This is a lovely article on Weierstrass and the early development of approximation theory. It begins with a short biography of Weierstrass. Two main themes stand out in his work: To set a new standard of rigor in analysis, and his love for power series or more generally for function series. The first theme is documented by his construction of a continuous, nowhere differentiable function which was shocking to the mathematical community at the time. Weierstrass presented this in his lectures since 1861 but published his example (using a cosine series) in 1872. Further history on that by Bolzano, Riemann, Takagi, and du Bois-Reymond is mentioned. The second theme is documented by the Fundamental Theorem of Approximation Theory: Algebraic polynomials are dense in \( C[a,b] \), where \( -\infty < a < \sigma < \infty \). This was published by Weierstrass in 1885 when he was 70 years old, and proved by representing \( f \in C[a,b] \) as a limit of integrals \( \int_{-\infty}^{\infty} \) depending on a parameter \( k \). Thus \( f \) is the uniform limit of a sequence of entire functions and hence of a sequence of polynomials. Weierstrass states and proves also the analogous theorem about the density of trigonometric polynomials.

The author then lists and analyses further proofs (before 1913) of the Fundamental Theorem. He puts them into three groups. In Group 1 there are proofs based on singular integrals (Weierstrass, Picard, Fejér, Landau), while those in Group 2 are based on the approximation of a particular function, like a polygonal function (Runge, Lebesgue, Mittag-Leffler, Lerch). Left over are those in Group 3 by Bernstein, Volterra, Lerch.

It is interesting to note that Runge proved (also in 1885!) that rational functions are dense in \( C[a,b] \) but overlooked the fact that this is true already for polynomials. Lebesgue reduces the Fundamental Theorem to the special case \( f(x) = |x| \), and he raises (1908) apparently for the first time questions about the speed of approximation, three years before Jackson’s dissertation appeared.

The last section deals with various generalizations: Müntz’s theorem, Hermite-Fejér interpolation, Carleman’s theorem, Stone-Weierstrass, and Bohman-Korovkin. All these theorems are given with full explanation, proofs, as well as historical notes. It is clear that this article is necessary reading for all approximators.

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