A PROBABILISTIC APPROACH TO THE LIOUVILLE PROPERTY FOR SCHRÖDINGER OPERATORS WITH AN APPLICATION TO INFINITE CONFIGURATIONS OF BALLS

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Abstract. Consider the equation

$$\frac{1}{2}\Delta u - Vu = 0 \text{ in } R^d, \tag{*}$$

for $d \geq 3$, where $V \gneq 0$. One says that the Liouville property holds if the only bounded solution to (*) is 0. Under a certain growth condition on V, we give a necessary and sufficient analytic condition for the Liouville property to hold. We then apply this to the case that V is the indicator function of an infinite collection of small balls.

1. Introduction

Consider the equation

$$\frac{1}{2}\Delta u - Vu = 0 \quad \text{in} \quad R^d, \tag{1.1}$$

for $d \geq 1$, where $V \gneq 0$. We will assume either that $V \in C^\alpha(R^d)$, for some $\alpha \in (0, 1]$, in which case solutions to (1.1) will be classical, or that V is bounded, in which case solutions to (1.1) will be weak solutions [7].

One says that the Liouville property holds if the only bounded solution of (1.1) is $u \equiv 0$. It is known [11], [2] that the following probabilistic condition is equivalent to the Liouville property:

$$\int_0^\infty V(B(t))dt = \infty \quad \text{a.s.}, \tag{1.2}$$

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where $B(t)$ is a d-dimensional Brownian motion. Since Brownian motion is recurrent when $d \le 2$, the Liouville property always holds if $d \le 2$. From now on we consider $d \ge 3$. Recall that the Green's function for $\frac{1}{2}\Delta$ in R^d, $d \ge 3$, is given by

$$G(x,y) = \frac{2}{(d-2)\omega_d}|y-x|^{2-d}, \tag{1.3}$$

where ω_d is the surface measure of the unit sphere in R^d. For Brownian motion starting from $x \in R^d$, denote probabilities by P_x and the corresponding expectation by E_x. For any nonnegative function ϕ, one has [10]

$$E_x \int_0^\infty \phi(B(s))ds = \int_{R^d} G(x,y)\phi(y)dy. \tag{1.4}$$

From (1.2)-(1.4) it follows that

$$\int_{R^d} \frac{V(x)}{|x|^{d-2}} dx = \infty \tag{1.5}$$

is a necessary condition for the Liouville property to hold. (A purely analytic proof of this can be found in [9, Proposition 3.4].)

The question of whether or not the Liouville property holds is of inherent interest and has been investigated by numerous authors. It also has useful applications. For example, recently, it was shown in [6] that if the Liouville property for (1.1) does not hold, and if u solves (1.1) and satisfies $0 < c_1 \le u \le c_2$, then for any $W \ge 0$ and any $j \ge 1$, the j-th negative eigenvalue of the operator $-\frac{1}{2}\Delta + V - W$ (if it exists) is bounded above and below respectively by the j-th negative eigenvalues of the operators $-\frac{1}{2}\Delta - \beta^{-1}W$ and $-\frac{1}{2}\Delta - \beta W$, where $\beta = (\frac{c_2}{c_1})^2$.

It is of interest to know for what classes of functions V the condition (1.5) is not only necessary, but also sufficient for the Liouville property to hold. In particular, this is true if $V \gneq 0$ is radially symmetric ([1], [2], [4], [11]). In [11] it was pointed out that results in [3] show that (1.5) is necessary and sufficient for the Liouville property to hold for the class of functions $V \gneq 0$ decaying at least quadratically: $V(x) \le \frac{c}{(1+|x|)^2}$, for some $c > 0$. Furthermore, it was shown that this result is sharp. Indeed, given any function ψ satisfying $\lim_{r\to\infty}\psi(r)=\infty$, one can find a function $V \gneq 0$ satisfying $V(x) \le \frac{\psi(|x|)}{(1+|x|)^2}$ and (1.5), but such that the Liouville property does not hold.

In this paper we present another class of potentials for which (1.5) is both necessary and sufficient for the Liouville property to hold. We then use this result to determine when the Liouville property holds in the case that V is the indicator function of a countable union of ϵ-separated balls.

Here is the class of potentials we will consider.

Assumption 1. There exists a $t > 1$ such that for $R_j = t^j$, one has

$$\sup_{j\ge 1}\ \sup_{R_j<|x|<R_{j+1}} \int_{R_j\le|y|<R_{j+1}} \frac{V(y)}{|x-y|^{d-2}}dy < \infty. \tag{1.6}$$

Theorem 1. *Let $V \gneq 0$ satisfy Assumption 1. Then the Liouville property holds for (1.1) if and only if (1.5) holds. Furthermore, Assumption 1 is sharp in the following sense: Let $\psi > 0$ be an increasing, unbounded function on $[0,\infty)$. Then there exists a potential V satisfying*

$$\sup_{R_j<|x|<R_{j+1}} \int_{R_j\le|y|<R_{j+1}} \frac{V(y)}{|x-y|^{d-2}}dy < \psi(j),\ \ j\ge 1,$$

and (1.5), where R_j is as in Assumption 1, but the Liouville property does not hold.

Remark 1. The “if and only if” part of the theorem (with $t = 2$) was proven in [8] using very different potential theoretic methods. Our proof uses probability and partial differential equations.

Remark 2. The case of quadratically decaying V, noted before the statement of the theorem, is a particular case of Theorem 1. Indeed, we have

$$\sup_{R_j<|x|<R_{j+1}}\int_{R_j\le|y|<R_{j+1}}\frac{V(y)}{|x-y|^{d-2}}dy$$
$$\le\frac{c}{(1+|R_j|)^2}\sup_{R_j<|x|<R_{j+1}}\int_{|y-x|<2R_{j+1}}\frac{1}{|x-y|^{d-2}}dy$$
$$=\frac{c}{(1+|R_j|)^2}(c_1R_{j+1}^2)\le c_2.$$

Let $B(x,r)$ denote the open ball of radius r centered at $x\in R^d$. Consider a collection $\{B(x_n,r_n)\}_{n=1}^\infty$ of balls. For $\epsilon>0$, we call the collection ϵ-separated if $|x_n-x_m|\ge 2\epsilon$ for all $n\neq m$.

Theorem 2. *Let $\{B(x_n,r_n)\}_{n=1}^\infty$ be a collection of ϵ-separated balls, for some $\epsilon>0$. Let $V=1_{\cup_{n=1}^\infty B(x_n,r_n)}$. Assume that*

$$\sup_{n\ge1} r_n^d|x_n|^2<\infty. \tag{1.7}$$

Then the Liouville property holds for (1.1) *if and only if* (1.5) *holds, or equivalently, if and only if*

$$\sum_{n=1}^\infty\frac{r_n^d}{|x_n|^{d-2}}=\infty. \tag{1.8}$$

Furthermore, condition (1.7) *is sharp in the following sense: Let $\psi(x)>0$ be an increasing, unbounded function on* $[0,\infty)$. *There exists an ϵ-separated collection of balls $\{B(x_n,r_n)\}_{n=1}^\infty$ such that $r_n^d|x_n|^2\le\psi(|x_n|)$, for $n\ge1$, and $\sum_{n=1}^\infty\frac{r_n^d}{|x_n|^{d-2}}=\infty$, yet the Liouville property does not hold for $V=1_{\cup_{n=1}^\infty B(x_n,r_n)}$.*

Remark. Recalling (1.2), the probabilistic import of Theorem 2 is that under condition (1.7), one has $\int_0^\infty 1_{\cup_{n=1}^\infty B(x_n,r_n)}(B(t))dt=\infty$ if and only if (1.8) holds; that is, the total occupation time of Brownian motion in $\cup_{n=1}^\infty B(x_n,r_n)$ is infinite if and only if (1.8) holds. A related problem was considered in [5]; namely to determine when an ϵ-separated collection of balls $\cup_{n=1}^\infty B(x_n,r_n)$ satisfies the condition $P_x(B(t)\notin\cup_{n=1}^\infty B(x_n,r_n),$ for all $t\ge$

$0) > 0$, for $x \notin \cup_{n=1}^{\infty} \bar{B}(x_n, r_n)$. When this occurs, the collection of balls is called *avoidable.* In [5] it was shown that if the balls are ϵ-separated and if $\sup_{n\geq 1} r_n^{d-2}|x_n|^2 < \infty$ (compare to (1.7)), then a necessary and sufficient condition for avoidability is that $\sum_{n=1}^{\infty}(\frac{r_n}{|x_n|})^{d-2} < \infty$ (compare to (1.8)). The authors also show that the condition $\sup_{n\geq 1} r_n^{d-2}|x_n|^2 < \infty$ is sharp with regard to the convergence of the above sum being the necessary and sufficient condition. We point out that it is possible to have a situation where the balls are unavoidable, but the total time spent in them is almost surely finite. For example, for $\epsilon > 0$ sufficiently small, and for each $k = 1, 2, \cdots$, one can construct k^{d-1} ϵ-separated balls centered on the sphere of radius k and with common radius r satisfying $r^{d-2}k^2 = 1$. Then one has $\sum_n(\frac{r_n}{x_n})^{d-2} = \sum_n \frac{1}{|x_n|^d} = \sum_k \frac{1}{k^d} \times k^{d-1} = \sum_k \frac{1}{k} = \infty$, so that the balls are unavoidable, whereas the occupation time is almost surely finite since $\sum_n \frac{r_n^d}{|x_n|^{d-2}} = \sum_n(\frac{r_n}{|x_n|})^{d-2} r_n^2 = \sum_k \frac{1}{k^d} k^{-\frac{4}{d-2}} \times k^{d-1} < \infty$.

We prove Theorem 1 in section 2 and Theorem 2 in section 3.

2. Proof of Theorem 1

The second part of the theorem, concerning the sharpness of the condition, follows from the sharpness result in Theorem 2. To prove the rest of the theorem, we need to show that under condition (1.6), the Liouville property holds for (1.1) if and only if (1.5) holds. In the introduction we noted that $\int_0^{\infty} V(B(s))ds = \infty$ a.s. is equivalent to the Liouville condition. As a consequence, it was pointed out there that (1.5) is a necessary condition for the Liouville property to hold. Conversely, suppose that (1.5) holds. To complete the proof of the theorem, we will show that $\int_0^{\infty} V(B(s))ds = \infty$ a.s..

Let $R_0 = 0$ and, for $t > 1$, let $R_j = t^j$, $j \geq 1$. The Green's function for $\frac{1}{2}\Delta$ in $B(0, R_j)$ with the Dirichlet boundary condition will be denoted by $G_{R_j}(x, y)$. We extend $G_{R_j}(x, \cdot)$ to all of R^d by defining $G_{R_j}(x, y) = 0$ for $y \in R^d - \bar{B}(0, R_j)$. (Recall that for $x \in B(0, R_j)$, one has $\int_{R^d} G_{R_j}(x, y)\phi(y)dy =$

$E_x \int_0^{\tau_j} \phi(B(t))dt$, for nonnegative ϕ, where $\tau_j = \inf\{t \geq 0 : |B(t)| = R_j\}$.) Let $I(S,T) = \{x \in R^d : S \leq |x| \leq T\}$.

Lemma 1. *Let $1 < \gamma < t$. There exists a positive constant $c = c(d,t,\gamma)$ such that $G_{R_{j+1}}(x,y) \geq cG(x,y)$, for all $x, y \in I(R_j, \gamma R_j)$ and all $j \geq 1$.*

Proof. The proof is an immediate consequence of Brownian scaling and the probabilistic representation of the Green's functions. □

Let $I_j = I(R_j, t^{\frac{1}{2}} R_j)$ and $J_j = I(t^{\frac{1}{2}} R_j, R_{j+1})$, for $j \geq 1$. Let $V_j = V\mathbf{1}_{I_j}$ and let $\hat{V} = \sum_{j=1}^{\infty} V_j$. Also, let $W_j = V\mathbf{1}_{J_j}$ and $\hat{W} = \sum_{j=1}^{\infty} W_j$. Since $\hat{V}, \hat{W} \leq V$, it is enough to prove that $\int_0^\infty \hat{V}(B(s))ds = \infty$ a.s. or that $\int_0^\infty \hat{W}(B(s))ds = \infty$ a.s.. Since $\int_{R^d} \frac{\hat{V}(y)}{|y|^{d-2}} dy = \sum_{j=1}^\infty \int_{I_j} \frac{V(y)}{|y|^{d-2}} dy$ and $\int_{R^d} \frac{\hat{W}(y)}{|y|^{d-2}} dy = \sum_{j=1}^\infty \int_{J_j} \frac{V(y)}{|y|^{d-2}} dy$, it follows from (1.5) that $\int_{R^d} \frac{\hat{V}(y)}{|y|^{d-2}} dy + \int_{R^d} \frac{\hat{W}(y)}{|y|^{d-2}} dy = \int_{R^d - B(0,R_1)} \frac{V(y)}{|y|^{d-2}} dy = \infty$. Thus, either $\int_{R^d} \frac{\hat{V}(y)}{|y|^{d-2}} dy = \infty$ or $\int_{R^d} \frac{\hat{W}(y)}{|y|^{d-2}} dy = \infty$. We will complete the proof assuming the former equality,

$$\int_{R^d} \frac{\hat{V}(y)}{|y|^{d-2}} dy = \sum_{j=1}^\infty \int_{I_j} \frac{V(y)}{|y|^{d-2}} dy = \infty, \tag{2.1}$$

and proving that $\int_0^\infty \hat{V}(B(s))ds = \infty$ a.s.. We choose $t > 1$ sufficiently close to one so that Assumption 1 holds. (The proof of the theorem assuming the latter equality is completely analogous. One just replaces $\hat{V}$ by $\hat{W}$ in the steps below. The implementation of Assumption A in (2.9) below is also valid in this latter case.)

One has that $\int_0^\infty \hat{V}(B(s))ds = \infty$ a.s. if and only if $E_0 \exp(-\int_0^\infty \hat{V}(B(s))ds) = 0$, and one has

$$E_0 \exp(-\int_0^\infty \hat{V}(B(s))ds) = \lim_{n\to\infty} E_0 \exp(-\int_0^{\tau_n} \hat{V}(B(s))ds). \tag{2.2}$$

Using conditional expectation and the strong Markov property, one has for $n = 1, 2, \cdots$,

$$E_0 \exp(-\int_0^{\tau_n} \hat{V}(B(s))ds) \quad \leq \quad \prod_{j=0}^{n-1} \sup_{|x|=R_j} u_j(x), \tag{2.3}$$

where $u_j(x) = E_x \exp(-\int_0^{\tau_{j+1}} \hat{V}(B(s))ds)$ for $j \geq 0$. Thus, by (2.2) and (2.3) it suffices to show that

$$\prod_{j=1}^{\infty} \sup_{|x|=R_j} u_j(x) = 0. \tag{2.4}$$

Since $\sup_{|x|=R_j} u_j(x) \in (0,1]$, (2.4) holds if and only if

$$\sum_{j=1}^{\infty} (1 - \sup_{|x|=R_j} u_j(x)) = \infty. \tag{2.5}$$

By the Feynman-Kac formula, $z_j(x) \equiv E_x \exp(-\int_0^{\tau_{j+1}} V_j(B(s))ds)$ is the solution of the equation

$$\begin{aligned} &\frac{1}{2}\triangle z_j - V_j z_j = 0,\ |x| < R_{j+1};\\ &z_j = 1,\ |x| = R_{j+1}. \end{aligned}$$

Since $z_j \geq u_j$, by (2.5) it is enough to prove that

$$\sum_{j=1}^{\infty} (1 - \sup_{|x|=R_j} z_j(x)) = \infty. \tag{2.6}$$

Let $w_j = 1 - z_j$. Then w_j solves the equation

$$\begin{aligned} &\frac{1}{2}\triangle w_j = -z_j V_j,\ |x| < R_{j+1};\\ &w_j = 0,\ |x| = R_{j+1}. \end{aligned} \tag{2.7}$$

By Chebyshev's inequality, for $\lambda > 0$,

$$P_x(\int_0^{\tau_{j+1}} V_j(B(s))ds \leq \lambda) \geq 1 - \frac{E_x \int_0^{\tau_{j+1}} V_j(B(s))ds}{\lambda}. \tag{2.8}$$

Also, using the strong Markov property for the first inequality below, using Assumption 1 for the last inequality below, and using (1.3) and (1.4) and the fact that V_j is supported in $I(R_j, R_{j+1})$, one has for $|x_0| \le R_{j+1}$,

(2.9)

$$
\begin{aligned}
&E_{x_0}\int_0^{\tau_{j+1}} V_j(B(s))ds \le \sup_{R_j\le|x|\le R_{j+1}} E_x\int_0^{\tau_{j+1}} V_j(B(s))ds\\
&\le \sup_{R_j\le|x|\le R_{j+1}}\int_{R^d} G(x,y)V_j(y)dy = \sup_{R_j\le|x|\le R_{j+1}}\int_{R_j\le|y|\le R_{j+1}} G(x,y)V(y)dy\\
&= \frac{2}{(d-2)\omega_d}\sup_{R_j\le|x|\le R_{j+1}}\int_{R_j\le|y|\le R_{j+1}}\frac{V(y)}{|x-y|^{d-2}}dy \le M,
\end{aligned}
$$

for some $M > 0$ independent of j. Taking $\lambda = 2M$, it follows from (2.8) and (2.9) that $P_x(\int_0^{\tau_{j+1}} V_j(B(s))ds \le 2M) \ge \frac{1}{2}$, for all $|x| \le R_{j+1}$. Thus,

$$
\begin{aligned}
z_j(x) &= E_x \exp(-\int_0^{\tau_{j+1}} V_j(B(s))ds)\\
&\ge P_x(\int_0^{\tau_{j+1}} V_j(B(s))ds \le 2M)e^{-2M} \qquad (2.10)\\
&\ge \frac{1}{2}\exp(-2M) \equiv \alpha > 0, \text{ for all } j.
\end{aligned}
$$

Let $\hat{w}_j$ be the solution of the equation

$$
\begin{aligned}
&\frac{1}{2}\triangle\hat{w}_j = -\alpha V_j,\ |x| < R_{j+1};\\
&\hat{w}_j = 0,\ |x| = R_{j+1}.
\end{aligned} \qquad (2.11)
$$

From (2.10), we have $z_j \ge \alpha$. Hence, comparing (2.7) and (2.11), the maximum principle allows one to conclude that $\hat{w}_j \le w_j$. Thus, by (2.6) and the fact that $w_j = 1 - z_j$, it is enough to prove that

$$
\sum_{j=1}^{\infty} \inf_{|x|=R_j} \hat{w}_j(x) = \infty. \qquad (2.12)
$$

The solution $\hat{w}_j$ to (2.11) is given by

$$
\hat{w}_j(x) = \alpha\int_{B(0,R_{j+1})} G_{R_{j+1}}(x,y)V_j(y) = \int_{I_j} G_{R_{j+1}}(x,y)V(y)dy. \qquad (2.13)
$$

By Lemma 1,

$$
\int_{I_j} G_{R_{j+1}}(x,y)V(y)dy \ge c\int_{I_j} G(x,y)V(y)dy, \text{ for } |x| = R_j. \qquad (2.14)
$$

Thus, (2.13), (2.14) and (1.3) give

$$\sum_{j=1}^{\infty} \inf_{|x|=R_j} \hat{w}_j(x) \geq c \sum_{j=1}^{\infty} \inf_{|x|=R_j} \int_{I_j} \frac{V(y)}{|x-y|^{d-2}} dy, \tag{2.15}$$

for some $c > 0$. For $|x| = R_j$ and $y \in I_j$, one has $|x - y| \leq 2|y|$. Thus (2.15) gives

$$\sum_{j=1}^{\infty} \inf_{|x|=R_j} \hat{w}(x) \geq c_1 \sum_{j=1}^{\infty} \int_{I_j} \frac{V(y)}{|y|^{d-2}} dy, \tag{2.16}$$

for some $c_1 > 0$. Now (2.12) follows from (2.16) and (2.1). □

3. Proof of Theorem 2

By the ϵ-separation assumption, at most a finite number of balls overlap other balls; thus, it is clear that (1.5) and (1.8) are equivalent. We now prove the first part of the theorem; that under condition (1.7), the Liouville property holds if and only if (1.8) holds. By Theorem 1, it suffices to prove that Assumption 1 holds. Recall the definition of $I(S,T)$ before Lemma 1. We have trivially

$$\begin{aligned} \int_{I(R_j,R_{j+1})} \frac{V(y)}{|x-y|^{d-2}} dy &\leq \int_{B(x,\frac{R_j}{2})} \frac{V(y)}{|x-y|^{d-2}} dy \\ &+ \int_{I(R_j,R_{j+1})-B(x,\frac{R_j}{2})} \frac{V(y)}{|x-y|^{d-2}} dy. \end{aligned} \tag{3.1}$$

To prove that Assumption 1 holds, we will show that each of the terms on the right hand side of (3.1) is bounded independent of j and of $x \in I(R_j, R_{j+1})$.

Consider the first term on the right hand side of (3.1). Fix $x \in I(R_j, R_{j+1})$ and define $T_k(x) = \{y \in R^d : k \leq |x-y| < k+1\}$, for $k = 0, 1, \cdots$. Let $S(x) = \cup_{k=1}^{[\frac{R_j}{2}]} T_k(x)$. Letting c denote a positive constant whose value may change from line to line, and using the ϵ-separation condition for the second inequality below, we have

$$
\begin{aligned}
&\int_{B(x,\frac{R_j}{2})}\frac{V_j(y)}{|x-y|^{d-2}}dy\le\sum_{k=0}^{[R_j/2]}\int_{T_k(x)}\frac{V_j(y)}{|x-y|^{d-2}}dy\\
&\le c+c\sum_{k=1}^{[R_j/2]}\sum_{n:B(r_n,x_n)\cap T_k(x)\neq\emptyset}\frac{r_n^d}{k^{d-2}}\\
&\le c+c\,(\sup_{n:B(r_n,x_n)\cap S_k(x)\neq\emptyset}r_n^d)\sum_{k=1}^{[R_j/2]}((k+1)^d-k^d)\frac{1}{k^{d-2}} \qquad (3.2)\\
&\le c+c\,(\sup_{n:B(r_n,x_n)\cap S_k(x)\neq\emptyset}r_n^d)\sum_{k=1}^{[R_j/2]}\frac{k^{d-1}}{k^{d-2}}\\
&\le c\,(\sup_{n:B(r_n,x_n)\cap S_k(x)\neq\emptyset}r_n^d)R_j^2.
\end{aligned}
$$

By (1.7) and the definition of $S_k(x)$, there exists a constant c_1 such that $\sup_{n:B(r_n,x_n)\cap S_k(x)\neq\emptyset} r_n^d \le \frac{c_1}{R_j^2}$. Using this with (3.2) allows us to conclude that $\int_{B(x,R_j/2)}\frac{V(y)}{|x-y|^{d-2}}dy$ is bounded independent of j and of $x\in I(R_j,R_{j+1})$.

Consider now the second term on the right hand side of (3.1). For all $y\in I(R_j,R_{j+1})-B(x,\frac{R_j}{2})$, one has $|x-y|\ge R_j/2$. Thus letting c denote a positive constant whose value may change from line to line, and using the ϵ-separation condition in the second inequality below, we have

$$
\begin{aligned}
&\int_{I(R_j,R_{j+1})-B(x,\frac{R_j}{2})}\frac{V(y)}{|x-y|^{d-2}}dy\le cR_j^{2-d}\int_{I(R_j,R_{j+1})-B(x,\frac{R_j}{2})}V(y)dy\\
&\le c\,(\sup_{n:B(x_n,r_n)\cap I(R_j,R_{j+1})\neq\emptyset}r_n^d)R_j^{2-d}(R_{j+1}^d-R_j^d)\\
&\le c\,(\sup_{n:B(x_n,r_n)\cap I(R_j,R_{j+1})\neq\emptyset}r_n^d)R_j^2.
\end{aligned}
$$

Using (1.7) again shows that $\int_{I(R_j,R_{j+1})-B(x,\frac{R_j}{2})}\frac{V(y)}{|x-y|^{d-2}}dy$ is bounded independent of j and of $x\in I(R_j,R_{j+1})$. This completes the proof that Assumption 1 holds.

We now show that (1.7) is sharp in the sense described in the statement of the theorem. Consider a cluster of k^d balls each of radius $r\le\frac{1}{4}$, whose centers are more or less evenly distributed in a large ball that has radius k

and is centered at a distance m from the origin, where $m \geq 2k$. Note that these balls satisfy an ϵ-separation condition, for some $\epsilon > 0$ which can be chosen independently of k. Ignore all balls whose centers are at a distance less than m from the origin. This leaves approximately $k^d/2$ balls. We refer to this cluster of balls as $C(m,k,r)$. We will now build a sequence of such clusters. The union of all the balls in all the clusters will constitute the collection $\{B(x_i,s_i)\}_{i=1}^{\infty}$ appearing in the statement of the theorem. (We have used the index i instead of n and the letter s instead of r advisedly, because the sequence of clusters will be indexed by n and the common radius of all the balls in the n-th cluster will be denoted by r_n.)

Let ψ be as in the statement of the theorem. To each positive integer $n \geq 1$, choose m_n such that $\psi(m_n) \geq n^{2d}$ and $m_n \geq 2m_{n-1}$. Choose $k_n = m_n/n^2$ and let r_n satisfy $r_n^d = \frac{1}{2}\frac{\psi(m_n)}{{m_n}^2}$. Consider a cluster $C(m_n,k_n,r_n)$. Then for each ball $B(x,r_n) \in C(m_n,k_n,r_n)$, one has $|x|^2{r_n}^d \leq m_n^2(1+\frac{1}{n^2})^2r_n^d \leq \psi(m_n) \leq \psi(|x|)$. We can assume that the function ψ satisfies $\psi(x) \leq 4^{-d}x^2$, so that $r_n \leq 1/4$. We have

$$\sum_{n=1}^{\infty}\sum_{x:B(x,r_n)\in C(m_n,k_n,r_n)}\frac{r_n^d}{|x|^{d-2}} \geq c\sum_{n=1}^{\infty}k_n^d\frac{r_n^d}{{m_n}^{d-2}},$$

for some $c > 0$. Also,

$$k_n^d\frac{r_n^d}{{m_n}^{d-2}} = \frac{m_n^d}{n^{2d}}\frac{\frac{\psi(m_n)}{2{m_n}^2}}{m_n^{d-2}} \geq \frac{1}{2};$$

thus

$$\sum_{n=1}^{\infty}\sum_{x:B(x,r_n)\in C(m_n,k_n,r_n)}\frac{r_n^d}{|x|^{d-2}} = \infty,$$

and (1.8) holds.

To finish the proof, we now prove that the Liouville property does not hold, by showing that $\int_0^\infty V(B(s))ds < \infty$ a.s.. To show this, it is enough to show that for some n_0, the collection $\cup_{i=n_0}^{\infty}B(x_i,s_i)$ of balls is avoidable; that is, that

$$P_x(B(t) \notin \cup_{i=n_0}^{\infty}B(x_i,s_i),\ \text{for all } t \geq 0) > 0, \quad \text{for } x \notin \cup_{i=n_0}^{\infty}\bar{B}(x_i,s_i). \tag{3.3}$$

Indeed, if (3.3) holds, then $P_x(\int_0^\infty V(B(s))ds < \infty) > 0$ since the total occupation time of a finite number of balls is almost surely finite. But since the tail σ-field is trivial for Brownian motion, it then follows that $\int_0^\infty V(B(s))ds < \infty$ a.s..

To prove (3.3), we adapt the method in [5, Theorem 3]. Without loss of generality, assume that $0 \notin \cup_{i=1}^\infty \bar{B}(x_i, s_i)$. Let $w(x, E; D)$ denote the harmonic measure at x, with respect to the domain D, of $E \subset \partial D$. Then $w(x, E; D)$ is the smallest positive harmonic function in D which equals 1 on ∂E. Define

$$u_n(x) = \sum_{y:B(y,r_n)\in C(m_n,k_n,r_n)} \frac{1}{|x-y|^{d-2}}. \tag{3.4}$$

Note that u_n is harmonic on $R^d - \bar{C}(m_n, k_n, r_n)$. Let x be a point on the boundary of one of the balls in $C(m_n, k_n, r_n)$. For $1 \le i \le k_n$, let

$$Q_x(i) = \{B(y, r) \in C(m_n, k_n, r_n) : 2(i-1) \le |y - x| < 2i\}.$$

Then $\cup_{i=1}^{k_n} Q_x(i) = C(m_n, k_n, r_n)$, and for every integer $1 \le i \le [k_n/2]$, there are at least ai^{d-1} balls in $Q_x(i)$, for some universal constant $a > 0$. Thus,

$$u_n(x) \ge \sum_{i=1}^{[k_n/2]} \frac{1}{(2i)^{d-2}} ai^{d-1} \ge ck_n^2, \tag{3.5}$$

for some $c > 0$. Therefore, it follows from the the maximum principle that

$$\begin{aligned} w(0, \partial C(m_n, k_n, r_n), R^d - C(m_n, k_n, r_n)) \ &\le \frac{c}{k_n^2} u_n(0) \\ \le c_1 \frac{k_n^d}{m_n^{d-2}} \frac{1}{k_n^2} = c_1 \left(\frac{k_n}{m_n}\right)^{d-2}&, \end{aligned} \tag{3.6}$$

for some $c_1 > 0$.

Let $\Omega = R^d - \bigcup_{n=n_0}^\infty C(m_n, k_n, r_n)$, where n_0 will be fixed later. Then using the maximum principle for the first inequality below and using (3.6) for the second one, we have

$$w(0, \partial\Omega, \Omega) \leq \sum_{n=n_0}^{\infty} w(0, \partial C(m_n, k_n, r_n), R^d - C(m_n, k_n, r_n))$$

$$\leq \sum_{n=n_0}^{\infty} c_1 \left(\frac{k_n}{m_n}\right)^{d-2} = c_1 \sum_{n=n_0}^{\infty} \frac{1}{n^{2(d-2)}}.$$

By choosing n_0 sufficiently large the right hand side above will be less than 1. Now $w(0, \partial\Omega, \Omega)$ is the probability that a Brownian motion starting at 0 will ever hit $\cup_{n=n_0}^{\infty} C(m_n, k_n, r_n)$. Thus $\cup_{n=n_0}^{\infty} C(m_n, k_n, r_n)$ is avoidable. □

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