SPECTRAL ANALYSIS OF A FAMILY OF SECOND-ORDER ELLIPTIC OPERATORS WITH NONLOCAL BOUNDARY CONDITION INDEXED BY A PROBABILTY MEASURE

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Abstract. Let $D \subset R^d$ be a bounded domain and let

$$L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla$$

be a second order elliptic operator on D. Let ν be a probability measure on D. Denote by $\mathcal{L}$ the differential operator whose domain is specified by the following non-local boundary condition:

$$\mathcal{D}_{\mathcal{L}} = \{f \in C^2(\overline{D}) : \int_D f d\nu = f|_{\partial D}\},$$

and which coincides with L on its domain. Clearly 0 is an eigenvalue for $\mathcal{L}$, with the corresponding eigenfunction being constant. It is known that $\mathcal{L}$ possesses an infinite sequence of eigenvalues, and that with the exception of the zero eigenvalue, all eigenvalues have negative real part. Define the spectral gap of $\mathcal{L}$, indexed by ν, by

$$\gamma_1(\nu) \equiv \sup\{\text{Re } \lambda : 0 \neq \lambda \text{ is an eigenvalue for } \mathcal{L}\}.$$

In this paper we investigate the eigenvalues of $\mathcal{L}$ in general and the spectral gap $\gamma_1(\nu)$ in particular.

The operator $\mathcal{L}$ and its spectral gap $\gamma_1(\nu)$ have probabilistic significance. The operator $\mathcal{L}$ is the generator of a diffusion process with random jumps from the boundary, and $\gamma_1(\nu)$ measures the exponential rate of convergence of this process to its invariant measure.

1. Introduction and Statement of Results

Let $D \subset R^d$ be a bounded domain and let

$$L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla$$

be a second order elliptic operator on D. We will assume that $a = \{a_{ij}\}_{i,j=1}^d$ is positive definite with entries in $C^{2,\alpha}(\mathbb{R}^d)$ and that $b = (b_1, \dots, b_d)$ has entries in $C^{1,\alpha}(\mathbb{R}^d)$, for some $\alpha \in (0,1]$. Note that we have written the principal part of the operator L in divergence form. This has been done for convenience and, in light of the above conditions on the coefficients, without loss of generality. We will assume either that D has a $C^{2,\alpha}$-boundary or that $D = D_1 \times \cdots \times D_k$, and $L = \sum_{i=1}^k L_i$, where L_i is defined on D_i and D_i has a $C^{2,\alpha}$-boundary. This latter situation allows in particular for the case of $\frac{1}{2}\Delta$ on a cube.

Let ν be a probability measure on D. Denote by $\mathcal{L}$ the differential operator whose domain is specified by a non-local boundary condition as follows:

$$\mathcal{D}_{\mathcal{L}} = \{f \in C^2(\overline{D}) : \int_D f d\nu = f|_{\partial D}\},$$

and which coincides with L on its domain. (Non-local boundary conditions in the spirit of the one above in the context of parabolic operators can be found in the physics literature on "well-stirred" liquids. See [11] and [5].)

Clearly 0 is an eigenvalue for $\mathcal{L}$, with the corresponding eigenfunction being constant. It is known that $\mathcal{L}$ possesses an infinite sequence of eigenvalues, and that with the exception of the zero eigenvalue, all eigenvalues have negative real part (see Theorem BP below). Note that the operator $\mathcal{L}$ depends on the measure ν through its domain of definition. Define the spectral gap of $\mathcal{L}$, indexed by ν, by

$$\gamma_1(\nu) \equiv \sup\{\text{Re } \lambda : 0 \neq \lambda \text{ is an eigenvalue for } \mathcal{L}\}. \tag{1.1}$$

In this paper we investigate the eigenvalues of $\mathcal{L}$ in general and the spectral gap $\gamma_1(\nu)$ in particular. The operator $\mathcal{L}$ and its spectral gap $\gamma_1(\nu)$ have probabilistic significance which we now point out.

Let $G^D(x,y)$ denote the Green's function for L, defined by

$$G^D(x,y) = \int_0^\infty p^D(t,x,y)dt,$$

where $p^D(t,x,y)$ is the Dirichlet heat kernel for $L-\frac{d}{\partial t}$ in D, or equivalently, as a function of y, $p^D(t,x,y)$ is the transition subprobability density for the diffusion process $Y(t)$ in D corresponding to L, starting from $x \in D$ and killed upon exiting D. It was shown in [1] that there exists a Markov process $X(t)$ in D which coincides with the diffusion $Y(t)$ governed by L until it exits D, at which time it jumps to a point in the domain according to the distribution ν and starts the diffusion afresh. This same mechanism is repeated independently each time the process reaches the boundary. This process is called a *diffusion with random jumps from the boundary.* In light of the above probabilistic connection, from now on we will refer to the measure ν appearing in the definition of $\mathcal{L}$ as the jump measure. Denote expected values corresponding to this process starting from $x \in D$ by $\mathbb{E}_x$. Let $\mathcal{P}(D)$ denote the space of probability measures on D. Under the smoothness conditions stated above, the following theorem was proven in [1, Theorem 1 and the remark following it].

Theorem (BP)**.** *Let X be the diffusion with random jumps from the boundary corresponding to L and ν.*
i. There exists a unique invariant measure μ for the process. It has a density, also denoted by μ, which is given by

$$\mu(y) = \frac{\int_D G^D(x,y)d\nu(x)}{\int_D \int_D G^D(x,z)d\nu(x)dz}.$$

The map Inv: $\mathcal{P}(D) \to \mathcal{P}(D)$, defined by Inv$(\nu) = \mu$, is continuous in the topology of weak convergence of probability measures.
ii. The operator $\mathcal{L}$ possesses an infinite sequence of eigenvalues. Furthermore,

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{f\in L^\infty(D),\ ||f||_\infty \le 1} ||\mathbb{E}_x f(X(t)) - \int_D f d\mu||_\infty = \gamma_1(\nu) < 0,$$

where $\gamma_1(\nu)$, defined in (1.1)*, is the spectral gap of $\mathcal{L}$.*

Remark. Actually, part (ii) of Theorem BP was proved in [1] for a more general problem, where the jump measure from the boundary is allowed to depend on the boundary location.

We now turn to the analysis of the eigenvalues of $\mathcal{L}$ in general and of the spectral gap of $\mathcal{L}$ in particular. Note that by Theorem BP, the larger the spectral gap, the faster is the rate of convergence to equilibrium for the diffusion with random jumps.

We begin with a very special case of jump measure ν where the eigenvalues (and eigenfunctions) of $\mathcal{L}$ can be completely characterized in terms of those of L with the Dirichlet boundary condition. Recall that the operator L with the Dirichlet boundary condition possesses an infinite sequence of eigenvalues, all of which have negative real part. By the Krein-Rutman theorem, the *principal eigenvalue*–the eigenvalue with largest real part—is real and simple, and the corresponding eigenfunction does not change sign The same is true for $\tilde{L}$, the formal adjoint of L with the Dirichlet boundary condition. Furthermore, the principal eigenvalues of L and $\tilde{L}$ coincide. Let $\widetilde{\phi_0^D} > 0$ denote the principal eigenfunction corresponding to the principal eigenvalue for $\widetilde{L}$. Normalize it by $\int_D \widetilde{\phi_0^D}(x)dx = 1$. Abusing notation, we will also let $\widetilde{\phi_0^D}$ denote the measure with density $\widetilde{\phi_0^D}$. (The measure $\widetilde{\phi_0^D}$ is the so-called quasi-invariant distribution for the original diffusion corresponding to L with killing at the boundary. That is, one has $\mathbb{E}^D_{\widetilde{\phi_0^D}}(f(Y(t))|\tau_D > t) = \int_D f(x)\widetilde{\phi_0^D}(x)dx$, for all $t > 0$, where τ_D is the first exit time of the diffusion $Y(t)$ from D and $\mathbb{E}^D_{\widetilde{\phi_0^D}}$ denotes the expectation for the diffusion killed at the boundary and starting from the distribution $\widetilde{\phi_0^D}$.)

Theorem 1. *Consider the operator $\mathcal{L}$ in the case that the jump measure is given by $\nu = \widetilde{\phi_0^D}$, where $\widetilde{\phi_0^D}$ is the normalized principal eigenfunction for the formal adjoint $\tilde{L}$ of L with the Dirichlet boundary condition. Let $\{\lambda_n^D\}_{n=0}^\infty$ denote the eigenvalues for L with the Dirichlet boundary condition, labeled so that Re $\lambda_{n+1}^D \leq$ Re λ_n^D, and let $\{\phi_n\}_{n=0}^\infty$ be a corresponding sequence*

of eigenfunctions. Then the eigenvalues for $\mathcal{L}$ are 0 and $\{\lambda_n^D\}_{n=1}^\infty$ and a corresponding sequence of eigenfunctions is given by 1 and $\{\phi_n\}_{n=1}^\infty$. In particular,

$$\gamma_1(\widetilde{\phi_0^D}) = Re\ (\lambda_1^D).$$

Furthermore, $\widetilde{\phi_0^D}$ is the invariant probability measure for the diffusion with random jumps from the boundary corresponding to $\mathcal{L}$. In fact, $\widetilde{\phi_0^D}$ is the unique fixed point for the map Inv: $\mathcal{P}(D) \to \mathcal{P}(D)$ defined in Theorem BP.

In order to make the spectral analysis tractable when the jump measure ν is not the special measure considered in Theorem 1, we will need to assume that the operator L with the Dirichlet boundary condition is self-adjoint, although $\mathcal{L}$ will still not be self-adjoint, as we now explain. If the first-order term b in the operator L is of the form $b = a\nabla Q$, then the operator L can be written in the form $L = \frac{1}{2}\exp(-2Q)\nabla \cdot a\exp(2Q)\nabla$. Since we can replace Q by $Q + c$, where c is a constant, without changing L, we will assume without loss of generality that $\int_D \exp(2Q)dx = 1$. Let μ_{rev} denote the probability measure $\exp(2Q)dx$. In this case, the operator L with the Dirichlet boundary condition, considered as an operator on $L^2(D, \mu_{\text{rev}})$, is symmetric on the domain of smooth functions vanishing at the boundary and is self-adjoint on an appropriate domain of definition. (The diffusion process in D killed at the boundary, corresponding to L with the Dirichlet boundary condition, is reversible and the normalized reversible measure is μ_{rev}; whence the notation μ_{rev}.) The operator $\mathcal{L}$, on the other hand, will *never* be self-adjoint. Indeed, a straight forward calculation shows that the adjoint operator (with respect to Lebesgue measure) $\tilde{\mathcal{L}}$ of $\mathcal{L}$ is defined on a domain which includes $\{v \in C^2(D) \cap C(\bar{D}) : v = 0 \text{ on } \partial D\}$, and for such functions one has $\tilde{\mathcal{L}}v = \tilde{L}v - (\int_D \tilde{L}v)\nu$, where $\tilde{L} = \frac{1}{2}\nabla \cdot a\nabla - b\nabla - \nabla \cdot b$. In the case that L is self-adjoint, if one takes the adjoint of $\mathcal{L}$ with respect to μ_{rev}, then the adjoint is defined on the above class of functions by $\tilde{\mathcal{L}}v = Lv - (\int_D Lv)\nu$.

We will begin with a key theoretical result, which will be mined to obtain more concrete results. Before we can state the theorem, we need some additional notation. The eigenvalues of the self adjoint operator L are real and negative. We will denote them by $\{\lambda_n^D\}_{n=0}^\infty$, labelled in nonincreasing order. Denote the corresponding eigenfunctions by $\{\phi_n^D\}_{n=0}^\infty$, normalized by $\int_D \phi_n^D d\mu_{\rm rev} = 1$, $n \geq 0$, and $\phi_0^D > 0$. Let

$$F_n \equiv \int_D \phi_n^D d\mu_{\rm rev} \quad \text{and} \quad G_n(\nu) \equiv \int_D \phi_n^D d\nu. \tag{1.2}$$

Let $\{\Lambda_n^D\}_{n=0}^\infty$ denote the collection of distinct eigenvalues among $\{\lambda_n^D\}_{n=0}^\infty$, labelled in decreasing order. We will sometimes need the following assumption.

Assumption 1. *The Fourier series* $\sum_{n=0}^\infty \frac{F_n}{\lambda_n^D}\phi_n^D(x)$ *converges uniformly and absolutely.*

Theorem 2. *Assume that the operator L with the Dirichlet boundary condition is self-adjoint. Let*

$$E_\nu(\lambda) \equiv \sum_{n=0}^\infty \frac{F_n G_n(\nu)}{\lambda_n^D - \lambda}.$$

Let d_n denote the dimension of the eigenspace corresponding to the n-th distinct eigenvalue Λ_n^D of L. Assume either that ν possesses an $L^2(D, d\mu_{rev})$-density or that Assumption 1 on the operator L holds.

Then the set of nonzero eigenvalues of $\mathcal{L}$ and their multiplicities are given as follows:

i. *The set $\{\lambda : E_\nu(\lambda) = 0\} - \{\Lambda_n^D\}_{n=1}^\infty$ consists of simple eigenvalues;*

ii. *For each $n = 1, 2 \cdots$, the following rule determines whether Λ_n^D is an eigenvalue, and if so, specifies its multiplicity:*

If $d_n = 1$ and neither $F_m = 0$ nor $G_m(\nu) = 0$, for the m satisfying $\lambda_m^D = \Lambda_n^D$, then Λ_n^D is not an eigenvalue. Otherwise, Λ_n^D is an eigenvalue and its multiplicity is specified as follows:

If $G_m(\nu) \neq 0$ for some m such that $\lambda_m^D = \Lambda_n^D$ and $F_m \neq 0$ for some m such that $\lambda_m^D = \Lambda_n^D$, then the multiplicity is $d_n - 1$;

If $G_m(\nu) = 0$ for all m such that $\lambda_m^D = \Lambda_n^D$ and $F_m \neq 0$ for some m such that $\lambda_m^D = \Lambda_n^D$, or if $G_m(\nu) \neq 0$ for some m such that $\lambda_m^D = \Lambda_n^D$ and $F_m = 0$ for all m such that $\lambda_m^D = \Lambda_n^D$, then the multiplicity is d_n;
If $G_m(\nu) = 0$ for all m such that $\lambda_m^D = \Lambda_n^D$ and $F_m = 0$ for all m such that $\lambda_m^D = \Lambda_n^D$, then the multiplicity is d_n if $E_\nu(\Lambda_n^D) \neq 0$ and is d_{n+1} if $E_\nu(\Lambda_n^D) = 0$.

Furthermore, even without Assumption 1 or the density condition on ν, the set of eigenvalues of $\mathcal{L}$ includes those listed in (ii).

Note that the complete characterization of the spectrum in Theorem 2 always holds if the jump measure ν possesses an $L^2(D, \mu_{\text{rev}})$-density. If the operator L on D satisfies Assumption 1, then it holds for all jump measures $\nu \in \mathcal{P}(D)$. The following theorem collects some sufficient conditions for Assumption 1 to hold.

Theorem 3. *i. If $d = 1$, then Assumption 1 holds.*
ii. Let $d = 2$ and let $L = \frac{1}{2}\exp(-2Q)\nabla\cdot a\exp(2Q)\nabla$ satisfy $Q = \frac{1}{2}\log\sqrt{det(a^{-1})}$ (in which case L can be considered as $\frac{1}{2}\Delta_M$, where Δ_M is the Laplacian of a Riemannian manifold with metric a). Then Assumption 1 holds.
iii If $d \leq 3$ and the eigenfunctions $\{\phi_n^D\}_{n=0}^\infty$ are uniformly bounded, then Assumption 1 holds.

Remark. A direct calculation (see the proof of Proposition 1 below) shows that the eigenfunctions $\{\phi_n^D\}_{n=0}^\infty$ are uniformly bounded for $L = \frac{1}{2}\Delta$ in $D = (0,1)^d$; however such a bound does not hold if D is a sphere [2].

As a first application of Theorem 2, we identify a class of jump measures ν for which all the eigenvalues of $\mathcal{L}$ are real. The analysis of the spectrum in this case turns out to be more tractable.

Theorem 4. *Assume that the conditions of Theorem 2 are in force, and let F_n and $G_n(\nu)$ be as in* (1.2)*. Assume also that the jump measure ν satisifies one of the following two conditions:*

i. $F_nG_n(\nu) \geq 0$, *for all* $n \geq 1$, *or* $F_nG_n(\nu) \leq 0$, *for all* $n \geq 1$;

ii. $F_nG_n \neq 0$ *for at most two values of* n.

Then all the eigenvalues of $\mathcal{L}$ *are real.*

When the nonzero eigenvalue with the largest real part is real, we can prove an upper bound on the eigenvalue spectral gap, $\gamma_1(\nu)$, of $\mathcal{L}$.

Theorem 5. *If the jump measure* ν *is such that the nonzero eigenvalue of* $\mathcal{L}$ *with the largest real part is real, then*

$$\gamma_1(\nu) < \lambda_0^D.$$

Remark 1. Theorem 5 holds regardless of whether the operator L is self-adjoint; however, if L is not self-adjoint then we have no way of determining whether the nonzero eigenvalue of $\mathcal{L}$ with the largest real part is real.

Remark 2. As is well known, λ_0^D gives the exponential rate of decay in t of the probability that the diffusion $Y(t)$ in D corresponding to L has not yet hit the boundary by time t; that is, $\lim_{t\to\infty} \frac{1}{t} \log P_x(\tau_D > t) = \lambda_0^D$, where τ_D is the first exit time of the diffusion from D. Now since $\gamma_1(\nu)$ gives the exponential rate of convergence of the distribution of the diffusion with random jumps to its invariant measure, and since the jump mechanism only comes into affect after time τ_D, Theorem 5 might seem (at least at first blush) counter-intuitive.

The normalized reversible measure μ_{rev}, with respect to which L is self-adjoint, plays a distinguished role as the jump measure. In particular, in this case the spectral gap can be given by a variational formula.

Theorem 6. *Assume that the operator* L *with the Dirichlet boundary condition is self-adjoint. Let* $\{\lambda_n^D\}_{n=0}^{\infty}$ *denote the eigenvalues of* L *with the Dirichlet boundary condition. Let the jump measure be the normalized reversible measure* μ_{rev}.

i. *All the eigenvalues of* $\mathcal{L}$ *are real.*

ii.

$$\gamma_1(\mu_{rev}) = -\inf \frac{\frac{1}{2}\int_D (\nabla u a \nabla u) d\mu_{rev}}{\int_D u^2 d\mu_{rev}},$$

where the infimum is over functions $u \neq 0$ *satisfying* $u|_{\partial D} = \int_D u d\mu_{rev} = 0$. *The infimum is attained at a function* u_{min} *which satisfies the equation* $Lu = \gamma_1(\mu_{rev})u + C$, *for some constant* C, *and the eigenfunction* v_1 *for* $\mathcal{L}$ *corresponding to the eigenvalue* $\gamma_1(\mu_{rev})$ *is given by* $v_1 = u_{min} + \frac{C}{\gamma_1(\mu_{rev})}$.

iii.

$$\lambda_1^D \leq \gamma_1(\mu_{rev}) < \lambda_0^D.$$

More precisely, consider the function

$$E_{\mu_{rev}}(\lambda) = \sum_{n=0}^{\infty} \frac{F_n^2}{\lambda_n^D - \lambda},$$

which is increasing for $\lambda \in (\lambda_1^D, \lambda_0^D)$. *If the equation* $E_{\mu_{rev}}(\lambda) = 0$ *possesses a root in* $(\lambda_1^D, \lambda_0^D)$, *then* $\gamma_1(\mu_{rev})$ *is equal to this root. Otherwise,* $\gamma_1(\mu_{rev}) = \lambda_1^D$. *In particular, such a root will exist if* $F_j = \int_D \phi_j^D d\mu_{rev} \neq 0$, *for some* $j \in \{1, \cdots, k_0\}$, *where* $k_0 = max\{n : \lambda_n^D = \lambda_1^D\}$. *If* $F_j = \int_D \phi_j^D d\mu_{rev} = 0$, *for all* $j \in \{1, \cdots, k_0\}$, *then* $\gamma_1(\mu_{rev}) > \lambda_1^D$ *if and only if*

$$\frac{F_0^2}{\lambda_0^D - \lambda_1^D} < \sum_{n=k_0+1}^{\infty} \frac{F_n^2}{\lambda_1^D - \lambda_n^D}. \tag{1.3}$$

Remark. Consider the diffusion process corresponding to L as in Theorem 6 with *reflection* at the boundary in the conormal direction an, where n denotes the inward unit normal to D. The process is reversible and it corresponds to a self-adjoint operator on $L^2(D, \mu_{\text{rev}})$ which is an extension of L with the Neumann boundary condition $\nabla u \cdot an = 0$ on ∂D. For this process, μ_{rev} is the invariant measure, and the rate of convergence to μ_{rev} is given by the largest nonzero eigenvalue, λ_1^N. This eigenvalue is given by the variational formula in part (ii) of Theorem 6, but with the infimum being taken over functions u satisfying $\int_D u d\mu_{\text{rev}} = 0$ (without the additional

restriction that $u|_{\partial D} = 0$). The infimum is attained at the eigenfunction(s) corresponding to λ_1^N, and it is known that any such function does not vanish identically on ∂D. Thus, it follows from part (ii) of Theorem 6 that $\lambda_1^N > \gamma_1(\mu_{\text{rev}})$. Therefore, the rate of convergence to equilibrium is greater for the diffusion with random jumps with jump measure μ_{rev} than for the reflected diffusion, whose invariant measure is μ_{rev}.

Here is an application of condition (1.3) in part (iii) of Theorem 6.

Proposition 1. *Consider the operator $\frac{1}{2}\Delta$ in the d-dimensional unit cube, $D = (0,1)^d$, and let the jump measure be Lebesgue measure, l_d, on D. One has $\lambda_0^D = -\frac{d\pi^2}{2}$ and $\lambda_1^D = -\frac{(d+3)\pi^2}{2}$.*

i *If $d \leq 10$, then $\gamma_1(l_d) = \lambda_1^D$.*

ii. *If $d \geq 11$, then $\lambda_1^D < \gamma_1(l_d) < \lambda_0^D$.*

Remark. Note that $\gamma_1(l_d)$ decreases to $-\infty$ as $d \to \infty$. Thus, for Brownian motion in the d-dimensional cube with random jumps from the boundary with normalized Lebesgue measure as the jump measure, the rate of convergence to equilibrium becomes arbitrarily fast as the dimension increases. This is because starting from any point, the distribution of the hitting time of the boundary converges to the δ-measure at 0 as $d \to \infty$, which means that as $d \to \infty$, the process constantly gets redistributed according to Lebesgue measure after arbitrarily small intervals of time. In contrast to this, consider Brownian motion in the d-dimensional unit cube with normal reflection at the boundary. The rate of convergence to equilibrium is governed by the largest nonzero eigenvalue of the Neumann Laplacian, which is $\lambda_1^N = -\frac{\pi^2}{2}$, independent of d. Similarly, consider Brownian motion in the d-dimensional unit cube, conditioned never to hit the boundary [8]. This process corresponds to the h-transformed operator $(\frac{1}{2}\Delta - \lambda_0^D)^{\phi_0^D}$. The rate of convergence to equilibrium is governed by the largest nonzero eigenvalue, which is $\lambda_1^D - \lambda_0^D = -\frac{(d+3)\pi^2}{2} + \frac{d\pi^2}{2} = -\frac{3}{2}\pi^2$, independent of d.

We have the following result for the one-dimension Laplacian.

Proposition 2. *Consider the operator* $\frac{1}{2}\frac{d^2}{dx^2}$ *in the interval* $(0,1)$.

i. *If the jump measure is deterministic; that is,* $\nu = \delta_p$, *for some* $p \in (0,1)$, *then* $\gamma_1(\nu) = \lambda_1^D = -2\pi^2$;

ii. *If the jump measure* ν *is such that the nonzero eigenvalue of* $\mathcal{L}$ *with the largest real part is real, then*

$$\gamma_1(\nu) = \lambda_1^D = -2\pi^2.$$

Remark. Part (i) above was shown in [3] and [4]. (Actually, $-\frac{\pi^2}{2}$ was obtained in [3], because a certain cancellation was not taken into account. The correct result appears in [4].) Our proof is completely different. A direct calculation shows that $F_1 = 0$; thus, by Theorem 2, $\gamma_1(\nu) \geq \lambda_1^D = -2\pi^2$, for all ν. In a preprint version of this paper, we made the conjecture that $\gamma_1(\nu) = \lambda_1^D = -2\pi^2$ for all ν. This conjecture has now been established by combining part (ii) with a very recent result [7] which states that in the case of $\frac{1}{2}\frac{d^2}{dx^2}$ on an interval, all of the eigenvalues are real, for all jump measures ν.

The next result shows that for the Laplacian on a square in R^2, one can find a deterministic jump measure for which $\gamma_1(\nu) \neq \lambda_1^D$.

Proposition 3. *Consider the operator* $\frac{1}{2}\Delta$ *in the square* $(0,1)^2$. *Then there exists a jump measure* ν *of the form* δ_{x_0}, *for some* $x_0 \in (0,1)^2$, *for which* $\gamma(\nu) > \lambda_1^D = -\frac{5\pi^2}{2}$.

Remark. Consider $L = \frac{1}{2}\Delta$ in the cube $(0,1)^d$, $d \geq 1$. The proof of Proposition 1 shows that $F_1 = 0$; thus, by Theorem 2, $\gamma_1(\nu) \geq \lambda_1^D$. Combining Proposition 3 and Proposition 1, it follows that for $d \geq 11$ or $d = 2$, there exists a jump measure ν for which $\gamma_1(\nu) > \lambda_1^D$. Presumably, this holds for all $d \geq 2$. Conversely, by Theorem 1 it follows that for all $d \geq 1$ it is also always possible to find a ν for which $\gamma_1(\nu) = \lambda_1^D$.

In all of the examples given so far, $\gamma_1(\nu) \geq \lambda_1^D$. The following result shows that such is not always the case.

Proposition 4. *Let $k_0 = max\{n : \lambda_n^D = \lambda_1^D\}$. If $F_j \neq 0$, for some $j \in \{1, 2, \cdots, k_0\}$, then it is always possible to find a jump measure ν for which $\gamma_1(\nu) > \lambda_1^D$ and it is always possible to find a jump measure ν for which $\gamma_1(\nu) < \lambda_1^D$.*

Remark. One can check that $L = \frac{1}{2}\frac{d^2}{dx^2} + b\frac{d}{dx}$ on $(0, 1)$, where b is a nonzero constant, is an example where Proposition 4 is applicable.

We conclude this section with several open questions.

Question 1. In a preprint version of this paper, we asked whether all the eigenvalues of $\mathcal{L}$ are real in the case that L is self-adjoint. It the very recent paper [7] it was shown that for $L = \frac{1}{2}\Delta$ in a ball in R^3, there exist deterministic jump measures, that is measures of the form $\nu = \delta_x$, for which some of the eigenvalues are not real. However, these non-real eigenvalues do not have maximal real part. We still ask whether the eigenvalue with largest real part is real in the case that L is self-adjoint.

Question 2. Does the inequality $\gamma_1(\nu) < \lambda_0^D$ hold for all ν when L is self-adjoint? What about for general L?

Question 3. Does a lower bound exist for $\gamma_1(\nu)$ in terms of the eigenvalues $\{\lambda_n^D\}_{n=0}^{\infty}$ of L?

Question 4. What can be said about the continuity properties of $\gamma_1(\nu)$ as ν varies over $\mathcal{P}(D)$, the space of probability measures on D with the topology of weak convergence?

Remark. Note that if Question 1 is answered affirmatively, then Theorem 5 shows that the answer to Question 2 is affirmative in the case that L is self-adjoint.

The proofs of the results stated in this section are grouped thematically and proved in the sections that follow.

2. Proof of Theorem 1

We first prove the statement concerning the eigenvalues and eigenfunctions. Let ϕ_n^D, $n \geq 1$, denote an eigenfunction for L with the Dirichlet

boundary condition, corresponding to the eigenvalue λ_n^D. Integration by parts yields

$$\lambda_0^D \int_D \phi_n^D \widetilde{\phi_0^D} dx = \int_D \phi_n^D \widetilde{L} \widetilde{\phi_0^D} dx = \int_D L\phi_n^D \widetilde{\phi_0^D} dx = \lambda_n^D \int_D \phi_n^D \widetilde{\phi_0^D} dx,$$

from which it follows that $\int_D \phi_n^D \widetilde{\phi_0^D} dx = 0$. Since $\phi_n^D|_{\partial D} = 0$, it follows that ϕ_n^D is in the domain of $\mathcal{L}$, and we conclude that λ_n^D is an eigenvalue for $\mathcal{L}$. Of course the function 1 is an eigenfunction for $\mathcal{L}$ corresponding to the eigenvalue 0. Thus, to conclude the proof, it is enough to show that the equation $L\psi = \lambda\psi$ with $\int_D \psi \widetilde{\phi_0^D} dx = \psi|_{\partial D} = c \neq 0$, has a solution only when $\lambda = 0$. Let ψ be a solution to the above equation. Let $d\sigma$ denote the Lebesgue surface measure on ∂D. Integrating by parts and using the boundary condition, we have

$$\begin{aligned}
\lambda_0^D c &= \lambda_0^D \int_D \widetilde{\phi_0^D} \psi dx = \int_D \psi \widetilde{L} \widetilde{\phi_0^D} dx \\
&= \int_D \widetilde{\phi_0^D} L\psi dx + \int_{\partial D} \frac{1}{2} \psi a \nabla \widetilde{\phi_0^D} \cdot n d\sigma - \int_{\partial D} \psi \widetilde{\phi_0^D} b \cdot n d\sigma \\
&= \lambda \int_D \widetilde{\phi_0^D} \psi dx + c \int_D \widetilde{L} \widetilde{\phi_0^D} dx = c\lambda + c\lambda_0^D,
\end{aligned}$$

where n is the unit outward normal of D at ∂D. Therefore $\lambda = 0$.

We now turn to the statement concerning the invariant measure. We denote by G^D the operator from L^1 to L^1 given by $(G^D f)(x) = G^D(x, f)$. We denote its adjoint on L^∞ by $\widetilde{G^D}$. We have $(\widetilde{G^D} g)(y) = G^D(g, y)$. We now prove that $\widetilde{G^D}$ is compact. For $\epsilon > 0$, let K_ϵ denote the operator on L^1 defined by

$$(K_\epsilon f)(y) = \int_\epsilon^{\epsilon^{-1}} p^D(s, f, y) ds.$$

An argument similar but simpler than the one given in the proof of Lemma 1 in [1], based on the continuity of p^D on $[\epsilon, \epsilon^{-1}] \times \overline{D} \times \overline{D}$, shows that K_ϵ is compact. Now,

$$\begin{aligned}
\|\widetilde{G^D} f - K_\epsilon f\|_1 &= \int_D \Big| \int_0^\epsilon p^D(s, f, y) ds + \int_{\epsilon^{-1}}^\infty p^D(s, f, y) ds \Big| dy \\
&\leq \epsilon \|f\|_1 + \|f\|_1 \int_{\epsilon^{-1}}^\infty \sup_{x \in D} P_x^D(\tau_1 > s) ds. \qquad (2.1)
\end{aligned}$$

By [9, Theorem 3.6.1], $\lim_{s\to\infty}\frac{1}{s}\log\sup_{x\in D}P_x(\tau_1>s)=\lambda_0^D$. Thus, there exists a constant $C>0$ such that for all ϵ sufficiently small,

$$\int_{\epsilon^{-1}}^{\infty}\sup_{x\in D}P_x^D(\tau_1>s)ds\le Ce^{\frac{\lambda_0^D}{2}\epsilon^{-1}}.$$

Therefore, it follows from (2.1) that

$$\|\widetilde{G^D}-K_\epsilon\|_1\le\epsilon+Ce^{\frac{\lambda_0^D}{2}\epsilon^{-1}}\underset{\epsilon\to 0}{\to}0.$$

Consequently, $\widetilde{G^D}$ is compact.

Assume now that $m\in\mathcal{P}$ is a fixed point for Inv. Since $dm(y)=\frac{G^D(m,y)}{G^D(m,\mathbf{1})}dy=\frac{\widetilde{G}^Dm}{G^D(m,\mathbf{1})}$, it follows that m has density in L^1. Therefore we may consider m as an eigenfunction for $\widetilde{G^D}$, corresponding to the eigenvalue $\lambda\equiv G^D(m,\mathbf{1})$. Let $\phi_0^D>0$ denote the principal eigenfunction for L with the Dirichlet boundary condition corresponding to the eigenvalue λ_0^D. Since $G^D=(-L)^{-1}$, we have $G^D\phi_0=-(\lambda_0^D)^{-1}\phi_0^D$. Therefore,

$$\lambda\int_D m\phi_0^Ddx=\int_D\widetilde{G^D}m\phi_0^Ddx=\int_D mG^D\phi_0^Ddx=-(\lambda_0^D)^{-1}\int_D m\phi_0^Ddx.$$

Since m and ϕ_0^D are nonnegative, it follows that $\lambda=-(\lambda_0^D)^{-1}$. Since $\widetilde{G}^D$ is compact, the Krein-Rutman theorem guarantees that the eigenvalue $-(\lambda_0^D)^{-1}$ for $\widetilde{G}^D$ is simple. Since $\widetilde{\phi_0}$ and m are both eigenfunctions for $\widetilde{G^D}$ corresponding to the eigenvalue $-(\lambda_0^D)^{-1}$, and since $\int_D mdx=\int_D\widetilde{\phi_0}dx=1$, we conclude that $m=\widetilde{\phi_0}$.

□

3. Proofs of Theorem 2, Theorem 4, Theorem 6, Proposition 1 and Proposition 4

Proof of Theorem 2. A number $\lambda\in C-\{0\}$ will be an eigenvalue if and only there exists a function v satisfying $Lv=\lambda v$ and $v|_{\partial D}=\int_D vd\nu$. Let $u=v-c$, where $c=v|_{\partial D}$. Then u satisfies $Lu=\lambda u+K$, where $K=\lambda c$.

On $L^2(D, \mu_{\text{rev}})$, the function u can be represented in the form

$$u = \sum_{n=0}^{\infty} C_n \phi_n^D, \tag{3.1}$$

for unknown constants $\{C_n\}_{n=0}^{\infty}$, and the constant function 1 can be represented by

$$1 = \sum_{n=0}^{\infty} F_n \phi_n^D, \tag{3.2}$$

where $\{F_n\}_{n=0}^{\infty}$ is as in (1.2). Since u is a smooth function vanishing on ∂D, it is in the domain of the self-adjoint operator L acting on $L^2(D, \mu_{rev})$. Thus, from (3.1), it follows that

$$Lu = \sum_{n=0}^{\infty} \lambda_n^D C_n \phi_n^D. \tag{3.3}$$

From (3.1)-(3.3) along with the fact that $Lu = \lambda u + K$, we obtain

$$C_n \lambda_n^D = \lambda C_n + K F_n, \ n \geq 0. \tag{3.4}$$

We first show that the condition $E_\nu(\lambda) = 0$ is necessary and sufficient for $\lambda \notin \{\Lambda_n^D\}_{n=0}^{\infty}$ to be an eigenvalue. Since we are now assuming that λ is not in the spectrum of L, we may assume that $K \neq 0$. Indeed, if K were equal to 0, then v would vanish on ∂D and consequently it would be an eigenfunction for L. This would mean that $\lambda = \Lambda_n^D$, for some n. From (3.4) we obtain

$$C_n = \frac{K F_n}{\lambda_n^D - \lambda},$$

and conclude that

$$u = \sum_{n=0}^{\infty} \frac{K F_n}{\lambda_n^D - \lambda} \phi_n^D. \tag{3.5}$$

In order that v be an eigenfunction, v must satisfy $v|_{\partial D} = \int_D v \, d\nu = c$. Since $u = v - c$, we require that $\int_D u \, d\nu = 0$. If ν has an $L^2(D, \mu_{rev})$-density,

then using (3.5) and taking inner products shows that $\int_D u\, d\nu = 0$ if and only if

$$\sum_{n=0}^{\infty} \frac{F_n G_n(\nu)}{\lambda_n^D - \lambda} = 0, \tag{3.6}$$

where $G_n(\nu)$ is as in (1.2). Alternatively, if Assumption 1 holds, then the formula for u in (3.5) holds not only in $L^2(D, \mu_{rev})$, but also pointwise, and from the bounded convergence theorem it follows again that $\int_D u\, d\nu = 0$ if and only if (3.6) holds. We have thus shown that the condition $E_\nu(\lambda) = 0$ is necessary and sufficient for a nonzero $\lambda \notin \{\Lambda_n^D\}_{n=1}^{\infty}$ to be an eigenvalue. Furthermore, as the method uniquely specifies the corresponding eigenfunction (up to a multiplicative constant), it follows that the multiplicity of such an eigenvalue is 1.

We now consider the possibility that $\lambda = \Lambda_{n_0}^D$ is an eigenvalue, where n_0 is a nonnegative integer. Let S_{n_0} denote the d_{n_0}-dimensional eigenspace corresponding to the eigenvalue $\Lambda_{n_0}^D$ of L. Let $S_{n_0}^G(\nu) = \{w \in S_{n_0} : \int_D w d\nu = 0\}$ and let $S_{n_0}^F = \{w \in S_n : \int_D w d\mu_{\text{rev}} = 0\}$. Clearly, each of these latter two spaces is either $(d_{n_0} - 1)$-dimensional or d_{n_0}-dimensional.

Consider first the case that $S_{n_0}^F$ is $(d_{n_0} - 1)$-dimensional. There exists an m_0 such that $\lambda_{m_0}^D = \Lambda_{n_0}^D$ and $F_{m_0} = \int_D \phi_{m_0} d\mu_{\text{rev}} \neq 0$. But then (3.4) will hold with $n = m_0$ and $\lambda = \Lambda_{n_0}$ if and only if $K = 0$. But if $K = 0$, then $u = v$, $v|_{\partial D} = \int_D v d\nu = 0$ and $\Delta v = \Lambda_{n_0}^D v$. Thus, v belongs to $S_{n_0}^G(\mu)$. Consequently, the multiplicity of $\Lambda_{n_0}^D$ will be either $d_{n_0} - 1$ or d_{n_0}, depending on which of these numbers is the dimension of $S_{n_0}^G(\mu)$. In particular, if $n_0 = 0$, then $d_{n_0} = 1$ and $S_{n_0}^F = S_{n_0}^G(\nu) = \{0\}$ since $\phi_0^D > 0$. Thus, $\lambda_0^D = \Lambda_0^D$ can never be an eigenvalue.

Now consider the case that $S_{n_0}^F$ is d_{n_0}-dimensional. In this case, $F_m = 0$, for all m such that $\lambda_m^D = \Lambda_{n_0}^D$. We first look for eigenfunctions for which

$K \neq 0$. Solving (3.4) gives

$$\begin{cases} C_n = \frac{KF_n}{\lambda_n^D - \lambda}, \text{ for all } n \text{ such that } \lambda_n^D \neq \Lambda_{n_0}^D; \\ C_n \text{ is arbitrary, for all } n \text{ such that } \lambda_n^D = \Lambda_{n_0}^D. \end{cases}$$

Writing $C_n = Kc_n$, for n such that $\lambda_n^D = \Lambda_{n_0}^D$, and employing the same reasoning as in (3.5) and (3.6) yields

$$\sum_{n:\lambda_n^D \neq \Lambda_{n_0}^D} \frac{F_n G_n(\nu)}{\lambda_n^D - \Lambda_{n_0}^D} + \sum_{n:\lambda_n^D = \Lambda_{n_0}^D} c_n G_n(\nu) = 0. \tag{3.7}$$

There are two cases to consider—when $S_{n_0}^G(\nu)$ is $(d_{n_0} - 1)$-dimensional and when it is d_{n_0}-dimensional. In the latter case, $G_n(\nu) = 0$, for all n satisfying $\lambda_n^D = \Lambda_{n_0}^D$. Thus, (3.7) reduces to $E_\nu(\Lambda_{n_0}^D) = 0$. If this equation is satisfied, we obtain one eigenfunction with $K \neq 0$, and if it is not satisfied, we obtain no such eigenfunctions. Since $S_{n_0}^G(\nu)$ is d_{n_0}-dimensional, there are also d_{n_0} additional linearly independent eigenfunctions with $K = 0$. Thus, the multiplicity is either $d_{n_0} + 1$ or d_{n_0}, depending on whether or not $E_\nu(\Lambda_{n_0}^D) = 0$.

Now consider the case that $S_{n_0}^G(\nu)$ is $(d_{n_0} - 1)$-dimensional. Since we may choose the orthonormal basis $\{\phi_m^D\}_{\{m:\lambda_m^D = \Lambda_{n_0}^D\}}$ corresponding to the eigenspace S_{n_0} however we like, we may assume without loss of generality, that $G_m(\nu) = \int_D \phi_m d\nu = 0$, for all but one of the m for which $\lambda_m^D = \Lambda_{n_0}^D$. Denote the single m for which this is not true by m_0. Then (3.7) reduces to

$$\sum_{n:\lambda_n^D \neq \Lambda_{n_0}^D} \frac{F_n G_n(\nu)}{\lambda_n^D - \Lambda_{n_0}^D} + c_{m_0} G_{m_0}(\nu) = 0.$$

The above equation is uniquely solvable for c_{m_0}, and thus yields one eigenfunction with $K \neq 0$. Since $S_{n_0}^G(\nu)$ is $(d_{n_0} - 1)$-dimensional, there are also $d_{n_0} - 1$ additional linearly independent eigenfunctions with $K = 0$; thus the multiplicity is d_{n_0}. □

Proof of Theorem 4. By Theorem 2, a complex number $\lambda = \alpha + i\beta$, with $\beta \neq 0$ will be an eigenvalue for $\mathcal{L}$ if and only if

$$\sum_{n=0}^{\infty} \frac{F_n G_n(\nu)}{\lambda_n^D - \lambda} = 0.$$

We can rewrite this as

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{F_n G_n(\nu)(\lambda_n^D - \alpha)}{(\lambda_n^D - \alpha)^2 + \beta^2} = 0; \\ &\sum_{n=0}^{\infty} \frac{F_n G_n(\nu)\beta}{(\lambda_n^D - \alpha)^2 + \beta^2} = 0. \end{aligned} \tag{3.8}$$

Clearly, the two equations in (3.8) hold if and only if the following two equations hold:

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{F_n G_n(\nu)\lambda_n^D}{(\lambda_n^D - \alpha)^2 + \beta^2} = 0; \\ &\sum_{n=0}^{\infty} \frac{F_n G_n(\nu)}{(\lambda_n^D - \alpha)^2 + \beta^2} = 0. \end{aligned} \tag{3.9}$$

Since $F_0 G_0(\nu)$ is always positive, neither equation in (3.9) can hold if $F_n G_n(\nu) \geq 0$, for all $n \geq 1$. Consider now either the case that $F_n G_n(\nu) \leq 0$, for all $n \geq 1$, or alternatively, the case that $F_n G_n(\nu)$ is nonzero for no more than two values of n. Rewriting (3.9) as

$$\begin{aligned} &\frac{F_0 G_0(\nu)}{(\lambda_0^D - \alpha)^2 + \beta^2} + \sum_{n=1}^{\infty} \frac{F_n G_n(\nu)}{(\lambda_n^D - \alpha)^2 + \beta^2} \frac{\lambda_n^D}{\lambda_0^D} = 0; \\ &\frac{F_0 G_0(\nu)}{(\lambda_0^D - \alpha)^2 + \beta^2} + \sum_{n=1}^{\infty} \frac{F_n G_n(\nu)}{(\lambda_n^D - \alpha)^2 + \beta^2} = 0, \end{aligned}$$

it follows that the two equations in (3.9) cannot hold simultaneously. □

Proof of Theorem 6. *i.* Since $G_n(\mu_{rev}) = F_n$, it follows from part (i) of Theorem 4 that all the eigenvalues of $\mathcal{L}$ are real.

ii. By part (i), $\gamma_1 = \gamma_1(\mu_{\text{rev}})$ is itself an eigenvalue; let ϕ_1 denote a corresponding eigenfunction. Let $\psi_1 = \phi_1 - c$, where $c = \phi_1|_{\partial D} = \int_D \phi_1 d\mu_{\text{rev}}$. Then $\psi_1|_{\partial D} = \int_D \psi_1 d\mu_{\text{rev}} = 0$ and $L\psi_1 = \gamma_1\psi_1 + \gamma_1 c$. Multiplying this

equation by $\psi_1 \exp(2Q)$ and integrating by parts gives

$$\gamma_1 = -\frac{\frac{1}{2}\int_D (\nabla\psi_1 a \nabla\psi_1) d\mu_{\text{rev}}}{\int_D \psi_1^2 d\mu_{\text{rev}}}.$$

On the other hand, consider the quotient $\frac{\frac{1}{2}\int_D (\nabla u a \nabla u) d\mu_{\text{rev}}}{\int_D u^2 d\mu_{\text{rev}}}$. By standard methods, the infimum of this quotient over functions $0 \neq u \in H_0^1(D)$ satisfying $u|_{\partial D} = \int_D u d\mu_{\text{rev}} = 0$ exists. We denote this infimum by $-\Gamma > 0$. To identify the minimum, we use a Lagrange multiplier and vary the quantity $\frac{1}{2}\int_D (\nabla u a \nabla u) d\mu_{\text{rev}} + k\int_D u^2 d\mu_{\text{rev}}$ over functions u satisfying the above restriction, where k is a free parameter. A minimizer ψ must satisfy the equation $\int_D q(L\psi - k\psi)\, d\mu_{\text{rev}} = 0$, for all q satisfying the above restriction. From this one concludes that $L\psi = k\psi + C$, for some constant C. Multiplying this equation by ψ, integrating both sides with respect to $d\mu_{\text{rev}}$, and integrating by parts, one finds that $k = \Gamma$. Letting $\phi = \psi + \frac{C}{\Gamma}$, it follows that ϕ satisfies $L\phi = \Gamma\phi$ and $\phi|_{\partial D} = \int_D \phi d\mu_{\text{rev}}$.

iii. By part (i) and the definition of $\gamma_1(\mu_{\text{rev}})$, it follows that $\gamma_1(\mu_{\text{rev}})$ is the largest nonzero eigenvalue of $\mathcal{L}$. And then by Theorem 5 it follows that $\gamma_1(\mu_{rev}) < \lambda_0^D$. In Theorem 2, note that when $\nu = \mu_{rev}$, then $G_n(\mu_{rev}) = F_n$. Consequently $E_{\mu_{rev}}(\lambda) = \sum_{n=0}^{\infty} \frac{F_n^2}{\lambda_n^D - \lambda}$. Since $E_{\mu_{rev}}(\lambda)$ is continuous for $\lambda \in (\lambda_1^D, \lambda_0^D)$, since $E_{\mu_{rev}}((\lambda_0^D)^-) = \infty$ and since $E_{\mu_{rev}}((\lambda_1^D)^+) = -\infty$ holds if $F_j = \int_D \phi_j^D d\mu_{\text{rev}} \neq 0$, for some $j \in \{1, 2, \cdots, k_0\}$, it follows that $E_{\mu_{rev}}$ possesses a root in $(\lambda_1^D, \lambda_0^D)$ if $F_j \neq 0$ for some $j \in \{1, 2, \cdots, k_0\}$. It now follows from Theorem 2 that $\lambda_1^D \leq \gamma_1(\mu_{rev}) < \lambda_0^D$, with strict inequality if $F_j \neq 0$, for some $j \in \{1, 2, \cdots, k_0\}$. Furthermore, since $E_{\mu_{rev}}$ is increasing on $(\lambda_1^D, \lambda_0^D)$, if follows that in the case that $F_j = 0$ for all $j \in \{1, 2, \cdots, k_0\}$, the strict inequality will hold if and only if $E_{\mu_{rev}}(\lambda_1^D) < 0$. This inequality can be rewritten as (1.3). □

Proof of Proposition 1. By Theorem 6-iii, $\gamma_1(\mu_{rev}) < \lambda_0^D$. To prove the rest of the proposition, we apply (1.3) from Theorem 6. The complete, orthonormal sequence of eigenfunctions on $L^2(D, l_d)$ for $\frac{1}{2}\Delta$ on $D \equiv (0,1)^d$

with the Dirichlet boundary condition is given by $\{2^{\frac{d}{2}}\prod_{j=1}^{d}\sin n_j\pi x_j\}_{n_1,\cdots,n_d=1}^{\infty}$. The corresponding eigenvalues are $\{-\frac{\pi^2}{2}\sum_{j=1}^{d}n_j^2\}_{n_1,\cdots,n_d=1}^{\infty}$. We will denote these eigenfunctions and eigenvalues respectively by $\phi^D_{n_1,\cdots,n_d}$ and $\lambda^D_{n_1,\cdots,n_d}$. We have

$$F_{n_1,\cdots,n_d}\equiv\int_D\phi^D_{n_1,\cdots,n_d}\,dx=\begin{cases}\frac{2^{\frac{3}{2}d}}{\pi^d\prod_{j=1}^{d}n_j}, & \text{if } n_j \text{ is odd for all } j;\\ 0, & \text{otherwise.}\end{cases}\tag{3.10}$$

In the present context, the terms F_0,λ^D_0 and λ^D_1 appearing in Theorem 6 are given respectively by $F_{1,\cdots,1}=\frac{2^{\frac{3}{2}d}}{\pi^d}$, $\lambda_{1,\cdots,1}=-\frac{d\pi^2}{2}$ and $\lambda_{n_1,\cdots,n_d}=-\frac{(d+3)\pi^2}{2}$, where $(n_1\cdots,n_d)$ satisfies $\sum_{j=1}^{d}n_j=d+1$. From (3.10), we have $F_{n_1,\cdots,n_d}=\int_D\phi^D_{n_1,\cdots,n_d}\,d\mu_{\text{rev}}=0$, if $\sum_{j=1}^{d}n_j=d+1$. Thus, (1.3) is applicable.

Using $\{\lambda_{n_1,\cdots,n_d}\}$ and $\{F_{n_1,\cdots,n_d}\}$ in place of the labeling $\{\lambda_n\}$ and $\{F_n\}$ in the inequality (1.3), we find that after cancellations the inequality can be written as

$$\sum_{\substack{n_1,\cdots,n_d \text{ odd}\\(n_1,\cdots,n_d)\neq(1,\cdots,1)}}\frac{1}{\prod_{j=1}^{d}n_j^2\left(\sum_{j=1}^{d}n_j^2-d-3\right)}>\frac{1}{3}.\tag{3.11}$$

Thus, by (1.3), (3.11) is a necessary and sufficient condition in order that $\gamma_1(\mu_{\text{rev}})>\lambda^D_1$, and if the condition does not hold, then $\gamma_1(\mu_{\text{rev}})=\lambda^D_1$. Denote the left hand side of (3.11) by H_d. If one considers H_{d+1}, but restricts the summation to those multi-indices $(n_1,\cdots,n_{d+1})$ for which $n_{d+1}=1$, the resulting quantity is H_d. Thus the left hand side of (3.11) is monotone increasing in d. A direct calculation shows that the inequality in (3.11) does not hold if $d=1$. On the other hand, by considering the contribution to the left hand side of (3.11) only from those multi-indices satisfying $\sum_{j=1}^{d}n_j=d+2$, it is easy to check that the inequality in (3.11) holds if $d\geq 15$. From these observations we conclude that there exists a $d^*\in[2,15]$ such that $\gamma_1(\mu_{\text{rev}})>\lambda^D_1$, if $d\geq d^*$, and $\gamma_1(\mu_{\text{rev}})=\lambda^D_1$, if $d<d^*$. Numerical calculations shows that in fact $d^*=11$. □

Proof of Proposition 4. Without loss of generality, assume that $F_1 > 0$. Choose $\nu^{\pm}$ with density $\nu^{\pm}(x) = c_{\pm}(\phi_0^D(x) \pm \epsilon\phi_1^D(x))$, where $\epsilon > 0$ is sufficiently small so that $\nu^{\pm}(x) \geq 0$, for all $x \in D$, and where $c_{\pm} > 0$ is a normalizing constant so that $\nu^{\pm}$ is a probability density. (It is possible to choose such an $\epsilon > 0$ because the Hopf maximum principle guarantees that the normal derivative $\nabla\phi_0^D(x) \cdot n \neq 0$, for $x \in \partial D$.) Recall that $G_n(\nu^{\pm}) = \int_D \phi_n^D \, d\nu^{\pm}$. Thus, $G_0(\nu^{\pm}) = c_{\pm}$, $G_1(\nu^{\pm}) = \pm c_{\pm}\epsilon$ and $G_n(\nu^{\pm}) = 0$, for $n \geq 2$. From the definition of $E_{\nu^{\pm}}$ in Theorem 2, we have $E_{\nu^{\pm}}(\lambda) = \frac{c_{\pm}F_0}{\lambda_0^D - \lambda} \pm \frac{c_{\pm}\epsilon F_1}{\lambda_1^D - \lambda}$. Thus, $E_{\nu^-}(\lambda) \neq 0$ for $\lambda \in (\lambda_1^D, \lambda_0^D)$. By Theorems 4 and 5, $\gamma_1(\nu^-) < \lambda_0^D$. Thus, we conclude from Theorem 2 that $\gamma_1(\nu^-) < \lambda_1^D$. Since $E_{\nu^+}((\lambda_0^D)^-) = \infty$ and $E_{\nu^+}((\lambda_1^D)^+) = -\infty$, $E_{\nu^+}(\lambda)$ possesses a root in $(\lambda_1^D, \lambda_0^D)$. Thus, by Theorem 2. $\gamma_1(\nu^+) > \lambda_1^D$. □

4. Proof of Theorem 5

By assumption, $\gamma_1(\nu)$ is a real eigenvalue for $\mathcal{L}$. We need to show that $\gamma_1(\nu) < \lambda_0^D$. Let u denote a corresponding eigenfunction, and let $c = u|_D = \int_D u d\nu$. We first show that $\gamma_1(\nu) \neq \lambda_0^D$. Assume to the contrary. In this case, $c \neq 0$. Indeed, otherwise ϕ_0^D and u would both be eigenfunctions for the principal eigenvalue λ_0^D of L with the Dirichlet boundary condition. Furthermore, ϕ_0^D and u would be linearly independent since ϕ_0^D does not change sign, whereas $\int_D u d\nu = c = 0$. This would then contradict the simplicity of the principal eigenvalue λ_0^D. Integrating by parts twice, exploiting the form of the reversible operator L and the reversible measure, we have

$$\int_D \phi_0^D Lu \, d\mu_{rev} = \int_{\partial D} \phi_0^D a\nabla u \cdot n \exp(2Q) d\sigma$$
$$- \int_{\partial D} ua\nabla\phi_0^D \cdot n \exp(2Q) d\sigma + \int_D uL\phi_0^D \, d\mu_{rev},$$

which reduces to

$$\int_{\partial D} ua\nabla\phi_0^D \cdot n \exp(2Q) d\sigma = 0. \tag{4.1}$$

However,

$$\begin{aligned}\int_{\partial D} ua\nabla\phi_0^D \cdot n\exp(2Q)d\sigma &= c\int_{\partial D} a\nabla\phi_0^D \cdot n\exp(2Q)d\sigma \\ &= c\int_D L\phi_0^D d\mu_{rev} = c\lambda_0^D \int_D \phi_0^D d\mu_{rev}.\end{aligned} \tag{4.2}$$

Now (4.1) and (4.2) give $c = 0$, which is a contradiction.

Now we show that $\gamma_1(\nu) \not> \lambda_0^D$. Assume to the contrary. By the Feynman-Kac formula, $u(Y(t \wedge \tau_D))\exp(-\gamma_1(\nu)(t \wedge \tau_D))$ is a martingale. Thus,

$$\mathbb{E}_x^D u(Y(t \wedge \tau_D))\exp(-\gamma_1(\nu)(t \wedge \tau_D)) = u(x). \tag{4.3}$$

Since $\gamma_1(\nu) > \lambda_0^D$, we have $\mathbb{E}_x^D \exp(-\gamma_1(\nu)\tau_D) < \infty$ [9, chapter 3]. Thus letting $t \to \infty$ in (4.3) and applying the dominated convergence theorem gives

$$u(x) = c\mathbb{E}_x^D \exp(-\gamma_1(\nu)\tau_D). \tag{4.4}$$

It follows from (4.4) that $c \neq 0$. Integrating both sides of (4.4) against ν now gives

$$\mathbb{E}_\nu^D \exp(-\gamma_1(\nu)\tau_D) = 1, \tag{4.5}$$

which is a contradiction. □

Remark. If one does not assume that the nonzero eigenvalue with largest real part is real, the calculation in the above proof can be made with $\gamma_1(\nu)$ replaced by $\lambda_1(\nu)$, where $\lambda_1(\nu)$ is an eigenvalue for $\mathcal{L}$ whose real part is $\gamma_1(\nu)$. One arrives at (4.5) with $\gamma_1(\nu)$ replaced by $\lambda_1(\nu)$. However, since $\lambda_1(\nu)$ can be complex-valued, (4.5) no longer constitutes a contradiction.

5. Proof of Proposition 2

i. The eigenvalue problem for $\mathcal{L}$ is

$$\begin{cases} \frac{1}{2}u'' = \lambda u \text{ in } (0,1); \\ u(p) = u(0) = u(1). \end{cases} \tag{5.1}$$

Every solution u to the differential equation in (5.1) is given by

$$u(x) = A\cos\kappa x + B\sin\kappa x,$$

where $\lambda = -\frac{1}{2}\kappa^2$. In order that such a solution also satisfy the boundary condition in (5.1), the following system of linear equations must have a nontrivial solution:

$$\begin{aligned} A\,(1-\cos\kappa p) - B\sin\kappa p &= 0; \\ A\,(1-\cos\kappa) - B\sin\kappa &= 0. \end{aligned} \tag{5.2}$$

The determinant of the linear system above is

(5.3)

$$\begin{aligned} -(1-\cos\kappa p)\sin\kappa + \sin\kappa p(1-\cos\kappa) \\ &= \sin\kappa(1-p) - \sin\kappa + \sin\kappa p \\ &= 2\sin\frac{\kappa(1-p)}{2}\cos\frac{\kappa(1-p)}{2} + 2\cos\frac{\kappa(p+1)}{2}\sin\kappa(p-1)2 \\ &= 2\sin\frac{\kappa(1-p)}{2}\left(\cos\frac{\kappa(1-p)}{2} - \cos\frac{\kappa(p+1)}{2}\right) \\ &= 4\sin\frac{\kappa(1-p)}{2}\sin\frac{\kappa}{2}\sin\frac{\kappa p}{2}. \end{aligned}$$

Since $\sin x = 0$ if and only if $x = \pi n$ for some integer n, the solutions κ of (5.3) are all real and are given by

$$\frac{2\pi n}{1-p},\ 2\pi n, \frac{2\pi n}{p},\ n \in \mathbb{Z}.$$

Therefore, the eigenvalues for $\mathcal{L}$ are

$$-\frac{2\pi^2 n^2}{(1-p)^2},\ -2\pi^2 n^2,\ -\frac{2\pi^2 n^2}{p^2},\ n \in \mathbb{N}.$$

Thus, the non-zero eigenvalue with maximal real part is $-2\pi^2$.

ii. By assumption, the nonzero eigenvalue with largest real part is real, and we know that it is negative. Thus, $\gamma_1(\nu)$ is the largest negative number γ

for which there is a solution to the following problem:

$$\begin{cases} \frac{1}{2}u'' = \gamma u & \text{in } (0,1); \\ u(0) = u(1) = \int_0^1 u \, d\nu. \end{cases} \tag{5.4}$$

Every solution to the differential equation in (5.4) with $\gamma < 0$ is of the form

$$u(x) = A\cos\kappa x + B\sin\kappa x,$$

where $\gamma = -\frac{1}{2}\kappa^2$, for some $\kappa \in R - \{0\}$. In order that (5.4) have a solution, the following system of linear equations must have a nontrivial solution:

$$\begin{aligned} A\left(1 - \int_0^1 \cos\kappa x \, d\nu\right) - B\int_0^1 \sin\kappa x \, d\nu = 0; \\ A\left(1 - \cos\kappa\right) - B\sin\kappa = 0. \end{aligned} \tag{5.5}$$

The determinant of the linear system above is

$$\begin{aligned} &-(1 - \int_0^1 \cos\kappa x \, d\nu)\sin\kappa + (1-\cos\kappa)\int_0^1 \sin\kappa x \, d\nu \\ &= \int_0^1 \sin\kappa(1-x) \, d\nu - \sin\kappa + \int_0^1 \sin\kappa x \, d\nu \\ &= 2\int_0^1 \sin\frac{\kappa(1-x)}{2}\cos\frac{\kappa(1-x)}{2} \, d\nu + 2\int_0^1 \cos\frac{\kappa(x+1)}{2}\sin\frac{\kappa(x-1)}{2} \, d\nu \\ &= 2\int_0^1 \sin\frac{\kappa(1-x)}{2}\left(\cos\frac{\kappa(1-x)}{2} - \cos\frac{\kappa(x+1)}{2}\right) d\nu \\ &= 4\int_0^1 \sin\frac{\kappa(1-x)}{2}\sin\frac{\kappa}{2}\sin\frac{\kappa x}{2} \, d\nu. \end{aligned}$$

Note that $\sin\frac{\kappa(1-x)}{2}\sin\frac{\kappa}{2}\sin\frac{\kappa x}{2} > 0$, for $\kappa \in (0, 2\pi)$ and $x \in (0,1)$, while the reverse inequality holds for $\kappa \in (-2\pi, 0)$ and $x \in (0,1)$. Thus, it follows that (5.5) has no solution $\kappa \in (-2\pi, 2\pi) - \{0\}$. On the other hand, $\sin\frac{\kappa(1-x)}{2}\sin\frac{\kappa}{2}\sin\frac{\kappa x}{2} \equiv 0$, for $\kappa = \pm 2\pi$. Thus, it follows that $\gamma = -\frac{1}{2}(2\pi)^2 = -2\pi^2$ is the largest real nonzero solution to (5.4). $\square$.

6. Proof of Proposition 3

By Theorem 3 (part (ii) or part(iii)), Theorem 2 holds for all jump measures ν. We will show that for an appropriate ν one has $E_\nu(\lambda) = 0$, for some $\lambda > \lambda_1^D$.

We use the notation and the calculations in the proof of Proposition 1. Let $\nu_0 = \delta_{(\frac{1}{9},\frac{1}{9})}$. We have $G_{n_1,n_2}(\nu_0) = \int_D \phi^D_{n_1,n_2}\, d\nu_0 = \phi^D_{n_1,n_2}(\frac{1}{9},\frac{1}{9}) = 2\sin(\frac{n_1}{9}\pi)\sin(\frac{n_2}{9}\pi)$. Then from the definition of $E_{\nu_0}(\lambda)$ it follows that

$$E_{\nu_0}(\lambda) = C\sum_{m_1,m_2=0}^{\infty} \frac{\sin(\frac{2m_1+1}{9}\pi)\sin(\frac{2m_2+1}{9}\pi)}{(2m_1+1)(2m_2+1)\left((2m_1+1)^2+(2m_2+1)^2+\frac{2}{\pi^2}\lambda\right)}, \tag{6.1}$$

for an appropriate negative constant C. We will show that the equation $E_{\nu_0}(\lambda) = 0$ has a root $\lambda \in (\lambda_1^D, \lambda_0^D) = (-\frac{5}{2}\pi^2, -\pi^2)$. Note that $E_{\nu_0}((-\pi^2)^-) = \infty$. Thus, it suffices to show that $E_{\nu_0}(-\frac{5\pi^2}{2}) < 0$. This can be checked using a program such as Mathematica, or alternatively, by a page and a half of estimates which we refrain from reproducing here.

7. Proof of Theorem 3

For all three parts of the theorem, we will need the following comparison result. By the mini-max principle [10], one can compare the eigenvalues $\{\lambda_n^D\}_{n=0}^\infty$ of L in D to those of $\frac{1}{2}\Delta$ in $(0,1)^d$, and conclude that there exist $c_1, c_2 > 0$ (depending on L and D) such that $c_1\hat{\lambda}_n \le \lambda_n^D \le c_2\hat{\lambda}_n$, where $\{\hat{\lambda}_n\}_{n=0}^\infty$ are the eigenvalues for $\frac{1}{2}\Delta$ in $(0,1)^d$, labelled in nonincreasing order.

i. It is known that the eigenfunctions $\{\phi_n^D\}_{n=0}^\infty$ are uniformly bounded [6, pp.270-273]. Thus, using the Cauchy-Schwarz inequality, it is enough to show that $\sum_{n=0}^\infty \frac{1}{(\lambda_n^D)^2} < \infty$. By the comparison principle above, it suffices to show the above inequality in the case that $L = \frac{1}{2}\frac{d^2}{dx^2}$ in $D = (0,1)$. In this case, $\lambda_n^D = -\frac{(n+1)^2\pi^2}{2}$.

ii. When L is the Laplacian on a Riemannian manifold, it is known that $|\phi_n^D| \leq C|\lambda_n^D|^{\frac{d-1}{4}}$, for some $C > 0$ [2]. Using this and applying the Cauchy-Schwarz inequality, it follows that $\sum_{n=0}^{\infty} \frac{F_n}{\lambda_n^D}\phi_n^D(x)$, converges uniformly and absolutely when $d = 2$ if $\sum_{n=0}^{\infty} |\lambda_n^D|^{-\frac{3}{2}}$ converges. By the mini-max principle above, it suffices to show the above inequality in the case that $L = \frac{1}{2}\Delta$ on $(0,1)^2$. But this then follows from Weyl's asymptotic distribution of eigenvalues [10] which gives $\lambda_n \sim cn$.

iii. As in part (i), it is enough to show that $\sum_{n=0}^{\infty} \frac{1}{(\lambda_n^D)^2} < \infty$, and by the comparison principle above, it suffices to show the above inequality in the case that $L = \frac{1}{2}\Delta$ on $(0,1)^d$, $d \leq 3$. But this then follows from Weyl's asymptotic distribution of eigenvalues [10], which gives $\lambda_n \sim cn^{\frac{2}{d}}$.

□

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