ONE-DIMENSIONAL DIFFUSIONS THAT EVENTUALLY STOP DOWN-CROSSING

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Abstract. Consider a diffusion process corresponding to the operator $L = \frac{1}{2}a\frac{d^2}{dx^2} + b\frac{d}{dx}$ and which is transient to $+\infty$. For $c > 0$, we give an explicit criterion in terms of the coefficients a and b which determines whether or not the diffusion almost surely eventually stops making down-crossings of length c. As a particular case, we show that if $a = 1$, then the diffusion almost surely stops making down-crossings of length c if $b(x) \ge \frac{1}{2c}\log x + \frac{\gamma}{c}\log\log x$, for some $\gamma > 1$ and for large x, but makes down-crossings of length c at arbitrarily large times if $b(x) \le \frac{1}{2c}\log x + \frac{1}{c}\log\log x$, for large x.

1. Introduction and Statement of Results

Consider the one-dimensional diffusion process $X(t)$ on R corresponding to the operator $L = \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$, where a is continuous and b is locally bounded and measurable. The question we address in this note is as follows: for $c > 0$ and a given $a = a(x)$, how large a drift $b = b(x)$ is needed in order for the process to almost surely eventually stop making down-crossings of length c?

Let P_x denote the measure on $\Omega \equiv C([0,\infty), R)$, the space of continuous functions from $[0,\infty)$ to R, induced by the above diffusion starting from $x \in R$. Denote functions in Ω by $\omega = x(\cdot,\omega)$. For each $c > 0$, define the set-valued function S_c on Ω by

$$S_c(\omega) = \{x(t,\omega) : t \ge 0 \text{ and } \exists s > t \text{ such that } x(s,\omega) \le x(t,\omega) - c\}. \tag{1.1}$$

We refer to $S_c(\omega)$ as the *c-down-crossed range of* ω. Let

$$\sigma_c(\omega) = \inf\{t \ge 0 : \exists s > t \text{ such that } x(s,\omega) \le x(t,\omega) - c\}$$

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and let

$$l_c(\omega)=\begin{cases} x(\sigma_c(\omega),\omega), \text{ if } \sigma_c(\omega)<\infty \\ \infty, \text{ if } \sigma_c(\omega)=\infty \end{cases}$$

denote the *onset location* of $S_c(\omega)$. If the diffusion process $X(t)$ is recurrent, then trivially $P_x(S_c=R)=1$, while if the diffusion almost surely converges to $-\infty$, then clearly $P_x(S_c=(-\infty,\mu])=1$, were $\mu=\sup_{t\ge 0}X(t)$. For this reason, we will be interested in the case that the diffusion almost surely converges to $+\infty$. As is well-known [2], this occurs if and only if

(1.2)
$$\int_{-\infty} \exp(-\int_0^x \frac{2b}{a}(y)dy)dx=\infty \text{ and } \int^{\infty} \exp(-\int_0^x \frac{2b}{a}(y)dy)dx<\infty.$$

Define

$$u(x)=\int_0^x \exp(-\int_0^y \frac{2b}{a}(z)dz)dy. \tag{1.3}$$

Note that u is harmonic for L; that is $Lu=0$.

Theorem 1. *Let P_x be the measure on $C([0,\infty),R)$ induced by the diffusion process $X(t)$ starting from x and corresponding to the operator $L=\frac12 a\frac{d^2}{dx^2}+b\frac{d}{dx}$. Let $c>0$.*

i. The onset location l_c of the c-down-crossed range S_c satisfies

$$P_x(l_c>x+\gamma)=\exp(-\int_x^{x+\gamma} \frac{u'(y)}{u(y)-u(y-c)}dy),\ \gamma>0,$$

where u is as in (1.3).

ii. Assume that the diffusion is transient to $+\infty$; that is, assume that (1.2) holds. Then the c-down-crossed range S_c is almost surely bounded, or equivalently, the diffusion $X(t)$ almost surely eventually stops making c-down-crossings, if and only if

$$\int^{\infty} \frac{u'(x)}{u(x)-u(x-c)}dx<\infty. \tag{1.4}$$

As an application of Theorem 1, consider Brownian motion with a drift, corresponding to the operator $L=\frac12\frac{d^2}{dx^2}+b\frac{d}{dx}$. Then $X(t)$ satisfies the stochastic differential equation $X(t)=B(t)+\int_0^t b(X(s))ds$, where $B(t)$ is a Brownian motion. We have the following result.

Theorem 2. *Consider the diffusion process $X(t)$ corresponding to the operator $L = \frac{1}{2}\frac{d^2}{dx^2} + b\frac{d}{dx}$. Let $c > 0$.*
i. If $b(x) \geq \frac{1}{2c}\log x + \frac{\gamma}{c}\log\log x$, for some $\gamma > 1$ and for sufficiently large x, then the diffusion $X(t)$ a.s. eventually stops making c-down-crossings; that is, the c-down-crossed range S_c is bounded a.s.;
ii. If $b(x) \leq \frac{1}{2c}\log x + \frac{1}{c}\log\log x$ for sufficiently large x, then the diffusion $X(t)$ a.s. makes down-crossings for arbitrarily large t; that is, the c-down-crossed range S_c is unbounded a.s.;
iii. If $\liminf_{x\to\infty}\frac{b(x)}{\log x} = \infty$, then the diffusion $X(t)$ a.s. eventually stops making c-down-crossings for all $c > 0$; that is, the c-down-crossed range S_c is bounded for all $c > 0$ a.s.;
iv. If $\limsup_{x\to\infty}\frac{b(x)}{\log x} = 0$, then the diffusion $X(t)$ a.s. makes c-down-crossings for arbitrarily large t for all $c > 0$; that is, the c-down-crossed range S_c is unbounded for all $c > 0$ a.s..

Remark. It is interesting to contrast the above down-crossing behavior with the down-crossing behavior in the case that the drift is of "diffusion type," that is of the order $O(\frac{1}{x})$ as $x \to \infty$. Consider the Bessel process on $(0, \infty)$ corresponding to the operator $\frac{1}{2}\frac{d^2}{dx^2} + \frac{k-1}{2x}\frac{d}{dx}$, and assume that $k > 2$ so that the process is transient to $+\infty$. Of course, if k is an integer, then the process corresponds to the absolute value of k-dimensional Brownian motion. Let A_n^ρ be the event that after hitting $((n+1)!)^\rho$ for the first time, the process down-crosses the interval $[(n!)^\rho, ((n+1)!)^\rho]$ before hitting $((n+2)!)^\rho$. In [1] it was shown that $P_x(A_n^\rho \text{ i.o.}) = \begin{cases} 0, \text{ if } \rho > \frac{1}{k-2}; \\ 1, \text{ if } \rho \leq \frac{1}{k-2}. \end{cases}$

2. Proofs

Proof of Theorem 1. *(i)* Let $x \in R$ and let $\gamma > 0$. For n a positive integer, define $x_k^{(n)} = x + \frac{k\gamma}{n}$, $k = 0, 1\cdots$. Let $\tau_r = \inf\{t \geq 0 : X(t) = r\}$.

By the strong Markov property, one has

(2.1)

$$\prod_{k=0}^{n-1} P_{x_k^{(n)}}(\tau_{x_{k+1}^{(n)}} < \tau_{x_{k+1}^{(n)}-c}) \le P_x(l_c > x+\gamma) \le \prod_{k=0}^{n-1} P_{x_k^{(n)}}(\tau_{x_{k+1}^{(n)}} < \tau_{x_k^{(n)}-c}).$$

As is well-known, since u is harmonic, one has $P_w(\tau_z < \tau_y) = \frac{u(w)-u(y)}{u(z)-u(y)}$, for $y < w < z$. Thus, since u is uniformly Lipschitz on bounded intervals, one has

(2.2)

$$\log \prod_{k=0}^{n-1} P_{x_k^{(n)}}(\tau_{x_{k+1}^{(n)}} < \tau_{x_k^{(n)}-c}) = \sum_{k=0}^{n-1} \log \frac{u(x_k^{(n)}) - u(x_k^{(n)}-c)}{u(x_{k+1}^{(n)}) - u(x_k^{(n)}-c)}$$
$$= \sum_{k=1}^{n-1} \log \left(1 - \frac{u(x_{k+1}^{(n)}) - u(x_k^{(n)})}{u(x_{k+1}^{(n)}) - u(x_k^{(n)}-c)}\right) = -\sum_{k=1}^{n-1} \frac{u(x_{k+1}^{(n)}) - u(x_k^{(n)})}{u(x_{k+1}^{(n)}) - u(x_k^{(n)}-c)} + O(\frac{1}{n})$$
$$= -\frac{\gamma}{n} \sum_{k=1}^{n-1} \frac{u'(z_k^{(n)})}{u(x_{k+1}^{(n)}) - u(x_k^{(n)}-c)} + O(\frac{1}{n}) \text{ as } n \to \infty, \text{ where } x_k^{(n)} \le z_k^{(n)} \le x_{k+1}^{(n)}.$$

Letting $n \to \infty$ in (2.2), one obtains

$$\lim_{n\to\infty} \log \prod_{k=0}^{n-1} P_{x_k^{(n)}}(\tau_{x_{k+1}^{(n)}} < \tau_{x_k^{(n)}-c}) = -\int_x^{x+\gamma} \frac{u'(y)}{u(y)-u(y-c)} dy. \tag{2.3}$$

An almost identical calculation shows that the left hand expression in (2.1) also converges to the right hand side of (2.3) when $n \to \infty$. Thus, one concludes from (2.1) that $P_x(l_c > x+\gamma) = \exp(-\int_x^{x+\gamma} \frac{u'(y)}{u(y)-u(y-c)} dy)$.

(ii) First assume that $\int^\infty \frac{u'(x)}{u(x)-u(x-a)} dx = \infty$. Then by part (i) the onset location l_c of S_c is a.s. finite. Since the diffusion is transient to $+\infty$, after it makes a c-down-crossing from the level l_c, it will a.s. return to the level l_c. Starting anew from $l_c^{(1)} \equiv l_c$, by part (i) the diffusion will again a.s. make a c-down-crossing with some onset location $l_c^{(2)} > l_c^{(1)}$. Continuing in this way, it follows that the set S_c is a.s. unbounded.

Now assume that $\int^\infty \frac{u'(x)}{u(x)-u(x-a)} dx < \infty$. Under P_x, the probability of ever making a c-down-crossing is $q_x \equiv 1 - \exp(-\int_x^\infty \frac{u'(y)}{u(y)-u(y-c)} dy) \in (0,1)$. If a down-crossing is made, with onset location $l_c \ge x$, then since the diffusion is transient to $+\infty$, it will a.s. eventually return to l_c. Starting anew from l_c, the probability of making another c-down-crossing is $1-\exp(-\int_{l_c}^{\infty}\frac{u'(y)}{u(y)-u(y-c)}dy)\le q_x$. Continuing like this, it follows that the diffusion will a.s. stop making c-down-crossings. □

For the proof of Theorem 2, we will need a monotonicity result. If $L_i=\frac{1}{2}a(x)\frac{d^2}{dx^2}+b_i(x)\frac{d}{dx}$, for $i=1,2$, with $b_1\le b_2$, then a well-known coupling shows that the process corresponding to L_2 and starting at some x stochastically dominates the process corresponding to L_1 and starting from the same point x. It seems intuitive that in such a case, the number of c-down-crossings of the process corresponding to L_1 should stochastically dominate the number of c-down-crossings of the process corresponding to L_2, however an appropriate, simple coupling doesn't seem obvious. We obtain such a monotonicity result by Frechét differentiating the integrand in (1.4).

Proposition 1. *Let $L_i=\frac{1}{2}a(x)\frac{d^2}{dx^2}+b_i(x)\frac{d}{dx}$, $i=1,2$, with $b_1\le b_2$. If the diffusion corresponding to L_1 eventually stops c-down-crossing, then so does the diffusion corresponding to L_2.*

Proof. Let $H(b,x)=\frac{u'(x)}{u(x)-u(x-c)}$, where u is as in (1.3). By the assumption in the proposition, the diffusion corresponding to L_1 must be transient to $+\infty$. Since $b_2\ge b_1$ the same holds for the diffusion corresponding to b_2. From Theorem 1, it suffices to show that $H(b_1,x)\ge H(b_2,x)$. To show this, it suffices to show that if $q=q(x)\ge 0$, then the Frechét derivative $H_q(b,x)=\lim_{\epsilon\to 0}\frac{H(b+\epsilon q,x)-H(b,x)}{\epsilon}\ge 0$. One calculates that

$$H_q(b,x)=\frac{\exp(-\int_0^x\frac{2b}{a}(y)dy)\int_{x-c}^x\exp(-\int_0^y\frac{2b}{a}(z)dz)(\int_y^x\frac{2q}{a}(z)dz)dy}{(\int_{x-c}^x\exp(-\int_0^x\frac{2b}{a}(y)dy))^2}\ge 0.$$

□

Proof of Theorem 2. Parts (iii) and (iv) follow immediately from parts (i) and (ii). In light of Proposition 1, to prove parts (i) and (ii), it suffices to consider the integral in (1.4) with $b(x)=\frac{1}{2c}\log x+\frac{\gamma}{c}\log\log x$, for large

x, and show that this integral is infinite if $\gamma = 1$ and finite if $\gamma > 1$. (For b of this form (1.2) holds.)

We apply l'Hôpital's rule to the quotient

$$\frac{(x \log^{2\gamma-1} x)u'(x)}{u(x) - u(x-c)} = \frac{(x \log^{2\gamma-1} x) \exp(-\int_0^x 2b(y)dy)}{\int_{x-c}^x \exp(-\int_0^y 2b(z)dz)dy}. \tag{2.4}$$

It is clear that the numerator and denominator of the right hand side of (2.4) tend to 0 as $x \to \infty$. Differentiating and doing some algebra, we obtain

(2.5)
$$\frac{\big((x \log^{2\gamma-1} x) \exp(-\int_0^x 2b(y)dy)\big)'}{(\int_{x-c}^x \exp(-\int_0^y 2b(z)dz)dy)'} = \frac{2b(x)(x \log^{2\gamma-1} x) + \text{lower order terms}}{\exp(\int_{x-c}^x 2b(y)dy) - 1}.$$

Some standard analysis shows that

$$\begin{aligned} &\int_{x-c}^x \frac{1}{c} \log y dy = \log x + 2 + o(1), \text{ as } x \to \infty, \\ &\int_{x-c}^x \frac{2\gamma}{c} \log\log y dy = 2\gamma \log\log x + o(1), \text{ as } x \to \infty. \end{aligned} \tag{2.6}$$

Using (2.4)-(2.6) along with the fact that $2b(x) = \frac{1}{c}\log x + \frac{2\gamma}{c}\log\log x$, for large x, we obtain

$$\lim_{x\to\infty} \frac{\big((x \log^{2\gamma-1} x)u'(x)\big)'}{\big(u(x) - u(x-c)\big)'} = \frac{1}{ce^2}.$$

It then follows from l'Hôpital's rule that $\frac{u'(x)}{u(x)-u(x-c)} \sim \frac{1}{ce^2}\frac{1}{x\log^{2\gamma-1} x}$, as $x \to \infty$. Thus, $\int^\infty \frac{u'(x)}{u(x)-u(x-c)}dx$ is infinite if $\gamma = 1$ and finite if $\gamma > 1$. □

References

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