DECAY OF MASS FOR THE EQUATION $u_t = \Delta u - a(x)u^p|\nabla u|^q$

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Abstract. Consider the quasilinear Cauchy problem

$$u_t = \Delta u - a(x)u^p|\nabla u|^q, x \in R^d, t > 0$$
$$u(x,0) = \phi(x) \geq 0, x \in R^d,$$

where $a > 0$, where p and q satisfy

$$p \geq 0 \text{ and } q \geq 1 \text{ or } p > 1 \text{ and } q = 0,$$

and where

$$0 \lneqq \phi \in L_1(R^d) \cap C_b^{3,\alpha}(R^d).$$

This paper proves that the above equation possesses a unique positive classical solution, and then investigates whether or not $\gamma \equiv \lim_{t\to\infty} \int_{R^d} u(x,t)dx = 0$. In particular, it is shown that if a is on the order $|x|^m$ for large $|x|$, then $\gamma = 0$ if $dp + (d+1)q \leq d + 2 + m$. Under the assumption that

$$(*) \qquad ||\nabla u_{\epsilon\phi}(\cdot,t)||_\infty \leq \frac{\epsilon c_\phi}{(t+1)^\beta}, \text{ for some } \beta \geq 0, \text{ and for } \epsilon \in (0,1],$$

where $u_{\epsilon\phi}$ denotes the solution to the above equation with initial condition $\epsilon\phi$, it is shown that $\gamma > 0$ if $dp + 2\beta q > d + 2 + \max(m, -d)$. For a certain range of the parameters d, p, q, m, it is proved that (*) holds with $\beta = \frac{d+1}{2}$, and for many other parameter values it is proved that (*) holds with $\beta = \frac{d}{2}$. Note that if $\beta = \frac{d+1}{2}$, then the above condition for $\gamma > 0$ becomes $dp + (d+1)q > d + 2 + \max(m, -d)$, which in light of the parameter restrictions, is equivalent to $dp + (d+1)q > d + 2 + m$.

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In this paper we consider positive solutions to the equation

$$u_t = \Delta u - a(x)u^p|\nabla u|^q, x \in R^d, t > 0$$
(1.1)
$$u(x,0) = \phi(x) \geq 0, x \in R^d,$$

where $a > 0$ and where p and q satisfy

(1.2) $$p \geq 0 \text{ and } q \geq 1 \text{ or } p > 1 \text{ and } q = 0.$$

We assume that

(1.3) $$0 \lneqq \phi \in L_1(R^d) \cap C_b^{3,\alpha}(R^d).$$

In the case $a \equiv 1$ and $p = 0$, (1.1) was studied in [1] and [2]. In [1], it was shown that (1.1) possesses a unique, global, positive solution $u(x,t) \geq 0$ and that this solution satisfies $||u(\cdot,t)||_\infty \leq ||\phi||_\infty$ and $||\nabla u(\cdot,t)||_\infty \leq ||\nabla\phi||_\infty$. In [2] it was shown that $\gamma \equiv \lim_{t\to\infty} \int_{R^d} u(x,t)dx$ exists and satisfies $\gamma = 0$ if $q \leq \frac{d+2}{d+1}$ and $\gamma > 0$ if $q > \frac{d+2}{d+1}$. In the case $a \equiv 1$, $q = 0$ and $p \in (1,2]$, (1.1) has important consequences in the study of super Brownian motion, and it is known that $\gamma = 0$ if $p \leq \frac{d+2}{d}$ and $\gamma > 0$ if $p > \frac{d+2}{d}$ [4,6].

The main purpose of this paper is to prove similar decay results for (1.1). However, first we establish existence and uniqueness, a maximum principle, and a comparison principle for (1.1). We will use the notation

$$p(t,x,y) = (4\pi t)^{-\frac{d}{2}} \exp(-\frac{|y-x|^2}{4t})$$

It will be useful to make the following definition.

Condition A. i. $0 < a \in C^2(R^d)$;

ii. There exists a positive integer N such that $|a^{(k)}(x)| \leq (1+|x|^2)^N$, for $k = 0,1,2$;

iii. $\sup_{x\in R^d} \frac{|\nabla a(x)|}{a(x)} < \infty$;

iv. For all $t > 0$, $\sup_{x\in R^d} a(x)(\int_{R^d} p(t,x,y)\phi(y)dy)^p < \infty$.

We will prove the following existence and uniqueness result.

Theorem 1. *Let p, q be as in (1.2), let ϕ be as in (1.3), and let a satisfy Condition A. Then there exists a unique, global, positive solution u to (1.1). The solution $u \in C^{2,1}(R^d \times [0,\infty))$ and satisfies*

$$||u(\cdot,t)||_\infty \le ||\phi||_\infty \tag{1.4}$$

and

$$||\nabla u(\cdot,t)||_\infty \le \max\left(||\nabla\phi||_\infty, ||\frac{\nabla a}{pa}||_\infty \min(\frac{||\phi||_1}{(4\pi t)^{\frac{d}{2}}}, ||\phi||_\infty)\right). \tag{1.5}$$

Furthermore, if $u_i(x,t)$ is the solution corresponding to the initial data $\phi_i, i = 1,2$, and $\phi_1 \le \phi_2$, then $u_1 \le u_2$.

We now turn to the decay of $u(x,t)$. The solution u to (1.1) satisfies

(1.6)
$$u(x,t) = \int_{R^d} p(t,x,y)\phi(y)dy - \int_0^t \int_{R^d} p(t-s,x,y)a(y)u^p(y,s)|\nabla u|^q(y,s)dyds.$$

Integrating (1.6) gives

$$\int_{R^d} u(x,t)dx = \int_{R^d} \phi(x)dx - \int_0^t \int_{R^d} a(x)u^p(x,s)|\nabla u|^q(x,s)dxds. \tag{1.7}$$

From this it follows that $\gamma \equiv \lim_{t\to\infty} \int_{R^d} u(x,t)dx$ exists.

For Theorem 2 below, we will need a very mild additional assumption on the initial condition ϕ:

$$\int_{|x|>\lambda} \phi(x)dx \le (\log\lambda)^{-\nu}, \text{ for large } \lambda \text{ and some } \nu > 0. \tag{1.8}$$

Remark. In order for a positive function $\phi \in L_1(R^d)$ not to satisfy (1.8), it must vary rather wildly. Indeed, note that (1.8) holds if $\phi(x) \le C\frac{1}{(1+|x|)^d(\log(2+|x|))^{1+\nu}}$, for some $\nu > 0$, while if $\phi(x) \ge C\frac{1}{(1+|x|)^d \log(2+|x|)}$, then $\phi \notin L^1$.

Theorem 2. *Let $u(x,t)$ be the positive solution to (1.1) with initial condition ϕ satisfying (1.3), and if $q > 0$, assume that ϕ satisfies (1.8). Let a satisfy*

$$a(x) \geq c(1+|x|)^m, \text{ for some } c > 0 \text{ and } m \in R.$$

If $dp + (d+1)q \leq d + 2 + m$, then

$$\gamma = \lim_{t\to\infty} \int_{R^d} u(x,t)dx = 0.$$

The next theorem gives a condition insuring $\gamma > 0$. The condition depends not only on the parameters p, q, m, d, but also on a hypothesized decay rate for $||\nabla u(\cdot,t)||_\infty$. The verification of the hypothesis will be treated in Proposition 1 below.

Hypothesis 1. There exists a $\beta \geq 0$ such that for every compactly supported initial condition ϕ satisfying (1.3), there exists a constant c_ϕ such that

$$||\nabla u_{\epsilon\phi}(\cdot,t)||_\infty \leq \frac{\epsilon c_\phi}{(t+1)^\beta}, \text{ for } \epsilon \in (0,1], \tag{1.9}$$

where $u_{\epsilon\phi}$ denotes the solution to (1.1) with initial condition $\epsilon\phi$.

Theorem 3. *Let $u(x,t)$ be the positive solution to (1.1) with initial condition ϕ satisfying (1.3) and assume that $p > 0$. Let $a(x)$ satisfy*

$$a(x) \leq c(1+|x|)^m, \text{ for some } c > 0 \text{ and } m \in R.$$

Assume that Hypothesis 1 is satisfied with exponent $\beta \geq 0$. If $dp + 2\beta q > d + 2 + \max(m, -d)$, then

$$\gamma = \lim_{t\to\infty} \int_{R^d} u(x,t)dx > 0.$$

Before turning to a discussion and statement of results concerning the values of β we are able to obtain in (1.9), we conjecture the actual necessary and sufficient condition for $\gamma = 0$.

Conjecture: $\gamma = 0$ if and only if $dp + (d+1)q \le d + 2 + \max(m, -d)$.

Note that excluding the case that $d = 1$ and $q = 0$, if (1.9) holds with $\beta = \frac{d+1}{2}$ whenever $dp + (d+1)q > d + 2 + \max(m, -d)$, then Theorems 2 and 3 prove the conjecture, since the condition for $\gamma > 0$ in Theorem 3 now becomes $dp+(d+1)q > d+2+\max(m, -d)$, and in light of (1.2) and the above noted exclusion, the condition for $\gamma = 0$ in Theorem 2 is equivalent to $dp + (d+1)q \le d + 2 + \max(m, -d)$. We believe that (1.9) holds with $\beta = \frac{d+1}{2}$ whenever $dp+(d+1)q > d+2+\max(m, -d)$, and perhaps even without this restriction; however we have not been able to prove this. We can prove (1.9) with $\beta = \frac{d+1}{2}$ in certain cases, and in many other cases we can prove (1.9) with $\beta = \frac{d}{2}$. These results are stated in Proposition 1 below. We haven't even bothered to record the remaining cases where the best β we can obtain is less than $\frac{d}{2}$ (note that by (1.5), $\beta = 0$ always works in (1.9)).

Proposition 1. *Let $u(x,t)$ be the positive solution to (1.1). Let a satisfy*

$$a(x) \le c(1 + |x|)^m, \text{ for some } c > 0 \text{ and } m \in R,$$

and assume that the initial data has compact support if $m > 0$.

i. Assume that $\max(m, 0) < dp - 1$ *or that $a \equiv const$ and $dp + (d+1)q > d + 2$. Then (1.9) holds with $\beta = \frac{d+1}{2}$.*

ii. Assume that $1 \le q \le 2$ and $dp + dq > d + 2 + \max(m, 0)$, or that $q > 2$ and $d(p+1) > m$. Then (1.9) holds with $\beta = \frac{d}{2}$.

Remark 1. Note that if $a = const$, then by (1.5) we have $||\nabla u(\cdot, t)||_\infty \le ||\nabla \phi||_\infty$. If such a bound were shown to hold in general, or actually if we could just show that the exponent $\frac{d}{2}$ appearing in (1.5) could be replaced by $\frac{d+1}{2}$, then a check of its proof shows that Proposition 1(ii) would hold with $\beta = \frac{d+1}{2}$, thereby proving the conjecture for a much wider class of parameter values.

Remark 2. If $a = const$, in which case $m = 0$, it follows from Proposition 1(i) and Theorem 3 that if $dp + (d+1)q > d+2$, then $\gamma > 0$. In fact this result then continues to hold for bounded a by appealing to a comparison principle whose proof is similar to that of the comparison principle appearing in Theorem 1.

Remark 3. Note that $a(x) \equiv (1+|x|^2)^{\frac{m}{2}}$, with $m \in R$, satifies Condition A(i)-(iii). If $m \leq 0$, then a also satisfies Condition A(iv) without any restriction on ϕ. If $m > 0$, then it is easy to show that a satisfies Condition A(iv) if $\phi(x) \leq C(1+|x|)^{-\frac{m}{p}}$.

Remark 4. The proof we give for Theorem 3 is somewhat simpler than the corresponding proof in [2] when $p = 0$ and $a \equiv const$; however, it breaks down when $p = 0$. It may be possible to adapt the proof in [2] and prove Theorem 3 even if $p = 0$.

Remark 5. In the case that $q = 0$ and $p > 1$, Theorems 2 and 3 show that for $d \geq 2$, $\gamma = 0$ if and only if $dp \leq d + 2 + \max(m, -d)$. We believe this to be true also if $d = 1$, but Theorems 2 and 3 only give $\gamma = 0$ if $p \leq 3 + m$, and $\gamma > 0$ if $p > 3 + \max(m, -1)$, leaving a gap when $-2 < m < -1$. We point out that $dp \leq d + 2 + \max(m, -d)$ was also the necessary and sufficient condition obtained in [7] in order that the solution to $u_t = \Delta u + a(x)u^p$ with $u(x,0) = \phi(x) \gneq 0$ blow up in finite time for *every* choice of $\phi \gneq 0$. It would be interesting to determine if some underlying connection exists between these two phenomona.

Theorem 1 is proved in section 2, Theorem 2 is proved in section 3, and Theorem 3 and Proposition 1 are proved in section 4. Throughout the proofs, C will denote a positive constant whose value may change from term to term.

2. Proof of Theorem 1. We use the notation $\Omega_t = \{(x,s) : x \in R^d, s \in [0,t]\}$. Recall that the assumption on p and q in (1.2) is that either $p \geq 0$ and $q \geq 1$, or $p > 1$ and $q = 0$. We will prove the theorem for the case $p \geq 0$ and $q > 1$. The case $p > 1$ and $q = 0$ is well-known. At the end of the proof, we will discuss how to extend the proof to the case $p \geq 0$ and $q = 1$.

Let $\epsilon, \delta > 0$. Our first step will be to prove the existence of a unique, positive, global solution to the equation

$$
\begin{aligned}
&U_t = \Delta U - a(x)(1+\delta|x|^2)^{-N}(|U|+\epsilon)^p|\nabla U|^q, \\
&U(x,0) = \phi(x),
\end{aligned} \tag{2.1}
$$

where N is as in Condition A(ii), and to show that this solution satisfies a variant of (1.5), specified below in Lemma 1. (Actually, the introduction of ϵ is only necessary when $p < 1$.) We begin by proving local existence and uniqueness for (2.1). For the time being, we suppress the dependence on ϵ and δ. It will be convenient to define $\beta(x) = a(x)(1+\delta|x|^2)^{-N}$. Note that by Condition A(ii), β is bounded. Define $U_0(x,t) = \phi(x)$, $U_1(x,t) = \int_{R^d} p(t,x,y)\phi(y)dy$, and for $n \geq 1$, define

$$
U_{n+1}(x,t) = U_1(x,t) - \int_0^t \int_{R^d} p(t-s,x,y)\beta(y)(|U_n(y,s)|+\epsilon)^p|\nabla U_n|^q(y,s)dyds. \tag{2.2}
$$

Then

$$
\begin{aligned}
\nabla U_{n+1}(x,t) = &\int_{R^d} p(t,x,y)\nabla\phi(y)dy \\
&- \int_0^t \int_{R^d} \nabla_x p(t-s,x,y)\beta(y)(|U_n(y,s)|+\epsilon)^p|\nabla U_n(y,s)|^q dyds.
\end{aligned} \tag{2.3}
$$

Let $K_n(t) = \sup_{x\in R^d}|U_n(x,t)|$ and $L_n(t) = \sup_{x\in R^d}|\nabla U_n(x,t)|, n = 0, 1, ...$ Then from (2.2), we have

$$
K_{n+1}(t) \leq K_0(t) + C\int_0^t (K_n(s)+\epsilon)^p L_n^q(s)ds. \tag{2.4}
$$

Using the fact that $\int_{R^d}|\nabla_x p(t,x,y)|dy = \frac{C}{t^{\frac{1}{2}}}$, it follows from (2.3) that

$$
L_{n+1}(t) \leq L_0(t) + C\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}(K_n(s)+\epsilon)^p L_n^q(s)ds. \tag{2.5}
$$

From (2.4) and (2.5), we obtain

$$
\begin{aligned}
&(K_{n+1}(t)+L_{n+1}(t)+\epsilon)\\
&\le (K_0(t)+L_0(t)+\epsilon)+C\int_0^t (1+\frac{1}{(t-s)^{\frac{1}{2}}})\left((K_n(s)+\epsilon)^{p+q}+L_n^{p+q}(s)\right)ds \qquad (2.6)\\
&\le (K_0(t)+L_0(t)+\epsilon)+C\int_0^t (1+\frac{1}{(t-s)^{\frac{1}{2}}})(K_n(s)+L_n(s)+\epsilon)^{p+q}ds.
\end{aligned}
$$

Noting that $K_0(t)+L_0(t)+\epsilon = ||\phi||_\infty + ||\nabla\phi||_\infty+\epsilon$, and making the inductive hypothesis $K_n(t)+L_n(t)+\epsilon \le 2(||\phi||_\infty+||\nabla\phi||_\infty+\epsilon)$, it follows inductively from (2.6) that

$$
(2.7)\quad ||U_n(\cdot,t)||_\infty + ||\nabla U_n(\cdot,t)||_\infty \le 2(||\phi||_\infty+||\nabla\phi||_\infty),\ \text{for } 0\le t\le \hat{T}, n=1,2...,
$$

where $\hat{T}$ is the positive solution to $\hat{T}+2(\hat{T})^{\frac{1}{2}} = \frac{C}{(||\phi||_\infty+||\nabla\phi||_\infty+\epsilon)^{p+q-1}}$.

Let

$$
M_n(t)=\sup_{R^d}|U_n(x,t)-U_{n-1}(x,t)| \text{ and } N_n(t)=\sup_{R^d}|\nabla U_n(x,t)-\nabla U_{n-1}(x,t)|.
$$

Using the inequality

$$
\begin{aligned}
&|(|U_n|+\epsilon)^p|\nabla U_n|^q-(|U_{n-1}|+\epsilon)^p|\nabla U_{n-1}|^q|\\
&\le C(|U_n|+\epsilon)^p\left(|\nabla U_n|^{q-1}+|\nabla U_{n-1}|^{q-1}\right)(||\nabla U_n|-|\nabla U_{n-1}||)\\
&+C|\nabla U_{n-1}|^q\left((|U_n|+\epsilon)^{p-1}+(|U_{n-1}|+\epsilon)^{p-1}\right)|U_n-U_{n-1}|
\end{aligned}
$$

along with (2.7), (2.2) and (2.3), and noting that $||\nabla U_n|-|\nabla U_{n-1}||\le N_n(t)$, we obtain, similar to (2.6),

$$
\begin{aligned}
&M_{n+1}(t)+N_{n+1}(t)\le C_0\int_0^t(1+\frac{1}{(t-s)^{\frac{1}{2}}})(M_n(s)+N_n(s))ds, \qquad (2.8)\\
&\text{for } 0\le t\le\hat{T},\ \text{for some } C_0>0.
\end{aligned}
$$

We have

$$
\begin{aligned}
&M_1(t)+N_1(t)=\sup_{x\in R^d}|\int_{R^d}p(t,x,y)(\phi(y)-\phi(x))dy|+\\
&\sup_{x\in R^d}|\int_{R^d}p(t,x,y)(\nabla\phi(y)-\nabla\phi(x))dy|.
\end{aligned}
$$

If $f \in C_b^{2,\alpha}(R^d)$, then the Jacobian $D^2 f$ is equicontinuous, and writing f in a power series of the form $f(y) = f(x) + \nabla f(x) \cdot (y-x) + (y-x)D^2 f(x)(y-x) + (y-x)(D^2 f(x^*(y)) - D^2 f(x))(y-x)$, it follows easily that $\frac{1}{t}\int_{R^d} p(t,x,y)(f(y)-f(x))dy$ converges to Δf as $t \to \infty$, uniformly in R^d. Applying this with $f = \phi$ and $f = \nabla\phi$, and recalling (1.3), it follows that there exists a constant $C_1 > 0$ such that

$$M_1(t) + N_1(t) \le C_1 t, \text{ for } 0 \le t \le \hat{T}. \tag{2.9}$$

Now make the inductive hypothesis

$$M_n(t) + N_n(t) \le C_1 (3C_0)^{n-1} t^{\frac{n+1}{2}}, \text{ for } 0 \le t \le \hat{T} \wedge 4. \tag{2.10}$$

Note from (2.9) that (2.10) holds for $n = 1$. Using (2.8) and (2.10), we have

$$\begin{aligned}
&M_{n+1}(t) + N_{n+1}(t) \le C_0 \int_0^t (1 + \frac{1}{(t-s)^{\frac{1}{2}}}) C_1 (3C_0)^{n-1} s^{\frac{n+1}{2}} ds \le \\
&3^{n-1} C_1 C_0^n (\frac{2}{n+3}) t^{\frac{n+3}{2}} + 2 \cdot 3^{n-1} C_1 C_0^n t^{\frac{n+2}{2}} \le \\
&3^{n-1} C_1 C_0^n t^{\frac{n+2}{2}} (\frac{2t^{\frac{1}{2}}}{n+3} + 2) \le C_1 (3C_0)^n t^{\frac{n+2}{2}}, \text{ for } 0 \le t \le \hat{T} \wedge 4.
\end{aligned}$$

This verifies the inductive hypothesis (2.10). From (2.10), it follows that $\sum_1^\infty (M_n(t) + N_n(t))$ converges for $t \in [0, T^*]$, where $T^* = \frac{1}{10C_0^2} \wedge \hat{T} \wedge 4$. Thus, both U_n and ∇U_n converge uniformly in Ω_{T^*}. Set $U(x,t) = \lim_{n\to\infty} U_n(x,t)$.

From (2.2) and (2.3), we have

$$\begin{aligned}
U(x,t) = &\int_{R^d} p(t,x,y)\phi(y)dy - \\
&\int_0^t \int_{R^d} p(t-s,x,y)\beta(y)(|U(y,s)| + \epsilon)^p |\nabla U|^q (y,s) dyds,
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
\nabla U(x,t) = &\int_{R^d} p(t,x,y)\nabla\phi(y)dy \\
&- \int_0^t \int_{R^d} \nabla_x p(t-s,x,y)\beta(y)(|U(y,s)| + \epsilon)^p |\nabla U(y,s)|^q dyds.
\end{aligned} \tag{2.12}$$

From (2.7), it follows that U and $|\nabla U|$ are bounded on Ω_{T^*}; thus $\beta(y)(U(y,s)+\epsilon)^p|\nabla U|^q(y,s)$ is bounded on Ω_{T^*}. Using this in (2.12) gives

$$\begin{aligned}
&|\nabla U(x+z,t)-\nabla U(x,t)| \\
&\leq \int_{R^d} p(t,x,y)|\nabla\phi(y+z)-\nabla\phi(y)|dy+ \\
&C\int_0^t\int_{R^d}|\nabla_x p(s,x+z,y)-\nabla_x p(s,x,y)|dyds.
\end{aligned} \tag{2.13}$$

Using (2.13) along with (1.3) and the fact that there exists a K such that

$$\int_{R^d}|\nabla_x p(t,x+z,y)-\nabla_x p(t,x,y)|dy \leq Kt^{-\frac{1+\alpha}{2}}|z|^\alpha, \text{ for } \alpha\in(0,1)$$

(see [3]), it follows that ∇U is Hölder continuous in x, uniformly on $R^d\times[0,T^*]$.

We have now shown that the term $\beta(y)(|U(y,s)|+\epsilon)^p|\nabla U(y,s)|^q$, appearing in (2.12) is Hölder continuous in y on Ω_{T^*}, uniformly over $t\in[0,T^*]$. It then follows by a standard parabolic result [5, chapter 1, Theorem 4 or 9] that $U\in C^{2,1}(\Omega_{T^*})$, and that U solves (2.1).

By (2.7), for any $\lambda>0$, $U(x,t)-\lambda|x|^2-\lambda^{\frac{1}{3}}t$ attains its minimum in Ω_{T^*}, and by applying a maximum principle argument to this expression (see [1, proof of Theorem A, p.625] and recall that we have assumed that $q\geq 1$), it then follows that $U\geq 0$ in Ω_{T^*}. Since $U(x,t)=\int_{R^d}p(t,x,y)\phi(y)dy-\int_0^t\int_{R^d}p(t-s,x,y)a(y)(U(y,s)+\epsilon)^p|\nabla U|^q(y,s)dyds$, it follows that

$$0\leq U(x,t)\leq ||\phi||_\infty, \text{ in } \Omega_{T^*}. \tag{2.14}$$

In light of (2.14), we may remove the absolute value sign around U in (2.12). Therefore the term $\beta(y)(U(y,s)+\epsilon)^p|\nabla U(y,s)|^q$ appearing in (2.12) is in fact in $C^{1,\alpha}(\Omega_{T^*})$, for an appropriate $\alpha>0$. (It is here that we use the assumption that $q>1$.) It then follows by applying the method in [5, chapter 1, Theorem 4 or 9] that $U\in C^{3,1}(\Omega_{T^*})$.

We will now prove the following key estimate.

Lemma 1. *Let $0 \le U \in C^{3,1}(\Omega_T), T > 0$, be a positive solution to (2.1) satisfying $\sup_{0\le t\le T}(||U(\cdot,t)||_\infty + ||\nabla U(\cdot,t)||_\infty) < \infty$. Then for $0 \le t \le T$,*

$$||\nabla U(\cdot,t)||_\infty \le \max\left(||\nabla\phi||_\infty, \sup_{\Omega_t}(U+\epsilon)(\frac{|\nabla a|}{pa} + \frac{2\delta N}{p})\right). \tag{2.15}$$

An estimate of the type (2.15) will be used eventually to prove (1.5). Presently, (2.15) will allow us to extend the positive solution $U_{\epsilon,\delta}$ on Ω_{T^*} to a global positive solution. Indeed, recalling the definition of T^*, using (2.14) and (2.15), denoting the right hand side of (2.15) by μ, and defining α to be the positive solution to $\alpha^* + 2(\alpha^*)^{\frac{1}{2}} = \frac{C}{(||\phi||_\infty+\mu+\epsilon)^{p+q-1}}$, then the solution can now be extended up to time $t+\alpha$, and then by induction to $t+n\alpha, n = 2,3,...$

Proof of Lemma 1. Let $v_k = U_{x_k}$ and $v = \nabla U$. Since U is three times differentiable, we can differentiate (2.1), obtaining

$$\begin{aligned} &v_t = \Delta v - \nabla\beta(U+\epsilon)^p|v|^q - \beta p(U+\epsilon)^{p-1}|v|^q v - q\beta(U+\epsilon)^p|v|^{q-2}v\cdot\nabla v \\ &v(x,0) = \nabla\phi(x). \end{aligned} \tag{2.16}$$

Consider the function $|v|^2 - \lambda t - \lambda^2 h(x)$, for $\lambda > 0$, where h is a smooth function satisfying $h(x) = |x|$ for $|x| \ge 1$. This function attains a global maximum in Ω_T. Assume that a global maximum occurs at some point (x_0,t_0) with $t_0 \in (0,T]$. Then at the point (x_0,t_0), the following relations hold:

$$\begin{aligned} &0 \le (|v|^2)_t - \lambda = 2v\cdot v_t - \lambda; \\ &0 \ge \Delta(|v|^2) - \lambda^2\Delta h = 2v\cdot\Delta v + 2\sum_{i,j=1}^n v^2_{x_i,x_j} - \lambda^2\Delta h; \\ &0 = \nabla(|v|^2) - \lambda^2\nabla h = 2v\cdot\nabla v - \lambda^2\nabla h. \end{aligned} \tag{2.17}$$

Taking the inner product of each side of (2.16) with v, and using (2.17) and the fact that Δh is bounded, one finds that if λ is sufficiently small, then

$$-\nabla\beta\cdot v(U+\epsilon)^p|v|^q - \beta p(U+\epsilon)^{p-1}|v|^{q+2} - \frac{\lambda^2}{2}q\beta(U+\epsilon)^p|v|^{q-2}v\nabla h > 0 \text{ at } (x_0,t_0).$$

Thus

$$(2.18)\qquad |v|^3 \le (U+\epsilon)\left(\frac{|\nabla\beta|}{p\beta}|v|^2 + \frac{\lambda^2 q}{2p}|\nabla h|\right) \text{ at } (x_0,t_0).$$

It follows from (2.18) that either $|v(x_0,t_0)| \le (\lambda)^{\frac{1}{2}}$ or else, substituting $|v|^2(x_0,t_0)$ for one of the factors of λ in (2.18), that $|v| \le (U+\epsilon)(\frac{|\nabla\beta|}{p\beta} + \frac{\lambda q}{2p}|\nabla h|)$ at (x_0,t_0). Taking these two possibilities, together with the possibility that the maximum of $(|v|^2 - \lambda t - \lambda^2 h(x))$ occurs on the set $\{t=0\}$, we conclude that

$$\begin{aligned}&\sup_{\Omega_T}(|v|^2 - \lambda t - \lambda^2 h)\\ &\le \sup_{\Omega_T}\left(\max(|\nabla\phi|^2 - \lambda^2 h, (U+\epsilon)^2(\frac{|\nabla\beta|}{p\beta} + \frac{\lambda q}{2p}|\nabla h|)^2 - \lambda t - \lambda^2 h, \lambda - \lambda t - \lambda^2 h\right).\end{aligned}$$

Letting $\lambda \to 0$, noting that $|\nabla h|$ is bounded and that $\frac{|\nabla\beta|}{\beta} \le \frac{|\nabla a|}{a} + 2\delta N$, we obtain (2.15). □

From now on, we will denote the global, positive solution to (2.1) by $U_{\epsilon,\delta}$. Let $U^{(i)}_{\epsilon,\delta}, i=1,2,$ be positive solutions to (2.1) corresponding to initial data $\phi_i, i=1,2,$ with $\phi_2 \ge \phi_1$. Letting $W = U^{(2)}_{\epsilon,\delta} - U^{(1)}_{\epsilon,\delta}$, we have

$$(2.19)\qquad \begin{aligned}&W_t = \Delta W - a(x)f(x,t)W - a(x)g(x,t)(|\nabla U^{(2)}_{\epsilon,\delta}| - |\nabla U^{(1)}_{\epsilon,\delta}|)\\ &W(x,0) = \phi_2(x) - \phi_1(x),\end{aligned}$$

for appropriate positive functions f and g. Since $U^{(i)}_{\epsilon,\delta}(x,t) \le \int_{R^d} p(t,x,y)\phi_i(y)dy$, and $\phi_i \in L_1(R^d)$, if follows from the dominated convergence theorem that $\lim_{|x|\to\infty} U^{(i)}_{\epsilon,\delta}(x,t) = 0$. Thus, for each $T>0$, either $\inf W \ge 0$ in $R^d \times [0,T]$ or else W attains a negative minimum in $R^d \times [0,T]$. Noting that $|\nabla U^{(1)}_{\epsilon,\delta}| = |\nabla U^{(2)}_{\epsilon,\delta}|$ at a minimum point since $\nabla W = 0$ there, a standard maximum principle argument shows that this latter alternative is impossible. Thus, we conclude that $U^{(2)}_{\epsilon,\delta} \ge U^{(1)}_{\epsilon,\delta}$. In particular then, the positive solution to (2.1) is unique. A similar argument shows that $U_{\epsilon,\delta}$ is increasing in ϵ and decreasing in δ.

We have

(2.20)
$$U_{\epsilon,\delta}(x,t)=\int_{R^d}p(t,x,y)\phi(y)dy$$
$$-\int_0^t\int_{R^d}p(t-s,x,y)a(y)(1+\delta|y|^2)^{-N}(U_{\epsilon,\delta}(y,s)+\epsilon)^p|\nabla U_{\epsilon,\delta}|^q(y,s)dyds.$$

From (2.15), Condition A, and the fact that $U_{\epsilon,\delta}(x,t)\le\int_{R^d}p(t,x,y)\phi(y)dy$, it follows that for each $\delta>0$, the integrand $a(y)(1+\delta|y|^2)^{-N}(U_{\epsilon,\delta}(y,s)+\epsilon)^p|\nabla U_{\epsilon,\delta}|^q(y,s)$ is uniformly bounded over $\epsilon\in(0,1]$. Thus we can apply (2.13) and the argument that follows it to $U_{\epsilon,\delta}$, and conclude that for each $\delta>0$, $\nabla U_{\epsilon,\delta}$ is Hölder continuous in x, uniformly on Ω_T, for any $T>0$, and uniformly over $\epsilon\in(0,1]$. Thus, for each $\delta>0$, the integrand, $a(y)(1+\delta|y|^2)^{-N}(U_{\epsilon,\delta}(y,s)+\epsilon)^p|\nabla U_{\epsilon,\delta}|^q(y,s)$, in (2.20) is bounded and is uniformly Hölder continuous in the space variable on Ω_T, for any $T>0$, uniformly over $\epsilon\in(0,1]$. It then follows by [5, chapter 1, Theorem 4] that for each $\delta>0$ and each t, $\{U_{\epsilon,\delta}\}_{\epsilon\in(0,1]}$ is equicontinuous on Ω_t, and $\{\nabla U_{\epsilon,\delta}(x,t)\}_{\epsilon\in(0,1]}$ and $\{\nabla^2U_{\epsilon,\delta}(x,t)\}_{\epsilon\in(0,1]}$ are equicontinuous as functions of $x\in R^d$. From this and the fact that $U_{\epsilon,\delta}$ is monotone in ϵ, it follows that $u_\delta\equiv\lim_{\epsilon\to0}U_{\epsilon,\delta}$ exists, that $\nabla u_\delta=\lim_{\epsilon\to0}\nabla u_{\epsilon,\delta}$, and that (2.20) holds with $U_{\epsilon,\delta}$ replaced by u_δ and with $\epsilon=0$.

Since $u_\delta(x,t)\le\int_{R^d}p(t,x,y)\phi(y)dy$, it follows from Condition A(iv) that $a(y)(1+\delta|y|^2)^{-N}u_\delta^p(y,s)$ is bounded on Ω_T, uniformly over $\delta\in(0,1]$, for any $T>0$. Now the same argument applied above to $U_{\epsilon,\delta}$ to obtain u_δ can be applied to u_δ, allowing one to conclude that $u\equiv\lim_{\delta\to0}u_\delta$ exists and that

$$\begin{aligned}u(x,t)&=\int_{R^d}p(t,x,y)\phi(y)dy\\&-\int_0^t\int_{R^d}p(t-s,x,y)a(y)u^p(y,s)|\nabla u|^q(y,s)dyds.\end{aligned}\tag{2.21}$$

By [5, chapter 1, Theorem 4 or 9] again, it follows that $u\in C^{2,1}(R^d\times[0,\infty))$, and that u solves (1.1) This completes the proof of the existence of a positive solution to (1.1).

To prove uniqueness and the comparison principle, let u_i be a positive solution to (1.1) corresponding to the initial data $\phi_i, i = 1, 2$ with $\phi_1 \leq \phi_2$. Let $w = \phi_2 - \phi_1$. Then w solves (2.19) with

$$f(x,t) = p|\nabla u|^q(x,t)\left[r(x,t)u_1(x,t) + (1 - r(x,t))u_2(x,t)\right]^{p-1}$$

and

$$g(x,t) = qu_1^p(x,t)\left[s(x,t)|\nabla u_1|(x,t) + (1 - s(x,t))|\nabla u_2|(x,t)\right]^{q-1},$$

where r, s satisfy $0 < r, s < 1$. The argument following (2.19) now shows that $w \geq 0$.

Since u solves (1.6), it follows that (1.4) holds. Letting $\epsilon \to 0$ and then $\delta \to 0$ in (2.15) gives

$$\sup_{\Omega_T} ||\nabla u|| \leq \max(||\nabla\phi||_\infty, \sup_{\Omega_T} u\frac{|\nabla a|}{pa}). \tag{2.22}$$

Now (1.5) follows from (2.22) and the fact that

$$u(x,t) \leq \int_{R^d} p(t,x,y)\phi(y)dy \leq \min(\frac{||\phi||_1}{(4\pi t)^{\frac{d}{2}}}, ||\phi||_\infty).$$

This completes the proof of Theorem 1 in the case that $q > 1$.

It remains to discuss the case $q = 1$. The only point in the above proof that doesn't go through when $q = 1$ is the proof that $U_{\epsilon,\delta}$ has three space derivatives. This fact was needed in order to prove Lemma 1. If we only know that two space derivatives exist, then (2.16) still holds for v in the sense of distributions, and v has one space derivative. The required maximum principle can still be proved, similar to the proof in the appendix to [1]. □

3. Proof of Theorem 2. The proof builds on a clever idea used in [2] for proving the result when $p = 0$ and $a \equiv 1$.

Let $r(t) = t^{\frac{1}{2}} \log^k(2+t)$ and $s(t) = t^{\frac{1}{2}}(\log(2+t))^{-l}$, where k and l are positive constants to be fixed later. We need the following lemma.

Lemma 2. *Let u be the positive solution to (1.1).*

i.

$$\int_{|x|\geq r(t)} u(x,t)dx \leq \int_{|x|\geq \frac{r(t)}{2}} \phi(x)dx + O(\exp(-\log^k(2+t))), \ \ as\ t\to\infty.$$

ii.

$$\lim_{t\to\infty}\int_{|x|\leq s(t)} u(x,t)dx = 0.$$

Proof. By (1.6), it suffices to prove the lemma with $u(x,t)$ replaced by $w(x,t) \equiv \int_{R^d} p(t,x,y)\phi(y)dy$, The proof of (i) is like the proof of Lemma 2.1 in [2], but we provide the calculation for completeness. We have

$$\begin{aligned}\int_{|x|\geq r(t)} w(x,t)dx &= \int_{|x|\geq r(t)}\int_{R^d} p(t,x,y)\phi(y)dydx \\ &\leq \int_{|y|\geq \frac{r(t)}{2}}\phi(y)dy + \int_{|x|\geq r(t)}\int_{|y|\leq\frac{r(t)}{2}} p(t,x,y)\phi(y)dydx.\end{aligned} \tag{3.1}$$

Also, changing variables for the first and third inequalities below, we have

(3.2)

$$\begin{aligned}&\int_{|x|\geq r(t)}\int_{|y|\leq\frac{r(t)}{2}} p(t,x,y)\phi(y)dydx \\ &\leq \int_{|x|\geq r(t)}\int_{|v|\geq\frac{r(t)}{2}} (4\pi t)^{-\frac{d}{2}}\exp(-\frac{|v|^2}{4t})\phi(v+x)dvdx \\ &\leq ||\phi||_1 \int_{|v|\geq\frac{r(t)}{2}} (4\pi t)^{-\frac{d}{2}}\exp(-\frac{|v|^2}{4t})dv \\ &\leq ||\phi||_1(4\pi)^{-\frac{d}{2}}\int_{|x|\geq\frac{1}{2}\log^k(2+t)}\exp(-\frac{|x|^2}{4})dx \leq C\int_{\frac{1}{2}\log^k(2+t)}^{\infty} r^{d-1}\exp(-\frac{r^2}{4})dr \\ &\leq C\exp(-\frac{\log^{2k}(2+t)}{32})\int_{\frac{1}{2}\log^k(2+t)}^{\infty} r^{d-1}\exp(-\frac{r^2}{8})dr \leq C\exp(-\frac{\log^{2k}(2+t)}{32}).\end{aligned}$$

This proves part (i).

Part (ii) follows from the inequality

$$\int_{|x|\leq s(t)} w(x,t)dx \leq \int_{|x|\leq s(t)} \int_{R^d} (4\pi t)^{-\frac{d}{2}} \phi(y)dydx \leq C(s(t))^d t^{-\frac{d}{2}} ||\phi||_1.$$

□

Define $D(t) = \{x \in R^d : s(t) \leq |x| \leq r(t)\}$ and $D_1(t) = \{x \in R^d : s(t) \leq |x| \leq 2r(t)\}$. In light of Lemma 2, it suffices to prove that

$$\lim_{t\to\infty} \int_{D(t)} u(x,t)dx = 0. \tag{3.3}$$

From now on, we assume that $q \geq 1$. At the end of the proof we will show how to treat the much simpler case, $q = 0$, starting from (3.3). For any $r > 0$, we have

$$\begin{aligned} & u^{1+\frac{p}{q}}(x,t) = u^{1+\frac{p}{q}}(x + r\frac{x}{|x|}, t) \\ & - (1+\frac{p}{q}) \int_0^r (u^{\frac{p}{q}} \nabla u)(x + \lambda\frac{x}{|x|}, t) \cdot \frac{x}{|x|} d\lambda. \end{aligned} \tag{3.4}$$

Define the transformation $T_\lambda, \lambda > 0$ by $T_\lambda x = x + \lambda\frac{x}{|x|}$. Let JT_λ denote the Jacobian of T_λ. We claim that $|JT_\lambda| \geq 1$. One can easily verify this directly for the case $d = 2$, but for higher dimensions, direct verification is unwieldy. We argue as follows. It is easy to see from its definition that T_λ takes the annulus $\{x \in R^d : r_1 < |x| < r_2\}$ onto the annulus $\{x \in R^d : r_1 + \lambda < |x| < r_2 + \lambda\}$. In particular, note that the volume of the image of an annulus under T_λ is greater than the volume of the original annulus. Now assume that the inequality $JT_\lambda \geq 1$ does not hold. Then there exists an open set $B \subset R^d$ such that $|A| < |T_\lambda A|$, for all nonempty, open $A \subset B$. Thus, there exists a $\theta_0 \in S^{d-1}$, an $\epsilon_0 > 0$ and r_1, r_2 satisfying $0 < r_1 < r_2$ such that the set $A_{\theta_0,\epsilon_0,r_1,r_2} \equiv \{x \in R^d : r_1 < |x| < r_2, |\frac{x}{|x|} - \theta_0|_{S^{d-1}} < \epsilon_0\}$ satisfies $|A_{\theta_0,\epsilon_0,r_1,r_2}| < |T_\lambda A_{\theta_0,\epsilon_0,r_1,r_2}|$. (The metric $|\cdot|_{S^{d-1}}$ denotes geodesic distance on S^{d-1}.) But then it follows from the spherical symmetry of T_λ that $|A_{\theta,\epsilon_0,r_1,r_2}| < |T_\lambda A_{\theta,\epsilon_0,r_1,r_2}|$, for all $\theta \in S^{d-1}$. Since the sets $\{A_{\theta,\epsilon_0,r_1,r_2}\}_{\theta\in S^{d-1}}$ cover the annulus

$\{x \in R^d : r_1 < |x| < r_2\}$, we conclude that the image of this annulus under T_λ has greater volume than the original annulus, which is a contradiction.

Note that for $\lambda \in [0, r(t)]$, we have $T_\lambda(D(t)) \subset D_1(t)$, and for $\lambda = r(t)$ we have $T_{r(t)}(D(t)) \subset R^d - B(r(t))$, where $B(s)$ denotes the ball of radius s centered at the origin. Thus, using the fact that $JT_\lambda \geq 1$, we have for any $f \geq 0$, $\int_{D(t)} f(x + \lambda \frac{x}{|x|})dx \leq \int_{D_1(t)} f(x)dx$ and $\int_{D(t)} f(x + r(t)\frac{x}{|x|})dx \leq \int_{R^d - B(r(t))} f(x)dx$. Therefore, choosing $r = r(t)$ in (3.4) and integrating the inequality over $D(t)$, we obtain

$$\begin{aligned} &\int_{D(t)} u^{1+\frac{p}{q}}(x,t)dx \leq \\ &\int_{R^d - B(r(t))} u^{1+\frac{p}{q}}(x,t)dx + r(t)(1+\frac{p}{q}) \int_{D_1(t)} u^{\frac{p}{q}}(x,t)|\nabla u(x,t)|dx. \end{aligned} \tag{3.5}$$

Using (3.5), the inequality $(a+b)^l \leq C(a^l + b^l)$, for $a, b \geq 0$ and $l > 0$, and the fact that

$$\int_{D(t)} u(x,t)dx \leq |D(t)|^{\frac{p}{p+q}} (\int_{D(t)} u^{1+\frac{p}{q}}(x,t)dx)^{\frac{q}{p+q}},$$

we obtain

$$\begin{aligned} &\int_{D(t)} u(x,t)dx \leq \\ &C|D(t)|^{\frac{p}{p+q}} \left(\int_{R^d - B(r(t))} u^{1+\frac{p}{q}}(x,t)dx \right)^{\frac{q}{p+q}} + \\ &C|D(t)|^{\frac{p}{p+q}} \left(r(t) \int_{D_1(t)} u^{\frac{p}{q}}(x,t)|\nabla u(x,t)|dx \right)^{\frac{q}{p+q}}. \end{aligned} \tag{3.6}$$

We will show that the two terms on the right hand side of (3.6) converge to 0 as $t \to \infty$. We begin with the first term. Since $u(x,t) \leq \int_{R^d} p(t,x,y)\phi(y)dy \leq Ct^{-\frac{d}{2}}||\phi||_1$, for all $x \in R^d$, it follows from Lemma 2 that

$$\begin{aligned} &\int_{R^d - B(r(t))} u^{1+\frac{p}{q}}(x,t)dx \leq Ct^{-\frac{dp}{2q}} \int_{R^d - B(r(t))} u(x,t)dx \\ &\leq Ct^{-\frac{dp}{2q}} \left(\int_{|x| \geq \frac{r(t)}{2}} \phi(x)dx + O(\exp(-\log^k(2+t))) \right), \text{ as } t \to \infty. \end{aligned} \tag{3.7}$$

We now fix $k \in (0, \frac{\nu q}{dp})$, where ν is as in (1.8). Then from (3.7) and assumption (1.8) on ϕ, we obtain

(3.8)

$$|D(t)|^{\frac{p}{p+q}}(\int_{R^d - B(r(t))} u^{1+\frac{p}{q}}(x,t)dx)^{\frac{q}{p+q}}$$

$$\leq C(t^{\frac{1}{2}}\log^k(2+t))^{\frac{dp}{p+q}}\ t^{-\frac{dp}{2(p+q)}}\left(\int_{|x|>\frac{r(t)}{2}}\phi(x)dx + O(\exp(-\log^k(2+t)))\right)^{\frac{q}{p+q}}$$

$$\leq C\log^{\frac{kdp}{p+q}}(2+t)\left((\log t)^{-\frac{\nu q}{p+q}} + O(\exp(-\frac{q}{p+q}\log^k(2+t)))\right) \to 0,\ \text{as } t\to\infty.$$

To treat the second term on the right hand side of (3.6), we begin with the inequality

$$(3.9)\qquad \int_{D_1(t)} u^{\frac{p}{q}}(x,t)|\nabla u(x,t)|dx \leq |D_1(t)|^{\frac{q-1}{q}}(\int_{D_1(t)} u^p(x,t)|\nabla u(x,t)|^q dx)^{\frac{1}{q}}.$$

Recalling the assumption on a in the statement of the theorem, we have

(3.10)

$$\int_{D_1(t)} u^p(x,t)|\nabla u(x,t)|^q dx \leq \frac{1}{\inf_{x\in D_1(t)} a(x)}\int_{D_1(t)} a(x)u^p(x,t)|\nabla u(x,t)|^q dx$$

$$\leq \frac{C}{(s(t))^m}\int_{D_1(t)} a(x)u^p(x,t)|\nabla u(x,t)|^q dx.$$

From (1.7), it follows that $\int_0^\infty\int_{R^d} a(x)u^p(x,t)|\nabla u(x,t)|^q dxdt < \infty$. Thus, there exists a sequence $\{t_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$(3.11)\qquad \int_{R^d} a(x)u^p(x,t_n)|\nabla u(x,t_n)|^q dx \leq \frac{C}{t_n\log(2+t_n)},\ n=1,2,...$$

From (3.9)-(3.11), we obtain for the second term on the right hand side of (3.6),

(3.12)

$$|D(t_n)|^{\frac{p}{p+q}}(r(t_n)\int_{D_1(t_n)} u^{\frac{p}{q}}(x,t_n)|\nabla u(x,t_n)|dx)^{\frac{q}{p+q}}$$

$$\leq C|D(t_n)|^{\frac{p}{p+q}}(r(t_n))^{\frac{q}{p+q}}\left(\frac{|D_1(t_n)|^{\frac{q-1}{q}}}{((s(t_n))^{\frac{m}{q}}}(\int_{D_1(t_n)} a(x)u^p(x,t_n)|\nabla u(x,t_n)|^q dx)^{\frac{1}{q}}\right)^{\frac{q}{p+q}}$$

$$\leq C(t_n^{\frac{1}{2}}\log^k(2+t_n))^{\frac{dp+q}{p+q}}\frac{(t_n^{\frac{1}{2}}\log^k(2+t_n))^{\frac{d(q-1)}{p+q}}}{(t_n^{\frac{1}{2}}(\log(2+t_n))^{-l})^{\frac{m}{p+q}}}(\frac{1}{(t_n\log(2+t_n))})^{\frac{1}{p+q}}$$

$$\leq Ct_n^{\frac{dp+(d+1)q-(d+2+m)}{2(p+q)}}(\log(2+t_n))^{\frac{k(dp+q+dq-d)+lm-1}{p+q}}.$$

Since by assumption, $dp+(d+1)q \leq d+2+m$, it follows from (3.12) that if we pick $k,l>0$ sufficiently small, then

$$\lim_{n\to\infty} |D(t_n)|^{\frac{p}{p+q}} (r(t_n) \int_{D_1(t_n)} u^{\frac{p}{q}}(x,t_n)|\nabla u(x,t_n)|dx)^{\frac{q}{p+q}} = 0. \tag{3.13}$$

From (3.6), (3.8) and (3.13), we conclude that $\lim_{n\to\infty} \int_{R^d} u(x,t_n)dx = 0$. However, by (1.7), $\int_{R^d} u(x,t)dx$ is decreasing in t; thus $\lim_{t\to\infty} \int_{R^d} u(x,t)dx = 0$. This completes the proof of the theorem in the case $q \geq 1$.

Consider now the case $q=0$. We need to prove (3.3). By assumption, $p>1$. We have

$$\begin{aligned}\int_{D(t)} u(x,t)dx &\leq (\int_{D(t)} u^p(x,t)dx)^{\frac{1}{p}} |D(t)|^{\frac{p-1}{p}} \\ &\leq \frac{1}{\inf_{x\in D(t)} a^{\frac{1}{p}}(x)} (\int_{D(t)} a(x)u^p(x,t)dx)^{\frac{1}{p}} |D(t)|^{\frac{p-1}{p}}.\end{aligned}$$

Using this with (3.11), and arguing as above proves the theorem in this case. □

4. Proof of Theorem 3 and Proposition 1.

Proof of Theorem 3. In the proof of the theorem, c will denote a positive constant whose value may change from line to line. We will proof the theorem under the assumption that ϕ is compactly supported. The general case then follows by comparison. Recall that by assumption, $p>0$. Let u_ϵ denote the solution to (1.1) corresponding to initial data $\epsilon\phi$. By (1.7),

$$\lim_{t\to\infty} \int_{R^d} u_\epsilon(x,t)dx = \epsilon \int_{R^d} \phi(x)dx - \int_0^\infty \int_{R^d} a(x)u_\epsilon^p(x,t)|\nabla u_\epsilon|^q(x,t)dxdt. \tag{4.1}$$

It is enough to show that

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^\infty \int_{R^d} a(x)u_\epsilon^p(x,t)|\nabla u_\epsilon|^q(x,t)dxdt = 0. \tag{4.2}$$

Indeed, from (4.1) and (4.2), it follows that $\lim_{t\to\infty} \int_{R^d} u_\epsilon(x,t)dx > 0$, for sufficiently small ϵ, and then, by the comparison principle, also for $\epsilon = 1$. Thus, it remains to show (4.2).

From (1.5), it follows that $\frac{1}{\epsilon}|\nabla u_\epsilon|^q(x,t)$ is uniformly bounded for $\epsilon \in (0,1]$. Since $u_\epsilon(x,t) \le \epsilon \int_{R^d} p(t,x,y)\phi(y)dy$, and since for all large $|x|$, $|x-y|^2 \ge \frac{|x|^2}{2}$, for $y \in supp(\phi)$, it follows easily that

(4.3) $$u_\epsilon(x,t) \le \epsilon c(1+t)^{-\frac{d}{2}} \exp(-\frac{|x|^2}{8(t \vee 1)}).$$

Thus, since $a(x) \le c(1+|x|)^m$, the dominated convergence theorem gives

(4.4) $$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{R^d} a(x) u_\epsilon^p(x,t)|\nabla u_\epsilon|^q(x,t)dx = 0.$$

An exercise in advanced calculus shows that

(4.5) $$\int_{R^d} (1+|x|)^m (1+t)^{-\frac{d}{2}} \exp(-\frac{p|x|^2}{8t})dx \le c(1+t)^{\frac{1}{2}\max(m,-d)} g(t),$$

where $g(t) = 1$ if $m \ne -d$ and $g(t) = \log(2+t)$ if $m = -d$. Here is a sketch of the proof of (4.5). Making a change of variables gives

$$\int_{R^d} (1+|x|)^m (1+t)^{-\frac{d}{2}} \exp(-\frac{p|x|^2}{8t})dx = \int_{R^d} (1+|y|t^{\frac{1}{2}})^m (\frac{t}{1+t})^{\frac{d}{2}} \exp(-\frac{p|y|^2}{8})dy$$
$$\le \int_{R^d} (1+|y|t^{\frac{1}{2}})^m \exp(-\frac{p|y|^2}{8})dy.$$

From this, the estimate in (4.5) is clear if $m \ge 0$. Consider now the case $m < 0$. It's easy to see that $\int_{|y|>1} (1+|y|t^{\frac{1}{2}})^m \exp(-\frac{p|y|^2}{8})dy$ decays on the order $t^{\frac{m}{2}}$ as $t \to \infty$. To complete the proof of (4.5), it suffices to show that $\int_{|y|\le 1}(1+|y|t^{\frac{1}{2}})^m \exp(-\frac{p|y|^2}{8})dy$ is on the order $t^{\frac{1}{2}\max(m,-d)}g(t)$, where $g(t)$ is as described above. We have

$$\int_{|y|\le 1} (1+|y|t^{\frac{1}{2}})^m \exp(-\frac{p|y|^2}{8})dy \le \int_{|y|\le 1} (1+|y|t^{\frac{1}{2}})^m dy$$
$$= C\int_0^1 r^{d-1}(1+rt^{\frac{1}{2}})^m dr = Ct^{-\frac{d}{2}} \int_0^{t^{\frac{1}{2}}} u^{d-1}(1+u)^m du.$$

As $t \to \infty$, the term $\int_0^{t^{\frac{1}{2}}} u^{d-1}(1+u)^m du$ remains bounded if $m < -d$, is on the order $t^{\frac{d+m}{2}}$ if $m > -d$, and is on the order $\log t$ if $m = -d$.

Using (4.5) along with the estimate $\frac{||\nabla u_\epsilon(\cdot,t)||_\infty}{\epsilon} \le c(t+1)^{-\beta}$, where c is independent of $\epsilon \in (0,1]$, which follows from (1.9), we have for all $\epsilon \in (0,1]$,

$$\frac{1}{\epsilon}\int_{R^d} a(x)u_\epsilon^p(x,t)|\nabla u_\epsilon|^q(x,t)dx \le c(t+1)^{\frac{d}{2}-\frac{dp}{2}+\frac{\max(m,-d)}{2}-\beta q}g(t). \tag{4.6}$$

From (4.4), (4.6), and the assumption that $dp+2\beta q > d+2+\max(m,-d)$, we can apply the dominated convergence theorem again, obtaining (4.2). □

Proof of Proposition 1. Let ϕ be a compactly supported function satisfying (1.3). ϕ will be fixed now for the duration of the proof. Let u_ϵ denote the solution to (1.1) with initial condition $\epsilon\phi$, with $\epsilon \in (0,1]$. In the sequel, C will denote a positive constant (independent of ϵ) which may change from line to line. We begin by showing that

$$\sup_{x\in R^d} a(x)u_\epsilon^p(x,t) \le \epsilon^p C(1+t)^{\frac{m^+}{2}-\frac{pd}{2}},\ m^+ = \max(m,0). \tag{4.7}$$

If $m \le 0$, then (4.7) follows from the inequality

$$u_\epsilon(x,t) \le \epsilon\int_{R^d} p(t,x,y)\phi(y)dy \le \epsilon\min(\frac{||\phi||_1}{(4\pi t)^{\frac{d}{2}}}, ||\phi||_\infty). \tag{4.8}$$

If $m > 0$, then since we have assumed that ϕ has compact support, u_ϵ satisfies (4.3). Thus, $a(x)u^p(x,t) \le \epsilon^p C(1+|x|)^m(1+t)^{-\frac{dp}{2}}\exp(-\frac{p|x|^2}{8(t\vee 1)})$, and maximizing the right hand side as a function of $|x|$ gives (4.7).

Using (4.7) along with the fact that $\int_{R^d}|\nabla_x p(t,x,y)|dy = Ct^{-\frac{1}{2}}$, it follows from the equality

$$\begin{aligned}\nabla u_\epsilon(x,t+s) &= \int_{R^d}\nabla_x p(s,x,y)u_\epsilon(y,t)dy \\ &- \int_0^s\int_{R^d}\nabla_x p(s-r,x,y)a(y)u_\epsilon^p(y,t+r)|\nabla u_\epsilon|^q(y,t+r)dydr,\end{aligned}$$

that

$$\begin{aligned}&||\nabla u_\epsilon(\cdot,t+s)||_\infty \le C||u_\epsilon(\cdot,t)||_\infty\ s^{-\frac{1}{2}}+ \\ &C\epsilon^p(t+1)^{\frac{m^+}{2}-\frac{dp}{2}}\int_0^s(s-r)^{-\frac{1}{2}}||\nabla u_\epsilon(\cdot,t+r)||_\infty^q dr,\ \text{if } m^+-dp\le 0,\end{aligned} \tag{4.9}$$

and

(4.10)
$$\begin{aligned}&||\nabla u_\epsilon(\cdot,t+s)||_\infty \le C||u_\epsilon(\cdot,t)||_\infty\ s^{-\frac{1}{2}}+\\ &C\epsilon^p(t+s)^{\frac{m^+}{2}-\frac{dp}{2}}\int_0^s(s-r)^{-\frac{1}{2}}||\nabla u_\epsilon(\cdot,t+r)||_\infty^q dr,\ \text{if } m^+-dp>0.\end{aligned}$$

Suppose for the time being that

(4.11)
$$||\nabla u_\epsilon(t)||_\infty \le C\epsilon(t+1)^{-\alpha},$$

for some $\alpha \ge 0$. Then from (4.7)-(4.11), we have

(4.12)
$$||\nabla u_\epsilon(\cdot,t+s)||_\infty \le C\epsilon(t+1)^{-\frac{d}{2}}\ s^{-\frac{1}{2}}+C\epsilon^{p+q}(t+1)^{\frac{m^+}{2}-\frac{dp}{2}-q\alpha}s^{\frac{1}{2}},\ \text{if } m^+-dp\le 0,$$

and

(4.13)
$$||\nabla u_\epsilon(\cdot,t+s)||_\infty \le C\epsilon(t+1)^{-\frac{d}{2}}\ s^{-\frac{1}{2}}+C\epsilon^{p+q}(t+s)^{\frac{m^+}{2}-\frac{dp}{2}}(t+1)^{-q\alpha}s^{\frac{1}{2}},\ \text{if } m^+-dp>0.$$

Consider the case $m^+ < dp-1$ appearing in part (i) of the proposition. Then (4.12) is in effect and setting $s=t$ there gives

(4.14)
$$||\nabla u_\epsilon(\cdot,2t)||_\infty \le C\epsilon\max((t+1)^{-\frac{1}{2}(d+1)},(t+1)^{-\frac{1}{2}(2q\alpha+dp-m^+-1)}).$$

By (1.5), it follows that (4.11) holds with $\alpha=0$. Thus, since $dp-m^+-1>0$, (4.14) improves upon (4.11) with $\alpha=0$. If $dp-m^+-1\ge d+1$, then we have obtained (1.9) with $\beta=\frac{d+1}{2}$ as desired. Otherwise, we can iterate the above calculation starting from (4.11) with $\alpha=0$ replaced by $dp-m^+-1$. Letting $h(\alpha)=2q\alpha+dp-m^+-1$, and noting that for some n_0, the n_0-th iteration $h^{(n_0)}$ of h satisfies $h^{(n_0)}(0)\ge\frac{d+1}{2}$, we conclude that $||\nabla u_\epsilon(\cdot,n_0t)||_\infty \le C\epsilon(t+1)^{-\frac{(d+1)}{2}}$, which gives (1.9) with $\beta=\frac{d+1}{2}$.

We now turn to the proof of part (ii) of the proposition, saving for last the case $a\equiv const$ appearing in part (i), which will be proved by amending slightly the proof we are about to give. The wording of the argument we now give is slightly different depending on whether $m^+ - dp \le 0$ or $m^+ - dp > 0$, but the outcome is the same. Assume first that $m^+ - dp > 0$, so that (4.13) is in effect. Either the second term on the right hand side of (4.13) is less than or equal to the first term when $s = t+1$, or there exists an $s_0 < t+1$ at which the two terms on the right hand side of (4.13) are equal. In the first case, we obtain

$$||\nabla u_\epsilon(\cdot, 2t+1)||_\infty \le C\epsilon(t+1)^{-\frac{d+1}{2}}. \tag{4.15}$$

In the latter case, there exists an $s_0 < t+1$ at which the two terms on the right hand side of (4.13) are equal, and we have

$$s_0 = \epsilon^{1-p-q}(t+1)^{q\alpha-\frac{d}{2}}(t+s_0)^{\frac{dp}{2}-\frac{m^+}{2}} \ge \epsilon^{1-p-q}(t+1)^{q\alpha-\frac{d}{2}}(2t+1)^{\frac{dp}{2}-\frac{m^+}{2}}.$$

Substituting this lower bound for s_0 in the first term on the right hand side of (4.13), and using the fact that the two terms on the right hand side are equal for $s = s_0$, we obtain

$$||\nabla u_\epsilon(\cdot, t+s_0)||_\infty \le C\epsilon^{\frac{p+q+1}{2}}(t+1)^{-\frac{d}{4}-\frac{q\alpha}{2}}(2t+1)^{\frac{m^+}{4}-\frac{dp}{4}}. \tag{4.16}$$

From (1.7), $||u_\epsilon(\cdot, t||_1 \le \epsilon||\phi||_1$, for all $t > 0$. Using this along with (1.5), (1.4) and the uniqueness of solutions, we obtain

$$||\nabla u_\epsilon(\cdot, 3t+2)||_\infty \le \max(||\nabla u_\epsilon(\cdot, t+s_0)||_\infty, C\epsilon(2t+2-s_0)^{-\frac{d}{2}}). \tag{4.17}$$

From (4.16) and (4.17) we conclude that

$$||\nabla u_\epsilon(\cdot, 3t+2)||_\infty \le C\max(\epsilon^{\frac{p+q+1}{2}}(t+1)^{-\frac{1}{4}(2q\alpha+d+dp-m^+)}, \epsilon(t+1)^{-\frac{d}{2}}). \tag{4.18}$$

A similar analysis can be done in the alternative case that $m^+ - dp \le 0$, and it leads again to the two alternatives (4.15) and (4.18).

In summary then, what these calculations have shown is that if (4.11) holds, then either (4.15) or (4.18) holds. Since (4.11) holds with $\alpha = 0$, to prove part (ii) of the proposition, it is enough to show that there exists a $\delta > 0$ such that if (4.11) holds for some $\alpha \in [0, \frac{d}{2})$, then (4.11) also holds with α replaced by $\min(\alpha + \delta, \frac{d}{2})$. Thus, in light of (4.15) and (4.18), to complete the proof we will show that if p, q, m, d are as in the statment of part(ii), then

$$\delta \equiv \inf_{\alpha \in [0, \frac{d}{2}]} (\frac{1}{4}(2q\alpha + d + dp - m^+) - \alpha) > 0.$$

Let $h(\alpha) = (2q\alpha + d + dp - m^+) - 4\alpha, \alpha \in [0, \frac{d}{2}]$. Consider first the case that $q \in [1, 2]$, and recall that in this case the assumption of the proposition is that $dp + dq > d + 2 + m^+$. For q as above, h attains its minimum at $\alpha = \frac{d}{2}$, and we have $h(\frac{d}{2}) = d(q-2) + d + dp - m^+ = dp + dq - (d + 2 + m^+) > 0$. Now consider the case that $q > 2$, and recall that the assumption of the proposition in this case is that $m < d(p+1)$. For q in this range, h attains its minimum at $\alpha = 0$ and $h(0) = d + dp - m^+ > 0$. This completes the proof of part (ii).

It remains to consider the case that $a \equiv const$ from part (i). In this case, the assumption of the proposition is that $dp + (d+1)q > d + 2$. Under the assumption that $a \equiv const$, (1.5) reduces to $||\nabla u(\cdot, t)||_\infty \leq ||\nabla \phi||_\infty$, and thus we may replace the right hand side of (4.17) by $||\nabla u_\epsilon(\cdot, t + s_0)||_\infty$. Therefore, in (4.18), the term $(t+1)^{-\frac{d}{2}}$ may be replaced by $(t+1)^{-\frac{d+1}{2}}$. To complete the proof by the same argument as above, we must show that $\inf_{\alpha \in [0, \frac{d+1}{2}]} h(\alpha) > 0$. If $q \in [1, 2]$, then h attains its minimum at $\alpha = \frac{d+1}{2}$, and $h(\frac{d+1}{2}) = 2(q-2)(\frac{d+1}{2}) + d + dp = dp + (d+1)q - (d+2) > 0$. If $q > 2$, then h attains its minimum at 0, and $h(0) = d + dp > 0$. $\square$

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