hitting times a_i , $0 \leq i \leq L$. As shown in the proof of Theorem 4, using soft methods, these expected hitting times must all be finite.

For simplicity of notation we define $b_i = a_{L-i}$, $0 \leq i \leq L$. Rearranging slightly the system (3.17) then becomes

$$\begin{aligned} b_{i+1} &= 1 + (1-p)(a_0 + b_i), \quad 0 \leq i \leq L-1 \\ b_0 &= \frac{1}{q} + \left(\frac{1-q}{q}\right) a_0. \end{aligned}$$

Thus, for each $0 \leq i \leq L$, we have

$$b_i = u_i + v_i \cdot a_0$$

where the sequences $(u_i)_{i=0}^L$ and $(v_i)_{i=0}^L$ are defined recursively by

$$\begin{aligned} u_0 &= 1/q \quad \text{and} \quad u_{i+1} = 1 + (1-p)u_i, \quad 0 \leq i \leq L-1, \\ v_0 &= (1-q)/q \quad \text{and} \quad v_{i+1} = (1-p)(1+v_i), \quad 0 \leq i \leq L-1. \end{aligned}$$

By induction on i , we find that, for each $1 \leq i \leq L$,

$$\begin{aligned} u_i &= \frac{(1-p)^i}{q} + \sum_{j=0}^{i-1} (1-p)^j = \frac{1 + (p/q - 1)(1-p)^i}{p}, \\ v_i &= \frac{(1-p)^i}{q} + \sum_{j=1}^{i-1} (1-p)^j = \frac{1 - p + (p/q - 1)(1-p)^i}{p}. \end{aligned}$$

Substituting, first for the b_i 's and then for the a_i 's with $a_i = b_{L-i}$, one obtains (1.12) and (1.13).

B Proof of Lemma 1

Here we prove Lemma 1 from section 2.2. The three parts are proved separately. In each case, we prove only the first of the two statements, since the second follows by symmetry. The following notation will be used for the proofs.

- $T_x^{(i)}$ is the i -th hitting time of site x :

$$T_x^{(1)} = T_x \quad \text{and} \quad T_x^{(i+1)} = \inf\{n > T_x^{(i)} : X_n = x\},$$

with the convention $T_x^{(j)} = \infty$, for all $j > i$, if $T_x^{(i)} = \infty$.

- $m_i = \sup\{X_n : n \leq T_0^{(i)}\}$ is the maximum position of the random walk up to the i -th hitting time of site 0.
- For an initial environment ω and path $\zeta = (x_0, \dots, x_n)$, $\omega^{(\zeta)}$ is the environment induced at time n by following the path ζ starting in ω :

$$\{\omega_0 = \omega, X_0 = x_0, \dots, X_n = x_n\} \implies \omega_n = \omega^{(\zeta)}.$$

Proof of (ii): Clearly, $\mathbb{P}_\omega(X_n \rightarrow \infty) \leq \mathbb{P}_\omega(\liminf_{n \rightarrow \infty} X_n > -\infty)$. To show the reverse inequality also holds observe that, for any $k \in \mathbb{Z}$, $\mathbb{P}_\omega(\liminf_{n \rightarrow \infty} X_n = k) = 0$. Thus,

$$\mathbb{P}_\omega\left(\liminf_{n \rightarrow \infty} X_n > -\infty, X_n \not\rightarrow \infty\right) = \mathbb{P}_\omega\left(-\infty < \liminf_{n \rightarrow \infty} X_n < \infty\right) = 0.$$

Proof of (i): By (ii), $\mathbb{P}_\omega(X_n \rightarrow \infty) \geq \mathbb{P}_\omega(\mathcal{A}_0^+)$. Thus, $\mathbb{P}_\omega(X_n \rightarrow \infty) > 0$, if $\mathbb{P}_\omega(\mathcal{A}_0^+) > 0$. On the other hand, if $\mathbb{P}_\omega(X_n \rightarrow \infty) > 0$ then there exists some finite path $\zeta = (x_0, \dots, x_n)$, such that $x_0 = 0$, $x_n = 2$, and

$$\mathbb{P}_\omega(X_m > 1, \forall m \geq n | X_0 = x_0, \dots, X_n = x_n) > 0.$$

We construct from $\zeta = (x_0, \dots, x_n)$ the reduced path $\tilde{\zeta} = (\tilde{x}_0, \dots, \tilde{x}_{\tilde{n}})$ by setting $\tilde{x}_0 = x_0 = 0$, and then removing from the tail (x_1, \dots, x_n) all steps before the first hitting time of site 1 and all steps in any leftward excursions from site 1. For example,

$$\begin{aligned} \text{if } \zeta &= (0, \mathbf{-1}, \mathbf{0}, 1, 2, 1, \mathbf{0}, \mathbf{1}, 2, 1, \mathbf{0}, \mathbf{-1}, \mathbf{-2}, \mathbf{-1}, \mathbf{-2}, \mathbf{-1}, \mathbf{0}, \mathbf{1}, 2, 3, 2), \\ \text{then } \tilde{\zeta} &= (0, 1, 2, 1, 2, 1, 2, 3, 2) \end{aligned}$$

(where we denote the removed steps in bold for visual clarity). By construction, $\omega^{(\tilde{\zeta})}(x) = \omega^{(\zeta)}(x)$, for all $x \geq 2$. So, $\mathbb{P}_\omega(X_m > 1, \forall m \geq \tilde{n} | (X_0, \dots, X_{\tilde{n}}) = \tilde{\zeta}) = \mathbb{P}_\omega(X_m > 1, \forall m \geq n | (X_0, \dots, X_n) = \zeta) > 0$. Thus,

$$\mathbb{P}_\omega(\mathcal{A}_0^+) \geq \mathbb{P}_\omega((X_0, \dots, X_{\tilde{n}}) = \tilde{\zeta}) \cdot \mathbb{P}_\omega(X_m > 1, \forall m \geq \tilde{n} | (X_0, \dots, X_{\tilde{n}}) = \tilde{\zeta}) > 0.$$

Proof of (iii): Since we assume $\mathbb{P}_\omega(X_n \rightarrow -\infty) = 0$, it follows from (ii) that (a) T_x is \mathbb{P}_ω a.s. finite for each $x \geq 0$, and (b) every time the random walk steps left from a site x it will eventually return with probability 1. Now (b) implies that the probability that the walk is transient to $+\infty$, after first hitting a site $x \geq 0$, is independent of the trajectory taken to get to x . That is, $\mathbb{P}_\omega(X_n \rightarrow \infty | (X_0, \dots, X_n) = \zeta) = \mathbb{P}_\omega(X_n \rightarrow \infty | T_x < \infty)$, for any $x \geq 0$ and path $\zeta = (x_0, \dots, x_n)$ such that $x_0 = 0$, $x_n = x$, and $x_m < x$ for $m < n$. Combining this last observation with (a) shows that

$$\begin{aligned} \mathbb{P}_\omega(X_n \rightarrow \infty | T_0^{(i)} < \infty, m_i = x) &= \mathbb{P}_\omega(X_n \rightarrow \infty | T_0^{(i)} < \infty, m_i = x, T_{x+1} < \infty) \\ &= \mathbb{P}_\omega(X_n \rightarrow \infty | T_{x+1} < \infty) = \mathbb{P}_\omega(X_n \rightarrow \infty), \text{ for all } x \geq 0 \text{ and } i \geq 1. \end{aligned}$$

So, $\mathbb{P}_\omega(X_n \rightarrow \infty | T_0^{(i)} < \infty) = \mathbb{P}_\omega(X_n \rightarrow \infty)$, for all $i \geq 1$. Thus, by (ii),

$$\mathbb{P}_\omega(X_n \not\rightarrow \infty) = \mathbb{P}_\omega(X_n \not\rightarrow \infty | T_0^{(i)} < \infty) = \prod_{j=i}^{\infty} \mathbb{P}_\omega(T_0^{(j+1)} < \infty | T_0^{(j)} < \infty), \forall i \geq 1.$$

Since the LHS is independent of i , the product on the RHS is constant for $i \geq 1$. Thus, there are two possibilities: either the product is 0 (for all $i \geq 1$) or $\mathbb{P}_\omega(T_0^{(j+1)} < \infty | T_0^{(j)} < \infty) = 1$, for all $j \geq 1$. In the latter case, $\mathbb{P}_\omega(X_n \not\rightarrow \infty) = 1$, which contradicts the assumption that $\mathbb{P}_\omega(X_n \rightarrow \infty) > 0$. In the former case, $\mathbb{P}_\omega(X_n \rightarrow \infty) = 1$, as required. \square

C Proof of Lemma 7

The following strong law for sums of dependent random variables is a special case of [8, Theorem 1] with $w_i = 1$ and $W_i = i$.

Theorem 9. Let $(\xi_i)_{i \in \mathbb{N}}$ be a sequence of nonnegative random variables satisfying:

1. $\sup_i \mathbb{E}(\xi_i) < \infty$.
2. $\mathbb{E}(\xi_i^2) < \infty$, for each i .
3. $\sum_{j=1}^{\infty} \sum_{i=1}^j \frac{1}{j^2} \cdot \text{Cov}^+(\xi_i, \xi_j) < \infty$.

Then

$$\frac{1}{n} \sum_{i=1}^n (\xi_i - \mathbb{E}(\xi_i)) \xrightarrow{a.s.} 0, \text{ as } n \rightarrow \infty.$$

Using this theorem we will prove Lemma 7. Throughout our proof the initial environment ω is fixed, and all random variables are distributed according to the measure \mathbb{P}_ω , which we will abbreviate simply as \mathbb{P} . Also, $\beta > 0$ is the constant given in Corollary 2.

Proof of Lemma 7. By Corollary 3,

$$\mathbb{E}(N_x) \leq \frac{1}{\beta} \quad \text{and} \quad \mathbb{E}(N_x^2) \leq \frac{2-\beta}{\beta^2}, \quad \text{for each } x \in \mathbb{N}. \quad (\text{C.1})$$

Thus, by Theorem 9, it suffices to show that

$$\sum_{y=1}^{\infty} \sum_{x=1}^y \frac{1}{y^2} \text{Cov}^+(N_x, N_y) < \infty.$$

Since N_x and N_y are nonnegative integer valued random variables, $\text{Cov}(N_x, N_y)$ can be represented as the following absolutely convergent double sum:

$$\text{Cov}(N_x, N_y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left(\mathbb{P}(N_x \geq k, N_y \geq j) - \mathbb{P}(N_x \geq k) \mathbb{P}(N_y \geq j) \right). \quad (\text{C.2})$$

To bound this sum we will need the following two estimates for the differences $D_{k,j} \equiv \mathbb{P}(N_x \geq k, N_y \geq j) - \mathbb{P}(N_x \geq k) \mathbb{P}(N_y \geq j)$:

$$\text{For any } 1 \leq x < y \text{ and } k, j \in \mathbb{N}, D_{k,j} \leq (1 - \beta)^{\max\{j,k\}-1}. \quad (\text{C.3})$$

$$\text{For any } 1 \leq x < y \text{ and } k, j \in \mathbb{N}, D_{k,j} \leq (1 - \beta)^{y-x}. \quad (\text{C.4})$$

(C.3) follows from Corollary 3:

$$\begin{aligned} D_{k,j} &\equiv \mathbb{P}(N_x \geq k, N_y \geq j) - \mathbb{P}(N_x \geq k) \mathbb{P}(N_y \geq j) \\ &\leq \mathbb{P}(N_x \geq k, N_y \geq j) \leq \min\{\mathbb{P}(N_x \geq k), \mathbb{P}(N_y \geq j)\} \leq (1 - \beta)^{\max\{j,k\}-1}. \end{aligned}$$

To see (C.4) recall that N_x^y and N_y are independent for all $1 \leq x < y$, by Lemma 6. Thus, for any $1 \leq x < y$, we have

$$\begin{aligned} \mathbb{P}(N_x \geq k, N_y \geq j) &= \mathbb{P}(N_x^y \geq k, N_y \geq j) + \mathbb{P}(N_x^y < k, N_x \geq k, N_y \geq j) \\ &= \mathbb{P}(N_x^y \geq k) \mathbb{P}(N_y \geq j) + \mathbb{P}(N_x^y < k, N_x \geq k, N_y \geq j) \\ &\leq \mathbb{P}(N_x \geq k) \mathbb{P}(N_y \geq j) + \mathbb{P}(B_y \geq y - x) \\ &\leq \mathbb{P}(N_x \geq k) \mathbb{P}(N_y \geq j) + (1 - \beta)^{y-x} \end{aligned}$$

by Corollary 4.

Now, for given $1 \leq x < y$, let $n = y - x$ and let $N = \lfloor (1 - \beta)^{-n/4} \rfloor$. Breaking the (absolutely convergent) double sum in (C.2) into pieces and applying Fubini's Theorem gives

$$\begin{aligned} \text{Cov}(N_x, N_y) &= \sum_{j=1}^N \sum_{k=1}^N D_{k,j} + \sum_{j=1}^N \sum_{k=N+1}^{\infty} D_{k,j} + \sum_{k=1}^N \sum_{j=N+1}^{\infty} D_{k,j} \\ &+ \sum_{k=N+1}^{\infty} \sum_{j=k}^{\infty} D_{k,j} + \sum_{j=N+1}^{\infty} \sum_{k=j+1}^{\infty} D_{k,j}. \end{aligned}$$

By (C.4), the first term on the RHS of this equation is bounded above by $N^2(1 - \beta)^n$. Similarly, by (C.3):

- The second term is bounded by $N \cdot \sum_{k=N+1}^{\infty} (1 - \beta)^{k-1} = N(1 - \beta)^N / \beta$.
- The third term is bounded by $N \cdot \sum_{j=N+1}^{\infty} (1 - \beta)^{j-1} = N(1 - \beta)^N / \beta$.
- The fourth term is bounded by $\sum_{k=N+1}^{\infty} \sum_{j=k}^{\infty} (1 - \beta)^{j-1} = (1 - \beta)^N / \beta^2$.
- The fifth term is bounded by $\sum_{j=N+1}^{\infty} \sum_{k=j+1}^{\infty} (1 - \beta)^{k-1} = (1 - \beta)^{N+1} / \beta^2$.

The upper bound on the first term is at most $(1 - \beta)^{n/2}$, and the same is also true for the upper bounds on each of the other 4 terms for all sufficiently n , since N grows exponentially in n . Thus, there exists some $n_0 \in \mathbb{N}$ such that

$$\text{Cov}(N_x, N_y) \leq 5(1 - \beta)^{n/2}, \text{ whenever } y - x = n \geq n_0.$$

But, for any $1 \leq x \leq y$ such that $y - x = n < n_0$ we also have

$$\text{Cov}(N_x, N_y) \leq \mathbb{E}(N_x^2)^{1/2} \cdot \mathbb{E}(N_y^2)^{1/2} \leq \frac{2 - \beta}{\beta^2} \leq \left(\frac{2 - \beta}{\beta^2(1 - \beta)^{n_0 - 1}} \right) (1 - \beta)^n$$

by (C.1). Thus, for all $1 \leq x \leq y$,

$$\text{Cov}(N_x, N_y) \leq C(1 - \beta)^{n/2}, \text{ where } C \equiv \max \left\{ 5, \frac{2 - \beta}{\beta^2(1 - \beta)^{n_0 - 1}} \right\} \text{ and } n = y - x.$$

So,

$$\sum_{y=1}^{\infty} \sum_{x=1}^y \frac{1}{y^2} \text{Cov}^+(N_x, N_y) \leq \sum_{y=1}^{\infty} \sum_{x=1}^y \frac{1}{y^2} \cdot C(1 - \beta)^{(y-x)/2} < \infty.$$

□

D Proofs of Lemmas 9, 10, and 11

Proof of Lemma 9. Since $U(n) = \sum_{j=1}^n \Gamma_j$, it follows from the Markov chain representation of section 4.2 and the ergodic theorem for finite-state Markov chains along with (4.8) that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(U(n))}{n} = \lim_{j \rightarrow \infty} \mathbb{E}(\Gamma_j) = \langle \psi, E \rangle \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \Gamma_j = \langle \psi, E \rangle, \quad \text{a.s.}$$

By definition, Γ_j is the number of right jumps (i.e. 1's) in the jump sequence $(J_k^x)_{k \in \mathbb{N}}$ between the $(j-1)$ -th and j -th left jumps. So, this implies

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m \mathbb{1}\{J_k^x = 1\} = \lim_{n \rightarrow \infty} \left(\frac{\sum_{j=1}^n \Gamma_j}{n + \sum_{j=1}^n \Gamma_j} \right) = \frac{\langle \psi, E \rangle}{1 + \langle \psi, E \rangle}, \text{ a.s.}$$

On the other hand, as noted at the end of section 2.1.1,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m \mathbb{1}\{J_k^x = 1\} = \alpha, \text{ a.s.}$$

Since $\alpha = 1/2$, it follows that $\langle \psi, E \rangle = 1$. \square

Proof of Lemma 11. We consider separately the cases $L = 1$ and $L \geq 2$. In both cases, since $\alpha = 1/2$ we have $\mu = 1$, by Lemma 9. Thus, $\nu(n) = \mathbb{E}[(U(n) - n)^2]/n$.

Case 1: $L = 1$.

In this case $\omega^j = (q, 0)$ for all $j \geq 2$, regardless of the values of the Γ_j 's. Thus, $\Gamma_1, \dots, \Gamma_n$ are independent and $\Gamma_2, \dots, \Gamma_n$ are i.i.d. distributed as S_0 . So,

$$\liminf_{n \rightarrow \infty} \nu(n) = \liminf_{n \rightarrow \infty} \frac{\mathbb{E}[(U(n) - n)^2]}{n} \geq \liminf_{n \rightarrow \infty} \frac{\text{Var}(U(n))}{n} = \text{Var}(S_0) > 0.$$

Case 2: $L \geq 2$.

By construction ω^{j+1} is a deterministic function of ω^j and Γ_j . For $\lambda, \lambda' \in \Lambda$, we define $K_{\lambda, \lambda'} = \{k \geq 0 : \omega^{j+1} = \lambda', \text{ if } \omega^j = \lambda \text{ and } \Gamma_j = k\}$. We say a sequence of configurations $\vec{\lambda} = (\lambda_1, \dots, \lambda_{n+1}) \in \Lambda^{n+1}$ is *allowable* if $|K_{\lambda_i, \lambda_{i+1}}| > 0$ for all $1 \leq i \leq n$, and denote by G_{n+1} the set of all allowable length- $(n+1)$ configuration sequences. For each allowable configuration sequence $\vec{\lambda} \in G_{n+1}$ we define $(\Gamma_{j, \vec{\lambda}})_{j=1}^n$ to be independent random variables with distribution

$$\begin{aligned} \mathbb{P}(\Gamma_{j, \vec{\lambda}} = k) &= \mathbb{P}(\Gamma_j = k | \omega^j = \lambda_j, \omega^{j+1} = \lambda_{j+1}) \\ &= \mathbb{P}(\Gamma_j = k | \omega^j = \lambda_j, \Gamma_j \in K_{\lambda_j, \lambda_{j+1}}). \end{aligned}$$

Also, we define $U_{\vec{\lambda}}(n) = \sum_{j=1}^n \Gamma_{j, \vec{\lambda}}$.

By construction of the joint process (ω^j, Γ_j) , it follows that $U(n)$ conditioned on $(\omega^1, \dots, \omega^{n+1}) = \vec{\lambda}$ is distributed as $U_{\vec{\lambda}}(n)$. Thus, denoting $\vec{\omega} = (\omega^1, \dots, \omega^{n+1})$, we have

$$\begin{aligned} \mathbb{E}[(U(n) - n)^2] &= \sum_{\vec{\lambda} \in G_{n+1}} \mathbb{P}(\vec{\omega} = \vec{\lambda}) \cdot \mathbb{E}[(U(n) - n)^2 | \vec{\omega} = \vec{\lambda}] \\ &= \sum_{\vec{\lambda} \in G_{n+1}} \mathbb{P}(\vec{\omega} = \vec{\lambda}) \cdot \mathbb{E}[(U_{\vec{\lambda}}(n) - n)^2] \\ &\geq \sum_{\vec{\lambda} \in G_{n+1}} \mathbb{P}(\vec{\omega} = \vec{\lambda}) \cdot \text{Var}(U_{\vec{\lambda}}(n)) \\ &= \sum_{\vec{\lambda} \in G_{n+1}} \mathbb{P}(\vec{\omega} = \vec{\lambda}) \sum_{j=1}^n \text{Var}(\Gamma_{j, \vec{\lambda}}). \end{aligned} \tag{D.1}$$

The lemma follows easily from this since the pair $((p, 1), (p, 1))$ is a recurrent state for the Markov chain over configuration pairs $(\omega^j, \omega^{j+1})_{j \in \mathbb{N}}$ and the distribution of Γ_j conditioned on $\omega^j = \omega^{j+1} = (p, 1)$ is

non-degenerate. Indeed, denoting the variance in the distribution of Γ_j conditioned on $\omega^j = \omega^{j+1} = (p, 1)$ as $V_{(p,1),(p,1)}$ and the stationary probability of the pair $((p, 1), (p, 1))$ as $\psi_{(p,1),(p,1)}$, (D.1) implies

$$\liminf_{n \rightarrow \infty} \nu(n) = \liminf_{n \rightarrow \infty} \frac{\mathbb{E}[(U(n) - n)^2]}{n} \geq V_{(p,1),(p,1)} \cdot \psi_{(p,1),(p,1)} > 0.$$

□

We now proceed to the proof of Lemma 10. This is based on the following basic facts concerning large deviations of i.i.d. random variables and finite-state Markov chains:

Fact 1: If ξ is a random variable with exponential tails and ξ_1, ξ_2, \dots are i.i.d. random variables distributed as ξ , then there exist constants $b_1, b_2 > 0$ such that the empirical means $\bar{\xi}_n \equiv \frac{1}{n} \sum_{i=1}^n \xi_i$ satisfy:

$$\mathbb{P}(|\bar{\xi}_n - \mathbb{E}(\xi)| > \epsilon) \leq b_1 \exp(-b_2 \epsilon^2 n), \text{ for all } 0 < \epsilon \leq 1 \text{ and } n \in \mathbb{N}; \quad (\text{D.2})$$

$$\mathbb{P}(|\bar{\xi}_n - \mathbb{E}(\xi)| > \epsilon) \leq b_1 \exp(-b_2 \epsilon n), \text{ for all } \epsilon > 1 \text{ and } n \in \mathbb{N}. \quad (\text{D.3})$$

Fact 2: If $(\xi_n)_{n \in \mathbb{N}}$ is an irreducible Markov chain on a finite state space S with stationary distribution ϕ , then there exist constants $b_1, b_2 > 0$ such that the empirical state frequencies $\phi_n(s) \equiv \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\xi_i = s\}$ satisfy

$$\mathbb{P}_{s'}(|\phi_n(s) - \phi(s)| > \epsilon) \leq b_1 \exp(-b_2 \epsilon^2 n), \text{ for all } s, s' \in S, \epsilon > 0, \text{ and } n \in \mathbb{N}.$$

Here $\mathbb{P}_{s'}(\cdot) \equiv \mathbb{P}(\cdot | \xi_1 = s')$ is the probability measure for the Markov chain (ξ_n) started from state s' .

Fact 1 can be proved using the standard Chernoff-Hoeffding method for establishing large deviation bounds of independent random variables. Fact 2 follows from Fact 1, since for a finite-state, irreducible Markov chain the return times to a given state are i.i.d. with exponential tails.

Proof of Lemma 10. Throughout the proof we assume $\omega(x) = \lambda_0$, $x \geq 0$, for some $\lambda_0 \in \Lambda_0 = \{(p, 1), \dots, (p, L-1), (q, 0)\}$. The result for general $\lambda \in \Lambda$ follows directly from this since, for any initial state $\lambda \in \Lambda$, the Markov chain $(\omega^j)_{j \in \mathbb{N}}$ collapses to the recurrent state set Λ_0 with probability 1 after a single transition and the random variable Γ_1 has an exponential tail.

The bounds for small ϵ and large ϵ are established separately. Specifically, we will show that there exist constants $c_1, c_2, \epsilon_0 > 0$ and other constants $c'_1, c'_2, \epsilon'_0 > 0$ such that the empirical means $\bar{\Gamma}_n \equiv \frac{1}{n} \sum_{j=1}^n \Gamma_j$ satisfy:

$$\mathbb{P}(|\bar{\Gamma}_n - 1| > \epsilon) \leq c_1 \exp(-c_2 \epsilon^2 n), \text{ for all } 0 < \epsilon \leq \epsilon_0 \text{ and } n \in \mathbb{N}, \quad (\text{D.4})$$

$$\mathbb{P}(|\bar{\Gamma}_n - 1| > \epsilon) \leq c'_1 \exp(-c'_2 \epsilon n), \text{ for all } \epsilon \geq \epsilon'_0 \text{ and } n \in \mathbb{N}. \quad (\text{D.5})$$

Together (D.4) and (D.5) show that (4.2) and (4.3) hold, with $\mu = 1$ and $N = 1$, for some constants $C, c > 0$ depending on $c_1, c_2, c'_1, c'_2, \epsilon_0, \epsilon'_0$.

For the proofs in both cases below we use the following notation for states $\lambda \in \Lambda_0$.

- $\psi(\lambda) \equiv \psi_\lambda$ is the stationary probability of state λ , as defined in Section 4.2, and $\psi_n(\lambda) \equiv \frac{1}{n} \sum_{j=1}^n \mathbb{1}\{\omega^j = \lambda\}$ is the empirical frequency of state λ .
- $\Gamma_j(\lambda) \equiv \Gamma_{\tau_j(\lambda)}$, where $\tau_j(\lambda)$ is the j -th visit time to state λ for the Markov chain $(\omega^i)_{i \in \mathbb{N}}$: $\tau_{j+1}(\lambda) = \inf\{i > \tau_j(\lambda) : \omega^i = \lambda\}$ with $\tau_0(\lambda) \equiv 0$.
- $E(\lambda) \equiv \mathbb{E}(\Gamma_j(\lambda)) = \mathbb{E}(\Gamma_j | \omega^j = \lambda)$.

Proof of (D.4): For each $\lambda \in \Lambda_0$, $(\Gamma_j(\lambda))_{j \in \mathbb{N}}$ is a sequence of i.i.d. random variables with mean $E(\lambda)$ and exponential tails. Thus, by Fact 1, there exist constants $b_1, b_2 > 0$ such that for each $\lambda \in \Lambda_0$,

$$\mathbb{P}(|\bar{\Gamma}_n(\lambda) - E(\lambda)| > \epsilon) \leq b_1 \exp(-b_2 \epsilon^2 n), \text{ for all } 0 < \epsilon \leq 1, n \in \mathbb{N}. \quad (\text{D.6})$$

Also, by Fact 2, there exists constants $b_3, b_4 > 0$ such that for each $\lambda \in \Lambda_0$,

$$\mathbb{P}(|\psi_n(\lambda) - \psi(\lambda)| > \epsilon) \leq b_3 \exp(-b_4 \epsilon^2 n), \text{ for all } \epsilon > 0, n \in \mathbb{N}. \quad (\text{D.7})$$

Finally, using nonnegativity of the sequence $(\Gamma_j(\lambda))_{j \in \mathbb{N}}$ one may show that, for any $0 < \epsilon \leq 1/3$ and $n \in \mathbb{N}$, the following holds:

$$\begin{aligned} & \text{If } |\bar{\Gamma}_{j_{\min}}(\lambda) - E(\lambda)| \leq \epsilon \text{ and } |\bar{\Gamma}_{j_{\max}}(\lambda) - E(\lambda)| \leq \epsilon, \\ & \text{then } |\bar{\Gamma}_j(\lambda) - E(\lambda)| \leq \epsilon b_5, \text{ for all } n\psi(\lambda)(1 - \epsilon) \leq j \leq n\psi(\lambda)(1 + \epsilon), \end{aligned} \quad (\text{D.8})$$

where

$$\begin{aligned} j_{\min} &= j_{\min}(n, \lambda, \epsilon) \equiv \lceil n\psi(\lambda)(1 - \epsilon) \rceil, \\ j_{\max} &= j_{\max}(n, \lambda, \epsilon) \equiv \max\{j_{\min}, \lfloor n\psi(\lambda)(1 + \epsilon) \rfloor\}, \\ b_5 &\equiv \max_{\lambda \in \Lambda_0} \{3E(\lambda) + 2\}. \end{aligned}$$

Now, define $G_{n,\epsilon}$ to be the ‘‘good event’’ that for each $\lambda \in \Lambda_0$ the following two conditions are satisfied:

1. $|\psi_n(\lambda) - \psi(\lambda)| \leq \epsilon\psi(\lambda)$.
2. $|\bar{\Gamma}_j(\lambda) - E(\lambda)| \leq \epsilon b_5$, for all $n\psi(\lambda)(1 - \epsilon) \leq j \leq n\psi(\lambda)(1 + \epsilon)$.

By (D.6) and (D.8) together with the union bound, we have

$$\begin{aligned} & \mathbb{P}(|\bar{\Gamma}_j(\lambda) - E(\lambda)| > \epsilon b_5, \text{ for some } n\psi(\lambda)(1 - \epsilon) \leq j \leq n\psi(\lambda)(1 + \epsilon)) \\ & \leq 2b_1 \exp[-b_2 \epsilon^2 (n\psi(\lambda)(1 - \epsilon))] \leq 2b_1 \exp[-(2/3)b_2 \psi(\lambda) \epsilon^2 n] \end{aligned}$$

for each $\lambda \in \Lambda_0$, $n \in \mathbb{N}$, and $\epsilon \leq 1/3$. Thus, by (D.7) and the union bound,

$$\begin{aligned} \mathbb{P}(G_{n,\epsilon}^c) &\leq 2Lb_1 \exp(-(2/3)b_2 \psi_{\min} \epsilon^2 n) + Lb_3 \exp(-b_4 \psi_{\min}^2 \epsilon^2 n) \\ &\leq b_6 \exp(-b_7 \epsilon^2 n), \end{aligned} \quad (\text{D.9})$$

for all $n \in \mathbb{N}$ and $\epsilon \leq 1/3$, where

$$\psi_{\min} = \min_{\lambda \in \Lambda_0} \psi(\lambda), \quad b_6 = 2Lb_1 + Lb_3, \quad \text{and } b_7 = \min\{(2/3)b_2 \psi_{\min}, b_4 \psi_{\min}^2\}.$$

Since $\alpha = 1/2$, Lemma 9 implies $\sum_{\lambda \in \Lambda_0} \psi(\lambda)E(\lambda) = \langle \psi, E \rangle = 1$. Thus, on the event $G_{n,\epsilon}$, $\epsilon \leq 1/3$, we have

$$\begin{aligned} |\bar{\Gamma}_n - 1| &= \left| \sum_{\lambda \in \Lambda_0} \left(\sum_{j=1}^{n\psi_n(\lambda)} \frac{\Gamma_j(\lambda)}{n} - E(\lambda)\psi(\lambda) \right) \right| \\ &\leq \sum_{\lambda \in \Lambda_0} \left(\psi_n(\lambda) \left| \sum_{j=1}^{n\psi_n(\lambda)} \frac{\Gamma_j(\lambda)}{n\psi_n(\lambda)} - E(\lambda) \right| + E(\lambda) |\psi_n(\lambda) - \psi(\lambda)| \right) \\ &\leq \sum_{\lambda \in \Lambda_0} \left(\psi_n(\lambda) \cdot \epsilon b_5 + E(\lambda) \cdot \epsilon\psi(\lambda) \right) \\ &\leq b_8 \epsilon, \text{ where } b_8 \equiv b_5 + \max_{\lambda \in \Lambda_0} E(\lambda). \end{aligned} \quad (\text{D.10})$$

Together (D.9) and (D.10) show that, for any $0 < \epsilon \leq 1/3$ and $n \in \mathbb{N}$,

$$\mathbb{P}(|\bar{\Gamma}_n - 1| > b_8 \epsilon) \leq b_6 \exp(-b_7 \epsilon^2 n),$$

which is equivalent to (D.4), for $0 < \epsilon \leq \epsilon_0 \equiv b_8/3$, with $c_1 = b_6$ and $c_2 = b_7/b_8^2$.

Proof of (D.5): Let $r = \max\{p, q\}$ and let ξ be a geometric random variable with parameter $1 - r$ started from 0, i.e. $\mathbb{P}(\xi = k) = r^k(1 - r)$, $k \geq 0$. Then, S_0 and S_R are both stochastically dominated by ξ , so $\sum_{j=1}^n \Gamma_j$ is stochastically dominated by $\sum_{j=1}^n \xi_j$, for each $n \in \mathbb{N}$, where ξ_1, ξ_2, \dots are i.i.d. distributed as ξ . Further, by Fact 1, there exist constants $b_1, b_2 > 0$ such that for all $\epsilon \geq 1$ and $n \in \mathbb{N}$,

$$\mathbb{P}\left(\bar{\xi}_n - \frac{r}{1-r} > \epsilon\right) = \mathbb{P}(\bar{\xi}_n - \mathbb{E}(\xi) > \epsilon) \leq b_1 \exp(-b_2 \epsilon n).$$

Now, since $\alpha = 1/2$, either p or q must be at least $1/2$, so $r/(1-r) \geq 1$. Thus, for $\epsilon \geq \epsilon'_0 \equiv 2r/(1-r)$ we have

$$\mathbb{P}(\bar{\Gamma}_n - 1 > \epsilon) \leq \mathbb{P}(\bar{\xi}_n > \epsilon) \leq \mathbb{P}\left(\bar{\xi}_n - \frac{r}{1-r} > \frac{\epsilon}{2}\right) \leq b_1 \exp(-(b_2/2)\epsilon n).$$

On the other hand, for all $\epsilon \geq \epsilon'_0$ we also have

$$\mathbb{P}(\bar{\Gamma}_n - 1 < -\epsilon) = 0,$$

since $\bar{\Gamma}_n$ is nonnegative and $\epsilon'_0 > 1$. Hence, (D.5) holds with $c'_1 = b_1$ and $c'_2 = b_2/2$. \square

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