PERMUTATIONS AVOIDING A CERTAIN PATTERN OF LENGTH THREE UNDER MALLOWS DISTRIBUTIONS

ROSS G. PINSKY

Abstract. We consider permutations avoiding a certain pattern of length three under the family of Mallows distributions. In particular, we obtain rather precise bounds on the asymptotic probability as \( n \to \infty \) that a permutation \( \sigma \in S_n \) under the Mallows distribution with parameter \( q \in (0,1) \) avoids the pattern 312. The same result also holds for permutations avoiding 231. And by a duality, the same type of result also holds for the pattern 213 or 132 under the Mallows distribution with parameter \( q > 1 \).

1. Introduction and Statement of Results

We recall the definition of pattern avoidance for permutations. Let \( S_n \) denote the set of permutations of \( [n] := \{1, \ldots, n\} \). If \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n \) and \( \tau = \tau_1 \cdots \tau_m \in S_m \), where \( 2 \leq m \leq n \), then we say that \( \sigma \) contains \( \tau \) as a pattern if there exists a subsequence \( 1 \leq i_1 < i_2 < \cdots < i_m \leq n \) such that for all \( 1 \leq j, k \leq m \), the inequality \( \sigma_{i_j} < \sigma_{i_k} \) holds if and only if the inequality \( \tau_j < \tau_k \) holds. If \( \sigma \) does not contain \( \tau \), then we say that \( \sigma \) avoids \( \tau \). Denote by \( S_n(\tau) \) the set of permutation in \( S_n \) that avoid \( \tau \). (If \( \tau \in S_m \) and \( m > n \), then we say that every \( \sigma \in S_n \) avoids \( \tau \).)

It is well-known that \( |S_n(\tau)| = C_n \), for all six permutations \( \tau \in S_3 \), where \( C_n = \binom{2n}{n} \) is the \( n \)th Catalan number [1]. Consequently, under the uniformly random probability measure \( P_n \) on \( S_n \), one has

\[
P_n(S_n(\tau)) \sim \frac{(4e)^n}{\sqrt{2} \pi n^{n+2}}, \text{ for all } \tau \in S_3.
\]

2000 Mathematics Subject Classification. 60C05, 05A05.

Key words and phrases. Pattern-avoiding permutation, Mallows distribution, random permutation, pattern of length three.
More generally, the celebrated Stanley-Wilf conjecture, now proven [3], states that for any \( \tau \in S_m \), \( m \geq 2 \), there exists a constant \( C = C(\tau) \) such that \( C^n \) constitutes an upper bound on the growth rate of \( |S_n(\tau)| \) as \( n \to \infty \). Consequently, for any \( \tau \in S_m \), \( m \geq 2 \), there exists a constant \( C_1 = C_1(\tau) \) such that \( P_n(S_n(\tau)) \leq (\frac{C_1}{n})^n \).

In this paper, we investigate the probability of avoiding a certain pattern of length three under the family of Mallows distributions. For each \( q > 0 \), the Mallows distribution with parameter \( q \) is the probability measure \( P_n^q \) on \( S_n \) defined by

\[
P_n^q(\sigma) = \frac{q^{\text{inv}(\sigma)}}{Z_n(q)},
\]

where \( \text{inv}(\sigma) \) is the number of inversions in \( \sigma \), and \( Z_n(q) \) is the normalization constant, given by

\[
Z_n(q) = \prod_{k=1}^{n} [k]_q,
\]

where \([k]_q = \sum_{j=0}^{k-1} q^j = \frac{1-q^{k+1}}{1-q}\). Thus, for \( q \in (0,1) \), the distribution favors permutations with few inversions, while for \( q > 1 \), the distribution favors permutations with many inversions.

Recall that the reverse of a permutation \( \sigma = \sigma_1 \cdots \sigma_n \) is the permutation \( \sigma^{\text{rev}} := \sigma_n \cdots \sigma_1 \). The Mallows distributions satisfy the following duality between \( q > 1 \) and \( q < 1 \):

\[
P_n^q(\sigma) = P_n^{\frac{1}{q}}(\sigma^{\text{rev}}), \quad \text{for } q > 0, \sigma \in S_n \text{ and } n = 1, 2, \cdots .
\]

Consequently, it suffices to restrict our study to \( q \in (0,1) \), which we do from here on. For the above results and more on the Mallows distribution, see for example [4], [2].

For several of the proofs in this paper, we will use the following on-line construction of a random permutation in \( S_n \) distributed according to the Mallows distribution with parameter \( q \). Let \( \{X_j\}_{j=2}^{n} \) be independent random variables with \( X_j \) distributed as a geometric random variable with parameter \( q \) and truncated at \( j-1 \); that is, \( P(X_j = m) = \frac{(1-q)q^m}{1-q^j}, m = 0, \cdots , j-1 \). Consider a horizontal line on which to place the numbers in \([n]\). We begin by placing down the number 1. Then inductively, if we have already placed
down the numbers 1, 2, · · · , j − 1, the number j gets placed down in the position for which there are $X_j$ numbers to its right. Thus, for example, for $n = 4$, if $X_2 = 1$, $X_3 = 2$ and $X_4 = 0$, then we obtain the permutation 3214.

To see that this construction does indeed induce the Mallows distribution with parameter $q$, note that the number of inversions in the constructed permutation $\sigma$ is $\sum_{j=2}^{n} X_j$. Furthermore, noting that $\frac{1-q^i}{1-q} = \sum_{i=0}^{j-1} q^i = [j]_q$, we have $P(X_j = x_j, j = 2, \cdots, n) = \frac{1}{Z_n(q)} q^{\sum_{j=2}^{n} x_j} = \frac{q^{\text{inv}(\tau)}}{Z_n(q)}$.

We begin with a couple of rough results to set the stage. From the on-line construction, it follows that

\[ P_n(q)^n(\sigma = \text{id}) = P(X_j = 0, j = 1, \cdots, n) = \frac{(1-q)^n}{\prod_{j=1}^{n} (1-q^j)}. \]

Since the identity permutation avoids all patterns of length three except for the pattern 123, the above calculation yields the following result.

**Proposition 1.** For $q \in (0, 1)$, there exists a $C_q > 0$ such that

\[ P_n(q)^n(S_n(\tau)) \geq C_q(1-q)^n, \text{ for all } \tau \in S_3 \text{ except for } \tau = 123. \]

On the other hand, we will prove the following result regarding the pattern 123.

**Proposition 2.** For $q \in (0, 1)$,

\[ \limsup_{n \to \infty} \left( P_n(q)^n(S_n(123)) \right)^{\frac{1}{n}} \leq q^{\frac{1}{2}}. \]

Propositions 1 and 2 along with (1.1) show that under the Mallows distribution with $q \in (0, 1)$, $\tau$-avoiding permutations, for $\tau \in S_3 - \{123\}$, are overwhelmingly more likely than under the uniform distribution, whereas 123-avoiding permutations are overwhelmingly less likely than under the uniform distribution.

Since the permutation 123 has no inversions, the permutations 132 and 213 have one inversion, the permutations 312 and 231 have two inversions, and the permutation 321 has three inversions, it is natural to expect that $P_n(q)^n(S_n(123)) < P_n(q)^n(S_n(132)) = P_n(q)^n(S_n(213)) < P_n(q)^n(S_n(312)) = P_n(q)^n(S_n(231)) < P_n(q)^n(S_n(321))$, for all $n$. We will prove these relations with regard to four of the six patterns.
Proposition 3. Let $q \in (0,1)$. Then

$$P_n^q(S_n(312)) = P_n^q(S_n(231)) > P_n^q(S_n(213)) = P_n^q(S_n(132)), \text{ for } n \geq 3.$$  

Most of this paper concerns a rather precise analysis of the asymptotic behavior of $P_n^q(S_n(312))$ (or equivalently, of $P_n^q(S_n(231))$). We now describe the rest of the results in detail. We will prove the following proposition.

Proposition 4. Let $q \in (0,1)$. Then

$$\lim_{n \to \infty} \left( P_n^q(S_n(312)) \right)^{1/n} \text{ exists.}$$

Define

$$w_n = w_n(q) = \prod_{l=1}^{n} (1 - q^l), n = 0, 1, \cdots,$$

where we use the convention $w_0 = 1$.

Proposition 5. Let $q \in (0,1)$.

i. Define $d_n = d_n(q) = P_n^q(S_n(312))$ or $d_n = d_n(q) = P_n^q(S_n(231))$. Then

$$d_n = (1 - q) \sum_{k=1}^{n} q^{k-1} \frac{w_{k-1}w_{n-k}}{w_n} d_{k-1}d_{n-k}, n = 1, 2, \cdots.$$  

ii. Define $d_n = d_n(q) = P_n^q(S_n(213))$ or $d_n = d_n(q) = P_n^q(S_n(132))$. Then

$$d_n = (1 - q) \sum_{k=1}^{n} q^{(n-k+1)(k-1)} \frac{w_{k-1}w_{n-k}}{w_n} d_{k-1}d_{n-k}, n = 1, 2, \cdots.$$  

Proposition 3 is actually a direct corollary of Proposition 5.

Proof of Proposition 3. The proposition follows immediately by induction from (1.7) and (1.8) along with the fact that $P_n^q(S_n(\tau)) = 1$ for $n = 1, 2$ and all $\tau \in S_3$.  

We can use (1.7) to study $P_n^q(S_n(312))$ in depth. Let

$$\gamma_n = \gamma_n(q) = w_n(q)P_n^q(S_n(312)),$$

and define the generating function

$$G_q(t) = \sum_{n=0}^{\infty} \gamma_n(q)t^n.$$
Let \( r(q) \) denote the radius of convergence of \( G_q \). Since \( \lim_{n \to \infty} w_n(q) > 0 \), it follows that \( \limsup_{n \to \infty} \left( P_n^q(S_n(312)) \right)^{\frac{1}{n}} = \frac{1}{r(q)} \), and thus, in light of Proposition 4, that
\[
\lim_{n \to \infty} \left( P_n^q(S_n(312)) \right)^{\frac{1}{n}} = \frac{1}{r(q)},
\]
where \( r(q) \) is the radius of convergence of \( G_q \).

**Theorem 1.** The generating function \( G_q \) satisfies the functional equation
\[
G_q(t) = \frac{1}{1 - (1 - q)tG_q(qt)}, \quad 0 \leq t < r(q).
\]
Furthermore,
\[
\lim_{t \to r(q)} G_q(t) = \infty.
\]

**Remark.** Note, for example, that the Catalan sequence has generating function \( \frac{1}{1 - \sqrt{1 - 4t}} \) with radius of convergence \( \frac{1}{4} \), however as \( t \) approaches \( \frac{1}{4} \), the generating function remains bounded.

Using Theorem 1, we can prove the following result.

**Theorem 2.** Let
\[
F(x) = \begin{cases} 
\frac{1}{1-x}, & x \in [0, 1); \\
\infty, & x \in \mathbb{R} - [0, 1) 
\end{cases}.
\]
i. If for some \( c \in (0, 1) \), one has
\[
F(cF(cq \cdots F(cq^{N-1}(1 + cq^N)) \cdots)) = \infty, \quad \text{for some } N \in \mathbb{N},
\]
then
\[
\lim_{n \to \infty} \left( P_n^q(S_n(312)) \right)^{\frac{1}{n}} > \frac{1 - q}{c}.
\]

ii. If for some \( c \in (0, 1) \), one has
\[
F(cF(cq \cdots F(cq^N F(cq^{N+1}}) \cdots) < \infty, \quad \text{for some } N \in \mathbb{N},
\]
then
\[
\lim_{n \to \infty} \left( P_n^q(S_n(312)) \right)^{\frac{1}{n}} < \frac{1 - q}{c}.
\]

In fact, if for some fixed \( N \), (1.15) holds with \( c \) replaced by \( b \), with \( b < c \) arbitrarily close to \( c \), then (1.16) continues to hold with \( c \).
Remark. In (1.13), we understand the left hand side in the case $N = 1$ as $F(c(1 + cq))$.

The larger one chooses $N$ in (1.13) and (1.15), the better an estimate one obtains. This works nicely for fixed values of $q$, as we demonstrate below. However, if we are interested in bounds given as explicit functions of $q$, then we are limited in our choices of $N$. In (1.13), if we choose $N = 2$, we will get a quadratic inequality for $c$, which we can solve for in terms of $q$. Alternatively, since $F$ is increasing on $(0, 1)$, the condition $F(cF(cq \cdots cq^{N-2}F(cq^{N-1}) \cdots) = \infty$ also implies (1.14). Using this condition, we can choose $N = 4$ and obtain a quadratic inequality for $c$. This latter case turns out to give a larger lower bound. In (1.15), we obtain a quadratic inequality for $c$ if we choose $N = 2$. However, as the proof of Theorem 2 shows, the term $\frac{cq^{N+1}}{1-q}$ is obtained by making an approximation, and this approximation is only a good one when $\frac{cq^{N+1}}{1-q}$ is small. Because of this, it turns out that the upper bound using $N = 2$ is only reasonably accurate for $q \leq .6$. Here is our result.

**Theorem 3.** Let $q \in (0, 1)$. Then

\begin{equation}
(1.17)
LB(q) := \frac{2q^2(1-q)(1-q^3)}{1 - q^4 - \sqrt{(1-q^4)^2 - 4q^2(1-q)(1-q^3)}} < \lim_{n \to \infty} \left( P_n^q(S_n(312)) \right)^{\frac{1}{n}} < \frac{2q^2(q^2 + 1)(1-q)}{1 - \sqrt{1 - 4(1-q)q^2(q^2 + 1)}} := UB(q).
\end{equation}

Remark. The upper and lower bound functions, $UB(q)$ and $LB(q)$, virtually coincide for $q \in (0, .4]$ and differ by less than .01 for $q \in (0, .5]$. The upper and lower bound functions, $UB(q)$ and $LB(q)$, as well as the true value of $\lim_{n \to \infty} \left( P_n^q(S_n(312)) \right)^{\frac{1}{n}}$ (with error no more than ±.01) are plotted in Figure 1. See also the table below.
The rest of this paper is organized as follows. In section 2 we prove Proposition 4, in section 3 we prove Proposition 5, in sections 4-6 we prove Theorems 1-3 respectively and in section 7 we prove Proposition 2.

2. Proof of Proposition 4

Let \( d_n = d_n(q) = P_n^q(S_n(312)) \). We will show that there exists a constant \( c = c_q \in (0,1) \) such that

\[
(2.1) \quad d_{n+m} \geq cd_n d_m, \text{ for all } n, m \in \mathbb{N}.
\]

Thus \( D_n := -\log d_n \) satisfies \( D_{n+m} \leq D_n + D_m + |\log c| \). If the term \( |\log c| \) were absent, then the sequence \( \{D_n\}_{n=1}^{\infty} \) would be sub-additive, and thus
the existence of $\lim_{n \to \infty} \frac{D_n}{n^{\frac{1}{2}}}$, and consequently of $\lim_{n \to \infty} d_n^\frac{1}{2}$, would follow from the well-known result on sub-additive sequences. It is easy to check that the proof of this result continues to hold when the extra constant term $|\log c|$ is present. It remains to prove (2.1).

For $n, m \in \mathbb{N}$, let $A_{n,m} \subset S_{n+m}$ denote the event that $\sigma \in S_{n+m}$ satisfies $\{\sigma_1, \ldots, \sigma_n\} = \{1, \ldots, n\}$ (and thus also $\{\sigma_{n+1}, \ldots, \sigma_{n+m}\} = \{n+1, \ldots, n+m\}$). From the on-line construction of the Mallows distributions, appearing at the end of section 1, it follows that

$$P_n(A_{n,m}) = \prod_{l=1}^{m} P(X_{n+l} \leq l-1) = \prod_{l=1}^{m} \frac{1 - q^l}{1 - q^{n+l}}.$$

Since $\prod_{k=1}^{\infty} (1 - q^k) > 0$, it follows that $c := \inf_{n,m \in \mathbb{N}} P_n(A_{n,m}) > 0$.

Define the events $B_n, C_m \subset S_{n+m}$ by

$B_n = \{\sigma_1 \sigma_2 \cdots \sigma_n \text{ is 312-avoiding}\}; \quad C_m = \{\sigma_{n+1} \cdots \sigma_{n+m} \text{ is 312-avoiding}\}$.

Note that $S_{n+m}(312) \cap A_{n,m} = B_n \cap C_m \cap A_{n,m}$. Also, it is easy to see from the on-line construction that conditional on $A_{n,m}$ occurring, the events $B_n$ and $C_m$ are independent and have probabilities $d_n$ and $d_m$ respectively. This gives (2.1).}

3. PROOF OF PROPOSITION 5

Proof of part (i). Let $d_n = P_n(S_n(312))$. If $\sigma \in S_n(312)$ and $\sigma_k = 1$, then necessarily $\{\sigma_1, \ldots, \sigma_k - 1\} = [k] - \{1\}$ (and then of course also $\{\sigma_{k+1}, \ldots, \sigma_n\} = [n] - [k]$). Let $A_k \subset S_n$ be the event that $\sigma_1 = k$ and $\{\sigma_1, \ldots, \sigma_k - 1\} = [k] - \{1\}$. From the on-line construction,

$$P_n(A_k) = P(X_j \geq 1, \text{ for } j \in [k] - \{1\}; X_{k+l} \leq l-1, \text{ for } l \in [n-k]).$$

Using the fact that

$$P(X_j \geq 1) = \frac{q(1 - q^{j-1})}{1 - q^j}, \quad P(X_{k+l} \leq l-1) = \frac{1 - q^l}{1 - q^{k+l}},$$

we obtain

$$P_n(A_k) = q^{k-1} \frac{1 - q^k}{1 - q^k} \prod_{l=1}^{n-k} \frac{1 - q^l}{1 - q^{k+l}} = (1 - q) q^{k-1} \frac{w_{k-1} w_{n-k}}{w_n},$$
where $w_k$ is as in (1.6). Also from the on-line construction, it follows that

$$P_n^q(S_n(312)|A_k) = d_{k-1}d_{n-k}.$$ 

Thus, 

$$d_n = \sum_{k=1}^n P_n^q(S_n(312)|A_k)P_n^q(A_k) = (1 - q) \sum_{k=1}^n q^{k-1} \frac{w_{k-1}w_{n-k}}{w_n} d_{k-1}d_{n-k},$$ 

which is (1.7). We leave it to the reader to check that the same formula holds when one works with 231-avoiding permutations.

**Proof of part (ii).** Let $d_n = P_n^q(S_n(213))$. If $\sigma \in S_n(213)$ and $\sigma_k = 1$, then necessarily $\{\sigma_1, \ldots, \sigma_{k-1}\} = \{n-k+2, \ldots, n\}$ (and then of course also $\{\sigma_{k+1}, \ldots, \sigma_n\} = \{2, \ldots, n-k+1\}$). Let $B_k \subset S_n$ be the event that $\sigma_1 = k$ and $\{\sigma_1, \ldots, \sigma_{k-1}\} = \{n-k+2, \ldots, n\}$. From the on-line construction,

$$P_n^q(B_k) = P(X_j \leq j-2, \text{for } j \in [n-k+1]-\{1\}; X_l \geq n-k+1, \text{for } l \in [n]-[n-k+1]).$$

Using the fact that 

$$P(X_j \leq j-2) = \frac{1 - q^{j-1}}{1 - q^j}; \quad P(X_l \geq n-k+1) = \frac{q^{n-k+1} - q^l}{1 - q^l},$$

we obtain 

$$P_n^q(B_k) = \frac{1 - q}{1 - q^{n-k+1}} q^{(n-k+1)(k-1)} \prod_{l=n-k+2}^n \frac{1 - q^{l-n+k-1}}{1 - q^l}.$$ 

Also from the on-line construction, it follows that 

$$P_n^q(S_n(213)|B_k) = d_{k-1}d_{n-k}.$$ 

Continuing as in the proof of part (i), we obtain (1.8). We leave it to the reader to check that the same formula holds when one works with 132-avoiding permutations.

4. **Proof of Theorem 1**

Using (1.7) and recalling the definition of $\gamma_n$ in (1.9), we have 

$$\gamma_n = (1 - q) \sum_{k=1}^n q^{k-1}\gamma_{k-1}\gamma_{n-k}. $$
Let \( a_n = a_n(q) = q^n \gamma_n \), and define
\[
A_q(t) = \sum_{n=0}^{\infty} a_n(q) t^{n+1}.
\]

Using (4.1), we have
\[
G_q(t) A_q(t) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_{k-1} \gamma_{n-k} t^n = \sum_{n=1}^{\infty} q^{k-1} \gamma_{n-k} t^n = \frac{1}{1-q} \sum_{n=1}^{\infty} \gamma_n t^n.
\]

Since \( A_q(t) = tG_q(t) \), we conclude from (4.2) that
\[
G_q(t) = \frac{1}{1 - (1-q)tG_q(qt)}.
\]

We now prove that \( \lim_{t \to r(q)} G_q(t) = \infty \). Assume to the contrary. Since \( G_q \) is increasing, \( \lim_{t \to r(q)} G_q(t) \) exists and from (4.3), \( (1-q)rG_q(qr(q)) < 1 \). Consequently, for \( \delta > 0 \) sufficiently small, the function
\[
H(z) = \begin{cases} 
G(z), & |z| < r(q) - \delta; \\
\frac{1}{1-(1-q)zG_q(az)}, & r(q) - 2\delta < |z| < r(q) + 2\delta
\end{cases}
\]
extends analytically the analytic function \( G(z) \) in \( |z| < r(q) \) to the disk \( |z| < r(q) + 2\delta \). Consequently, the radius of convergence of \( G(z) \) must be at least \( r(q) + 2\delta \), which is a contradiction.

5. Proof of Theorem 2

Let \( F \) be as in the statement of the theorem. Then (1.11) can be written as \( G_q(t) = F((1-q)tG_q(qt)) \). Iterating this and using (1.12), we conclude that for any \( N \in \mathbb{N} \),
\[
F((1-q)tF(q(1-q)t \cdots F(q^{N-1}(1-q)tG_q(q^N t)) \cdots ) < \infty, \text{ if } t \in [0, r(q)).
\]

Now \( G_q(0) = 1, G'_q(0) = \gamma_1 = 1 - q \) and \( G_q \) is convex. Therefore, \( G_q(q^N t) > 1 + (1-q)q^N t \). Thus, since \( F \) is increasing on \( [0,1) \) and is equal to \( \infty \) elsewhere, it follows from (5.1) that if
\[
F((1-q)tF(q(1-q)t \cdots F(q^{N-1}(1-q)t(1+(1-q)q^N t)) \cdots ) = \infty,
\]
then \( r(q) < t \). The strict inequality \( r(q) < t \) follows from the strict inequality \( G(q^N t) > 1 + (1 - q) q^N t \). Substituting \( c = (1 - q)t \) above, and using (1.10), we obtain (1.13) and (1.14), proving part (i).

We now prove part (ii). Since \( \gamma_0 = 1 \) and \( \gamma_n < 1 \), for all \( n \geq 1 \), we have \( G_q(t) < \frac{1}{1-t} = F(t) \), for \( t \in (0, 1) \). Thus, since \( F \) is increasing on \([0, 1)\) and is equal to \( \infty \) elsewhere, it follows from (1.11) and (1.12) that (5.2)

\[
(1-q)t \cdot F(q(1-q)tF(q^N t) \cdots) < \infty, \text{ then } r(q) > t.
\]

Substituting \( c = (1 - q)t \) above, and using (1.10), we obtain (1.15) and (1.16). The final statement in part (ii) follows from comparing (5.1) and (5.2) and noting the strict inequality \( G_q(q^N t) < F(q^N t) \), for \( t > 0 \). □

6. Proof of Theorem 3

We begin with the lower bound. As noted in the paragraph preceding Theorem 3, part (i) of Theorem 2 continues to hold with the condition (1.13) replaced by the condition \( F(cF(qF(cqF(cq^2F(cq^3)))) \cdots) = \infty \). We use this condition with \( N = 4 \).

Consider the requirement \( F(cF(qF(cq^2F(cq^3)))) = \infty \), where we consider \( c \in (0, 1) \). In order for this to occur, one needs \( cF(cqF(cq^2F(cq^3)))) \geq 1 \). Since \( F(1 - c) = \frac{1}{1-c} \), in order for the above inequality to occur, one needs \( cqF(cq^2F(cq^3)) \geq 1 - c \). Since the range of \( F \) is \([1, \infty)\), this second inequality holds automatically if \( \frac{1-c}{cq} \leq 1 \), or equivalently, if

\[
c \geq \frac{1}{1+q}.
\]

Otherwise, since \( F(1 - \frac{cq}{1-c}) = \frac{1-c}{cq} \), in order for the second inequality to occur, one needs \( c q^2 F(cq^3)) \geq 1 - \frac{cq}{1-c} = \frac{1-c-cq}{1-c} \). This third inequality holds automatically if \( \frac{1-c-cq}{(1-c)cq} \leq 1 \), or equivalently, if

\[
c \geq \frac{1+q+q^2 - \sqrt{(1+q+q^2)^2 - 4q^2}}{2q^2}.
\]

(We don’t need to place an upper bound on \( c \) because we are restricting from the start to \( c \in (0, 1) \), and one can see that the inequality holds for
c = 1−.) Otherwise, since \( F\left(\frac{1-c-cq^2+q^3}{1-c-cq}\right) = \frac{1-c-cq}{(1-c)q^2} \), in order for the third inequality to hold, one needs \( cq^3 \geq \frac{1-c-cq^2+q^2}{1-c-cq} \), or equivalently,

\[
(q^2 + q^3 + q^4)c^2 - (1 + q + q^2 + q^3)c + 1 \leq 0.
\]

This gives

\[
(6.3) \quad c \geq \frac{(1-q^4) - \sqrt{(1-q^4)^2 - 4q^2(1-q)(1-q^3)}}{2q^2(1-q)}.
\]

(As before, we don’t need to place an upper bound on \( c \) because we are restricting to \( c \in (0, 1) \), and one can see that the inequality holds for \( c = 1− \).)

We conclude that (1.14) holds for \( c \) satisfying any one of (6.1)-(6.3). It turns out that the right hand side of (6.3) yields the smallest value for \( c \), so we choose \( c \) to be equal to the right hand side of (6.3). After doing a little algebra, one finds that \( \frac{1-c}{c} \) is given by the left hand side of (1.17).

We now turn to the upper bound. We consider (1.15) with \( N = 2 \); that is, we consider the inequality \( F(cF(cqF(cq^3F(\frac{q^3}{1-q})))) < \infty \), where we consider \( c \in (0, 1) \). In order for this to occur, one needs \( cF(cqF(cq^3F(\frac{q^3}{1-q})))) < 1 \). Since \( F(1-c) = \frac{1}{c} \), in order for this second inequality to occur, one needs \( cqF(cq^3F(\frac{q^3}{1-q})) < 1 - c \). If \( \frac{1-c}{cq} \leq 1 \), or equivalently, if (6.1) occurs, then this second inequality cannot occur. Otherwise, since \( F(1-c) = \frac{1-c}{c} \), in order for this second inequality to occur, one needs \( cq^2F(cq^3F(\frac{q^3}{1-q})) < 1 - \frac{c}{cq} = \frac{1-c-cq}{1-c-cq} \). If \( \frac{1-c-cq}{(1-c)q^2} \leq 1 \), or equivalently, if (6.2) occurs, then this third inequality cannot occur. Otherwise, since \( F(1-c-cq^2+q^3) = \frac{1-c-cq}{(1-c)q} \), in order for this third inequality to hold, one needs \( cq^2 \leq \frac{1-c-cq^2+q^3}{1-c-cq} \), or equivalently,

\[
(q^2 + q^3 + q^4)c^2 - (1 + q + q^2 + q^3)c + 1 > 0.
\]

Thus, we need

\[
(6.4) \quad c < \frac{1 - \sqrt{1 - 4q^2(1-q)(1+q^2)}}{2q^2(1+q^2)}.
\]

(We don’t need to place a lower bound because we can see that the above inequality holds for \( c = 0 \).)

We conclude that (1.15) holds if \( c \) is smaller than each of the three right hand sides, (6.1), (6.2) and (6.4). One can show that the right hand side of
(6.4) is the smallest of the three. Thus, we choose \( c \) to be equal to the right hand side of (6.4). Thus, \( \frac{1−q}{c} \) is given by the right hand side of (1.17). \( \square \)

7. Proof of Proposition 2

In the on-line construction, in order to obtain a 123-avoiding permutation, a necessary condition is that the position counting from right to left that the number \( j \) is placed in (namely \( X_j + 1 \)), must be such that all of the numbers appearing to its left at this stage form a decreasing sequence from left to right. Consider now the on-line construction up to the placement of the number \( \lfloor \frac{n}{2} \rfloor \). Looking at this sequence of \( \lfloor \frac{n}{2} \rfloor \) numbers as a sequence from left to right, let \( Y_n \) denote the largest \( k \) such that the first \( k \) of these numbers form a decreasing subsequence. It is easy to see that a necessary condition for \( \{ Y_n \geq k \} \) to occur is that

\[
|\{ j \in \{ 1, \cdots, \lfloor \frac{n}{2} \} : X_j = j - 1 \}| \geq k.
\]

Since \( P(X_j = j - 1) = \frac{2^{j-1}(1-q)}{1-q^j} \) is decreasing in \( j \), there exists a \( c_q > 0 \) such that

\[
P(Y_n \geq k) \leq \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq \lfloor \frac{n}{2} \rfloor} P(X_{i_j} = i_j - 1, \text{ for } j = 1, \cdots, k) \leq
\]

\[
\left( \frac{\lfloor \frac{n}{2} \rfloor}{k} \right) \prod_{j=1}^{k} P(X_j = j - 1) = \left( \frac{\lfloor \frac{n}{2} \rfloor}{k} \right) \frac{(1-q)^{k-1}}{\prod_{j=2}^{k} (1-q^j)} q^{(k-1)k} \leq
\]

\[
c_q2^n q^2(k-1)^k, \text{ for } k = 1, \cdots, \lfloor \frac{n}{2} \rfloor.
\]

Now it is easy to see that conditioned on \( \{ Y_n = k \} \), in order for \( \sigma \) to be 123-avoiding it is necessary that \( X_j \geq \lfloor \frac{n}{2} \rfloor - k \), for all \( j = \lfloor \frac{n}{2} \rfloor + 1, \cdots, n \). We have for some \( C_q > 0 \),

\[
P(X_j \geq \lfloor \frac{n}{2} \rfloor - k, j = \lfloor \frac{n}{2} \rfloor + 1, \cdots, n) =
\]

\[
(1-q)^{n-\lfloor \frac{n}{2} \rfloor} \frac{\prod_{j=\lfloor \frac{n}{2} \rfloor+1}^{n} (q^{\lfloor \frac{n}{2} \rfloor - k} - q^{j-1})}{\prod_{j=\lfloor \frac{n}{2} \rfloor+1}^{n} (1-q^j)} \leq C_q q^{(\lfloor \frac{n}{2} \rfloor - k)(n - \lfloor \frac{n}{2} \rfloor)}.
\]

The events \( \{ Y_n = k \} \) and \( \{ X_j \geq \lfloor \frac{n}{2} \rfloor - k, j = \lfloor \frac{n}{2} \rfloor + 1, \cdots, n \} \) are clearly independent for \( k = 1, \cdots, \lfloor \frac{n}{2} \rfloor - 1 \). They are also independent for \( k = \lfloor \frac{n}{2} \rfloor \).
since in this case the latter event has probability one. Thus, using (7.1) and (7.2) we have

\begin{align}
P_q^n(S_n(123)) & \leq \sum_{k=1}^{\lfloor n^2 \rfloor} P(Y_n = k) P(X_j \geq \lfloor \frac{n}{2} \rfloor - k, j = \lfloor \frac{n}{2} \rfloor + 1, \ldots, n) \\
& \leq c_q C_q^2 2^n \sum_{k=1}^{\lfloor n^2 \rfloor} q^{\lfloor \frac{n}{2} \rfloor - k + (\lfloor \frac{n}{2} \rfloor - k)(n - \lfloor \frac{n}{2} \rfloor)} \\
& \leq c_q C_q^2 2^n \frac{n}{2} q^{\frac{1}{2} (\lfloor \frac{n}{2} \rfloor - 1)(\lfloor \frac{n}{2} \rfloor)}.
\end{align}

Now (1.3) follows from (7.3).

\begin{flushright}
\square
\end{flushright}

**REFERENCES**


**DEPARTMENT OF MATHEMATICS, TECHNION—ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA, 32000, ISRAEL**

_E-mail address:_ pinsky@math.technion.ac.il

_URL:_ http://www.math.technion.ac.il/~pinsky/