FINITE TIME BLOW-UP FOR THE INHOMOGENEOUS EQUATION $u_t = \Delta u + a(x)u^p + \lambda\phi$ IN R^d

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Abstract. We consider the inhomogeneous equation

$$u_t = \Delta u + a(x)u^p + \lambda\phi(x) \text{ in } R^d, t \in (0,T)$$
$$u(x,0) = f(x),$$

where $a, \phi \gneqq 0$, $\lambda > 0$ and $f \geq 0$, and give criteria on p, d, a, and ϕ which determine whether for all λ and all f the solution blows up in finite time or whether for λ and f sufficiently small, the solution exists for all time.

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1. Introduction and Statement of Results. In this paper, we investigate the existence and nonexistence of nonnegative global solutions for the equation

$$(1.1)\qquad \begin{aligned} &u_t = \Delta u + a(x)u^p + \lambda\phi(x) \text{ in } R^d, t \in (0,T) \\ &u(x,0) = f, \end{aligned}$$

where $0 \lneqq a, \phi \in C^\alpha(R^d)$, a and ϕ grow no faster than polynomially, $p > 1$, $\lambda > 0$, and $f \geq 0$.

Under these conditions, it is well known [4] that there exists a unique, bounded, nonnegative solution to (1.1), defined on a maximal time interval $[0, T^*)$, and that if $T^* < \infty$, then $\lim_{t\to T^*} \sup_{x\in R^d} u(x,t) = \infty$.

In a recent paper [5], we studied the same question for the homogeneous equation ($\lambda = 0$) with nonzero initial data ($f \gneqq 0$). It was shown that if $d \geq 2$ and $a(x)$ behaves on the order $|x|^m$ as $|x| \to \infty$, then if $1 < p \leq 1 + \frac{2+m}{d}$, every solution to the homogeneous equation with nonzero initial data blows up in finite time, while if $p > 1 + \frac{(2+m)^+}{d}$, then for sufficiently small initial data, there exist global solutions. (Results were also obtained in one dimension, but the formulas are a little different; in particular, if $d = 1$ and $1 < p \leq 2$, then all solutions blow up in finite time even if a has compact support.)

Our initial interest in (1.1) stemmed from its importance in the study of large deviations for super Brownian motion in the case that $p = 2$, $a(x) = 1$, and ϕ has compact support [2]. In any case, the study of (1.1) seems a natural follow-up to the study of the homogeneous case described above.

Note that if $u(x,t)$ solves (1.1), then by the maximum principle, for $\epsilon > 0$, the function $v(x,t) \equiv u(x, t+\epsilon)$ is larger than the solution to the homogeneous equation with initial data $u(x,\epsilon) \gneqq 0$. Thus, if d, p, and m fall in the blowup range for the homogeneous problem, they must also fall in the blowup range for the inhomogeneous one, for any choice of ϕ and λ. We will prove the following theorem.

Theorem. *a. Let $d \leq 2$ or let $d \geq 3$ and*

$$\int_{R^d} \frac{\phi(y)}{|y|^{d-2}} dy = \infty.$$

Then for any $a \gneqq 0$, any $\lambda > 0$, and any $f \geq 0$, the solution to (1.1) blows up in finite time.

b. Let $d \geq 3$.

1-i. Assume that

$$a(x) \geq c|x|^m, \text{ for large } |x| \text{ and } c > 0,$$

that

$$\phi(x) \geq c|x|^{-q}, \text{ for large } |x| \text{ and } c > 0, \text{ where } q \in (2,d),$$

and that

$$1 < p < 1 + \frac{2+m}{q-2}.$$

Then for any $\lambda > 0$ and any $f \geq 0$, the solution to (1.1) blows up in finite time.

1-ii. Assume that

$$a(x) \geq c|x|^m, \text{ for large } |x| \text{ and } c > 0,$$

that

$$0 \lneqq \phi(x) \leq c|x|^{-d}, \text{ for some } c > 0,$$

and that

$$1 < p \leq 1 + \frac{2+m}{d-2}.$$

Then for every $\lambda > 0$ and any $f \geq 0$, the solution to (1.1) blows up in finite time.

2. Assume that

$$0 \lneqq a(x) \leq c|x|^m, \text{ for large } |x| \text{ and } c > 0,$$

that

$$(1.2) \qquad 0 \lneqq \phi(x) \leq c|x|^{-q}, \text{ for large } |x| \text{ and } c > 0, \text{ where } q \in (2, d],$$

and that

$$p > 1 + \frac{(2+m)^+}{q-2}.$$

Then for sufficiently small $\lambda > 0$ and for sufficiently small $f \geq 0$, the solution to (1.1) exists for all time. More specifically, for sufficiently small $\lambda > 0$ and each $\epsilon > 0$, there exists a constant $c > 0$ such that if

$$(1.3) \qquad 0 \leq f(x) \leq c(1 + |x|)^{-\frac{(2+m)^+}{p-1} - \epsilon},$$

then the solution to (1.1) exists for all time.

Futhermore, if $f \equiv 0$, then $U(x) \equiv \lim_{t\to\infty} u(x,t)$ exists, $U \in C^2(R^d)$, and $\Delta U + a(x)U^p + \lambda\phi(x) = 0$ in R^d.

Remark 1. Note that if $\phi(x) \leq c|x|^{-d}$ and $m > -2$, then the critical case, $p = 1 + \frac{2+m}{d-2}$, belongs to the blow-up regime. However, when $c_1|x|^{-q} \leq \phi(x) \leq c_2|x|^{-q}$, for large $|x|$, with $q \in (2, d)$, and $m > -2$, we have been unable to prove that blow-up occurs in the critical case, $p = 1 + \frac{2+m}{q-2}$. We expect that blow-up does occur in this case as we are unaware of examples where blow-up has been proven not to occur in the critical case. [3].

Remark 2. Note that if $p > 1 + \frac{(2+m)^+}{q-2}$, then $q - 2 > \frac{(2+m)^+}{p-1}$; thus, in particular, global existence occurs in part (b-2) when $\lambda > 0$ and $c > 0$ are sufficiently small and $f(x) \leq c(1 + |x|)^{2-q}$.

Remark 3. A referee has brought to my attention the paper [6] which includes the result of this paper in the case that $a(x) \equiv 1$, $\phi(x) \leq c|x|^{-d-\epsilon}$, for large $|x|$ and some $\epsilon > 0$, and $f(x) \leq c_1(1 + |x|)^{-d-\epsilon}$. More specifically, it is proved in [6] that if ϕ and f satisfy the above bounds, and $p > \frac{d}{d-2}$, then for sufficiently small $\lambda > 0$ and sufficiently small $c_1 > 0$, the solution to (1.1) exists for all time, while if $\phi \gneqq 0$

and $p \le \frac{d}{d-2}$, then for all $\lambda > 0$ and all $f \ge 0$, the solution to (1.1) blows up in finite time.

In the sequel, we will use the notation $P(t,x,y) = (4\pi t)^{-\frac{d}{2}} \exp(-\frac{|y-x|^2}{4t})$.

2. Proof of Theorem. *Proof of part a.* Assume to the contrary that $T^* = \infty$. Let w denote the solution to (1.1) when $a(x) \equiv 0$. By the maximum principle, $u(x,t) \ge w(x,t) = \int_0^t \int_{R^d} P(s,x,y)\phi(y)dyds$. Thus, under the condition in (a), it follows that

$$\lim_{t\to\infty} u(x,t) = \infty, \text{ for all } x \in R^d. \tag{2.1}$$

Choose a smooth bounded domain $D \in R^d$ and a $\delta > 0$ such that $\inf_{x\in D} a(x) \ge \delta$. Let $\mu > 0$ denote the principal eigenvalue of $-\Delta$ in D, and let ψ, normalized by $\int_D \psi(x)dx = 1$, denote the corresponding positive eigenfunction. Define

$$F(t) = \int_D u(x,t)\psi(x)dx.$$

Using (1.1), integrating by parts, and using Jensen's inequality, we obtain

$$\begin{aligned} F'(t) &= \int_D u_t(x,t)\psi(x)dx = \int_D (\Delta u(x,t) + a(x)u^p(x,t) + \phi(x))\psi(x)dx \\ &\ge -\mu F(t) + \delta F^p(t) + \int_D \phi(x)\psi(x)dx. \end{aligned} \tag{2.2}$$

By (2.1), it follows that $\lim_{t\to\infty} F(t) = \infty$. Since the right hand side of (2.2) is positive and increasing as a function of F for $F > (\frac{\mu}{\delta})^{\frac{1}{p-1}}$, it follows that $F(t)$ blows up in finite time, contradicting the assumption that $T^* = \infty$. □

Proof of part b-1.

We may assume that $m > -2$ since otherwise there is nothing to prove.

We begin with a very simple proof which works in the case that $1 < p < 1 + \frac{2+m}{q-2}$ and

$$\phi(x) \ge c|x|^{-q}, \text{ for large } |x|, \text{ with } q \in (2,d] \text{ and } c > 0. \tag{2.3}$$

Assume to the contrary that $T^* = \infty$. Define $D_n = \{n < |x| < 2n\}$, let $\mu_n > 0$ denote the principal eigenvalue for $-\Delta$ in D_n, and let ψ_n, normalized by $\int_{D_n} \psi_n(x)dx = 1$, denote the corresponding positive eigenfunction. Note that by scaling, we have $\mu_n = \frac{\mu_1}{n^2}$ and $\psi_n(x) = n^{-d}\psi_1(\frac{x}{n})$. We will assume that n is sufficently large so that $a(x) \ge c|x|^m$, for $|x| \ge n$ and some $c > 0$, and so that (2.3) holds for $|x| \ge n$. Define

$$F_n(t) = \int_{D_n} u(x,t)\psi_n(x)dx.$$

Using (1.1), integrating by parts, and using Jensen's inequality, we obtain

$$F_n'(t) \ge -\frac{\mu_1}{n^2} F_n(t) + c_1 n^m F_n^p(t) + \int_D \phi(x)\psi_n(x)dx, \tag{2.4}$$

where $c_1 > 0$. By elementary calculus, we have

$$-\frac{\mu_1}{n^2}F_n(t) + c_1 n^m F^p(t) \geq -\gamma n^{\frac{-m-2p}{p-1}}, t \geq 0, \tag{2.5}$$

for some $\gamma > 0$. Also, by (2.3), we have

$$\int_{D_n} \phi(x)\psi_n(x)dx = \int_{D_1} \phi(nx)\psi_1(x)dx \geq c_2 n^{-q}, \tag{2.6}$$

for some $c_2 > 0$. By assumption, $p < 1 + \frac{2+m}{q-2}$, or equivalently, $-q > \frac{-m-2p}{p-1}$. Thus, it follows from (2.4), (2.5) and (2.6) that for sufficiently large n, the right hand side of (2.4) is positive and bounded away from 0 in t. Consequently, it follows that $F_n(t)$ blows up in finite time, contradicting the assumption that $T^* = \infty$.

We now turn to a more involved proof which works in all cases under consideration in the theorem. In the sequel, c will denote a positive constant whose value may change from expression to expression. We assume to the contrary that $T^* = \infty$ and define F_n as above. Ignoring the last term on the right hand side of (2.4), we have

$$F_n'(t) \geq -\frac{\mu_1}{n^2}F_n(t) + c_1 n^m F_n^p(t). \tag{2.7}$$

The function $g(z) = -\frac{\mu_1}{n^2}z + c_1 n^m z^p$ is positive and increasing for $z > cn^{\frac{-2-m}{p-1}}$, for an appropriate $c > 0$. Therefore, if for large n, we find a t_n and a γ_n satisfying

$$F_n(t_n) \geq \gamma_n n^{\frac{-2-m}{p-1}} \text{ and } \lim_{n\to\infty} \gamma_n = \infty, \tag{2.8}$$

it will follow from (2.7) that for n sufficiently large, $F_n(t)$ blows up in finite time, contradicting the assumption that $T^* = \infty$. Thus, to complete the proof, it remains to demonstrate (2.8).

We begin with two lemmas.

Lemma 1. *i. Let $\phi(x) \geq c|x|^{-q}$, for large $|x|$ and some $c > 0$, where $q < d$. Then for any $r \geq 0$, there exists a $c_1 > 0$ such that*

$$\int_{|x|\geq r} P(t,0,x)\phi(x)dx \geq c_1(1+t)^{-\frac{q}{2}}.$$

ii. Let $\phi(x) \geq c|x|^{-d}$ for large $|x|$ and some $c > 0$. Then for any $r \geq 0$, there exists a $c_1 > 0$ such that

$$\int_{|x|\geq r} P(t,0,x)\phi(x)dx \geq c_1(1+t)^{-\frac{d}{2}}\log(2+t).$$

iii. Let $\phi(x) \gneqq 0$. Then for any $r \geq 0$, there exists a $c_1 > 0$ such that

$$\int_{|x|\geq r} P(t,0,x)\phi(x)dx \geq c_1(1+t)^{-\frac{d}{2}}.$$

Proof. We leave this exercise in advanced calculus to the reader. □.

Lemma 2. *i. Let $\phi(x) \geq c|x|^{-q}$ for large $|x|$ and some $c > 0$, where $q \in (2, d)$. Then there exists a $c_1 > 0$ such that*

$$\int_0^t \int_{R^d} P(s, x, y)\phi(y)dyds \geq c_1|x|^{2-q} \exp(-\frac{|x|^2}{t}), \text{ for } |x| \geq 1 \text{ and } t \geq 2.$$

ii. Let $\phi(x) \gneqq 0$. Then there exists a $c_1 > 0$ such that

$$\int_0^t \int_{R^d} P(s, x, y)\phi(y)dyds \geq c_1|x|^{2-d} \exp(-\frac{|x|^2}{t}), \text{ for } |x| \geq 1 \text{ and } t \geq 2.$$

Proof. Define

$$\begin{aligned} &\gamma = q, \text{ if } \phi \text{ satisfies the condition in part (i) of Lemma 2} \\ &\gamma = d, \text{ if } \phi \text{ satisfies the condition in part (ii) of the Lemma 2.} \end{aligned} \tag{2.9}$$

Using Lemma 1 and the fact that $P(s, x, y) \geq c \exp(-\frac{|x|^2}{2s})P(\frac{s}{2}, 0, y)$, we have

$$\begin{aligned} &\int_0^t \int_{R^d} P(s, x, y)\phi(y)dyds \geq c \int_0^t ds \exp(-\frac{|x|^2}{2s}) \int_{R^d} P(\frac{s}{2}, 0, y)\phi(y)dy \\ &\geq c \int_0^t \exp(-\frac{|x|^2}{2s})(1 + s)^{-\frac{\gamma}{2}} ds. \end{aligned}$$

Making the change of variables, $r = \frac{|x|^2}{s}$, on the right hand side above, we obtain

$$\int_0^t \int_{R^d} P(s, x, y)\phi(y)dyds \geq c|x|^2 \int_{\frac{|x|^2}{t}}^{\infty} \exp(-\frac{r}{2})(r + |x|^2)^{-\frac{\gamma}{2}} r^{\frac{\gamma}{2}-2} dr. \tag{2.10}$$

As in the statement of the lemma, we assume that $|x| \geq 1$ and $t \geq 2$. If in addition, $\frac{|x|^2}{t} \leq 1$, then from (2.10), it is clear that the statement of the lemma holds. Now consider the case that $\frac{|x|^2}{t} \geq 1$. Since $r^{\frac{\gamma}{2}-2} \exp(-\frac{r}{2}) \geq c \exp(-\frac{2r}{3})$, for $r \geq 1$, we have

$$\int_0^t \int_{R^d} p(s, x, y)\phi(y)dyds \geq c|x|^2 \int_{\frac{|x|^2}{t}}^{\frac{3|x|^2}{2t}} \exp(-\frac{2r}{3})(r+|x|^2)^{-\frac{\gamma}{2}} dr \geq c|x|^{2-\gamma} \exp(-\frac{|x|^2}{t}).$$

□

As is well known, the solution $u(x, t)$ to (1.1) may be obtained as the pointwise limit of successive iterations $\{u_n(x, t)\}_{n=0}^{\infty}$, where $u_0 = 0$ and

$$u_{n+1}(x, t) = \int_0^t \int_{R^d} P(t - s, x, y) \left(a(y)u_n^p(y, s) + \lambda\phi(y)\right) dyds. \tag{2.11}$$

By construction, the sequence $\{u_n\}_{n=0}^{\infty}$ is nondecreasing; thus, in particular,

$$u(x, t) \geq u_2(x, t). \tag{2.12}$$

Also, by Lemma 2,

$$u_1(x,t) \geq c_1|x|^{2-\gamma}\exp(-\frac{|x|^2}{t}), \text{ for } |x| \geq 1 \text{ and } t \geq 2, \tag{2.13}$$

where γ is as in (2.9).

By assumption, we can choose an $r \geq 1$ and a $c > 0$ such that $a(y) \geq c|y|^m$, for $|y| \geq r$. Using this, and substituting (2.13) into (2.11), we obtain

$$u_2(x,t) \geq c\int_2^t \int_{|y|\geq r} P(t-s,x,y)|y|^{p(2-\gamma)+m}\exp(-\frac{p|y|^2}{s})dyds, \text{ for } t > 2. \tag{2.14}$$

We have

(2.15)

$$P(t-s,x,y)\exp(-\frac{p|y|^2}{s}) \geq c(t-s)^{-\frac{d}{2}}\exp(-\frac{|x|^2}{2(t-s)})\exp(-\frac{(s+2p(t-s))}{2s(t-s)}|y|^2)$$
$$= c(t-s)^{-\frac{d}{2}}(\frac{s(t-s)}{s+2p(t-s)})^{\frac{d}{2}}\exp(-\frac{|x|^2}{2(t-s)})P(\frac{s(t-s)}{2(s+2p(t-s))},0,y).$$

Recall that we are assuming that either $\gamma = q \in (2,d)$ and $1 < p < 1 + \frac{2+m}{q-2}$, or that $\gamma = d$ and $1 < p \leq 1 + \frac{2+m}{d-2}$. It then follows that the exponent $p(2-\gamma)+m$ appearing in (2.14) satisfies $p(2-\gamma)+m > -d$, except when $\gamma = d$ and $p = 1+\frac{2+m}{d-2}$, in which case $p(2-\gamma)+m = -d$. Substituting (2.15) into (2.14) and using Lemma 1 along with the above observation concerning the exponent $p(2-\gamma)+m$, we obtain

(2.16)

$$u_2(x,t) \geq c\int_2^t \exp(-\frac{|x|^2}{2(t-s)})(t-s)^{-\frac{d}{2}}\left(\frac{s(t-s)}{s+2p(t-s)}\right)^{\frac{d}{2}+\frac{p}{2}(2-\gamma)+\frac{m}{2}}ds, \text{ for } t > 2,$$
$$\text{if } \gamma = q \in (2,d) \text{ and } 1 < p < 1 + \frac{2+m}{q-2}, \text{ or if } \gamma = d \text{ and } 1 < p < 1 + \frac{2+m}{d-2},$$

and

$$\begin{aligned} &u_2(x,t) \geq c\int_2^t \exp(-\frac{|x|^2}{2(t-s)})(t-s)^{-\frac{d}{2}}\log(\frac{s(t-s)}{s+2p(t-s)})ds, \text{ for } t > 2, \\ &\text{if } q = d \text{ and } p = 1 + \frac{2+m}{d-2}. \end{aligned} \tag{2.17}$$

Substituting $s = ut$ in (2.16) and integrating up to $\frac{t}{2}$ instead of up to t in order to replace the term $\exp(-\frac{|x|^2}{2(t-s)})$ by $\exp(-\frac{|x|^2}{t})$, we obtain

(2.18)

$$u_2(x,t) \geq ct^{\frac{p}{2}(2-\gamma)+\frac{m}{2}+1}\exp(-\frac{|x|^2}{t}), \text{ for } t > 5,$$
$$\text{if } \gamma = q \in (2,d) \text{ and } 1 < p < 1 + \frac{2+m}{q-2}, \text{ or if } \gamma = d \text{ and } 1 < p < 1 + \frac{2+m}{d-2}.$$

Applying this same procedure to (2.17) gives

$$\begin{aligned} &u_2(x,t) \geq ct^{-\frac{d}{2}+1}(\log t)\exp(-\frac{|x|^2}{t}), \text{ for } t > 5, \\ &\text{if } \gamma = d \text{ and } p = 1 + \frac{2+m}{d-2}. \end{aligned} \tag{2.19}$$

From (2.9), (2.12), (2.18), and (2.19), we conclude that
(2.20)

$$u(x,n^2) \geq cn^{p(2-q)+m+2}, \text{ for } |x| \leq n, \text{ if } \phi \text{ satisfies the condition in Lemma 2-i}$$
$$\text{or if } \phi \text{ satisfies the condition in Lemma 2-ii and } 1 < p < 1 + \frac{2+m}{d-2},$$

and that
(2.21)

$$u(x,n^2) \geq cn^{-d+2} \log n \text{ for } |x| \leq n, \text{ if } \phi \text{ satisfies the condition in Lemma 2-ii}$$
$$\text{and } p = 1 + \frac{2+m}{d-2}.$$

With the estimates (2.20) and (2.21), we can verify (2.8) and thereby conclude the proof. If ϕ, p, d, and m satisfy the conditions of (2.20), then letting $t_n = n^2$, we have $F_n(t_n) \geq cn^{p(2-q)+m+2}$. The condition on p, d, q and m in (2.20) insures that $p(2-q)+m+2 > \frac{-2-m}{p-1}$; therefore (2.8) holds. If ϕ, p, d, and m satisfy the conditions of (2.21), then letting $t_n = n^2$, we have $F_n(t_n) \geq cn^{-d+2} \log n$. The condition on p, d, q and m in (2.21) gives $\frac{-2-m}{p-1} = -d+2$; therefore (2.8) holds. This completes the proof of part b-1. □

Proof of part b-2. We employ a technique used in [2] for the case $p = 2$ and $a(x) = 1$. We will prove that the following inequality holds: there exists a $c_1 > 0$ and for all $\delta > 0$, a $C_\delta > 0$ such that

$$c_1(1+|x|)^{2-q} \leq \int_{R^d} \frac{(1+|y|)^{-q}}{|x-y|^{d-2}} dy \leq C_\delta(1+|x|)^{2-q+\delta}, \text{ where } q \in (2,d]. \tag{2.22}$$

We will now use (2.22) to prove global existence. Then we will return to prove (2.22). Let

$$v(x,t) \equiv v(x) = \mu \int_{R^d} \frac{(1+|y|)^{-q}}{|x-y|^{d-2}} dy, \text{ where } \mu > 0. \tag{2.23}$$

Since $a(x) \leq c_1(1+|x|)^m$ and $\phi(x) \leq c_2(1+|x|)^{-q}$, for some $q \in (2,d]$, where $c_1, c_2 > 0$, and since the Green's function for $-\Delta$ on R^d satisfies $G(x,y) = \frac{\gamma_d}{|x-y|^{d-2}}$, for an appropriate constant $\gamma_d > 0$, it follows from (2.23), from the right hand inequality in (2.22), and from the fact that $p > 1 + \frac{(2+m)^+}{q-2}$, that there exists an $\delta > 0$ such that

$$\Delta v - v_t + a(x)v^p + \lambda\phi \leq \left(-\frac{\mu}{\gamma_d} + c_1 c^p \mu^p + \lambda C_\delta\right)(1+|x|)^{-q}. \tag{2.24}$$

By first choosing μ sufficiently small so that $c_1 c^p \mu^p \leq \frac{1}{2}\frac{\mu}{\gamma_d}$, and then choosing λ sufficiently small, we can make the right hand side of (2.24) negative. Thus, by the maximum principle and by the left hand inequality in (2.22), it follows that for sufficiently small $\lambda > 0$ and for sufficiently small $c > 0$, the solution $u(x,t)$ to (1.1) with $f(x) \leq c(1+|x|)^{2-q}$ satisfies

$$u(x,t) \leq v(x), \tag{2.25}$$

and is thus a global solution. We will now show how to increase the exponent $(2-q)$ appearing in the bound for f above to the exponent $\frac{(2+m)^+}{p-1}$ appearing in (1.3) in the statement of the theorem.

We argue as follows. By assumption, $p > 1 + \frac{(2+m)^+}{q-2} = \frac{q+\max(m,-2)}{q-2}$. Note that the right hand side of the above inequality is decreasing for $q \in (2, d]$. Define $q_0 < q$ by $p = \frac{q_0+\max(m,-2)}{q_0-2}$. Then

$$2 - q_0 = -\frac{(2+m)^+}{p-1}. \tag{2.26}$$

Let $q_1 \in (q_0, q)$ and note that q_1 satisfies $p > 1 + \frac{(2+m)^+}{q_1-2}$. Thus, by the above results, it follows that if $\phi(x) \le c|x|^{-q_1}$, for large $|x|$ and some $c > 0$, then for sufficiently small $\lambda > 0$ and sufficiently small $c_1 > 0$, the solution to (1.1) with $f(x) \le c_1(1+|x|)^{2-q_1}$ is global. But then, by the maximum principal, the same holds true if we decrease ϕ so that it satisfies the original bound, $\phi(x) \le c|x|^{-q}$. Thus we have proven that (1.1) possesses a global solution if $\lambda > 0$ and $c_1 > 0$ are sufficiently small and $f(x) \le c_1(1+|x|)^{2-q_1}$. Since q_1 can be chosen arbitrarily close to q_0, it follows from (2.26) that a global solution exists if $f(x) \le c_1(1+|x|)^{-\frac{(2+m)^+}{p-1}-\epsilon}$, for some $\epsilon > 0$ and for $c_1 > 0$ sufficiently small. This completes the proof of global existence.

We now return to prove (2.22). We emphasize that c will denote a positive constant whose value may change from expression to expression. For the right hand inequality, we will show that for any $\delta \in (0, q-2)$,

$$\int_{R^d} \frac{(1+|x|)^{q-2-\delta}}{|x-y|^{d-2}(1+|y|)^q} dy$$

is bounded in x. We split the integral into three parts. Let $D_1 = \{y \in R^d : |x-y| < \frac{|x|}{2}\}$, $D_2 = \{y \in R^d : |x-y| > \frac{|x|}{2}, |x-y| < \frac{|y|}{2}\}$, and $D_3 = \{y \in R^d : |x-y| > \frac{|x|}{2}, |x-y| > \frac{|y|}{2}\}$. We have

$$\int_{D_1} \frac{(1+|x|)^{q-2-\delta}}{|x-y|^{d-2}(1+|y|)^q} dy \le c(1+|x|)^{-2-\delta} \int_{D_1} |x-y|^{2-d} dy \le c(1+|x|)^{-2-\delta}|x|^2 \le c,$$

where the first inequality uses the fact that $|y| \ge \frac{|x|}{2}$ on D_1. Also,

$$\int_{D_2} \frac{(1+|x|)^{q-2-\delta}}{|x-y|^{d-2}(1+|y|)^q} dy \le c \int_{D_2} |x-y|^{-d-\delta} dy \le c.$$

Finally, we consider $\int_{D_3} \frac{(1+|x|)^{q-2-\delta}}{|x-y|^{d-2}(1+|y|)^q} dy$. It is clear that

$$\int_{D_3 \cap \{|x-y| \le 1\}} \frac{(1+|x|)^{q-2-\delta}}{|x-y|^{d-2}(1+|y|)^q} dy \le c.$$

Letting $D_4 = D_3 \cap \{|x-y| > 1\}$, and noting that $q-d-\delta < 0$, we have

$$\int_{D_4} \frac{(1+|x|)^{q-2-\delta}}{|x-y|^{d-2}(1+|y|)^q} dy \le c \int_{D_4} \frac{|x-y|^{q-d-\delta}}{(1+|y|)^q} dy \le c \int_{D_4} \frac{|y|^{q-d-\delta}}{(1+|y|)^q} dy \le c.$$

We now prove the left hand inequality in (2.22). Since $|y| < \frac{3}{2}|x|$ on D_1, we have

$$\int_{R_d} \frac{(1+|y|)^{-q}}{|x-y|^{d-2}} dy \geq c(1+|x|)^{-q} \int_{D_1} (|x-y|^{2-d} dy \geq c(1+|x|)^{-q}|x|^2. \tag{2.27}$$

Since it is clear that $\int_{R_d} \frac{(1+|y|)^{-q}}{|x-y|^{d-2}} dy$ is bounded away from zero on $\{|x| < 1\}$, the left hand inequality in (2.22) follows from (2.27)

It remains to prove the final statement of the theorem. Since the initial condition in (1.1) is 0, it follows by comparison that $u(x,t)$ is monotone in t. Thus $U(x) \equiv \lim_{t\to\infty} u(x,t)$ exists. By (2.22), (2.23), and (2.25), it follows that for any $\delta > 0$, $U(x) \leq C_\delta(1+|x|)^{2-q+\delta}$. Using the integral equation for $u(x,t)$, namely (2.11) with u_n and u_{n+1} replaced by u, and letting $t \to \infty$, gives

$$U(x) = \int_{R^d} G(x,y)(aU^p + \lambda\phi)(y)dy,$$

where $G(x,y) = \gamma_d \frac{1}{|y-x|^{d-2}}$ is the Green's function for Δ on R^d. Note that from the above bound on U, along with the conditions on a, ϕ, and q, and the assumption that $p > 1 + \frac{(2+m)^+}{q-2}$, it follows that $(aU^p + \lambda\phi)(y) \leq c(1+|y|)^{-q}$, and $q > 2$. It then follows from standard theory [1] that $U \in C^2(R^d)$ and $\Delta U + a(x)U^p + \lambda\phi(x) = 0$ in R^d.

□

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