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Variations on the Chebyshev and L^q theories of best approximation.

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For $f \in C[0,1]$ and $q \in [1,\infty)$ put $|||f|||_q = \sup(\int_a^b |f(x)|^q)^{1/q}$ where the supremum is taken over all intervals [a,b]$\subseteq [0,1]$ for which $f(x) > 0$ on (a,b) or $f(x) < 0$ on (a,b). The functional $|||\cdot|||_{q^*}$ is defined similarly with the modification that the supremum is taken over all intervals [a,b]$\subseteq [0,1]$, for which f(x)$\geq$ 0 on (a,b) or f(x)$\leq$ 0 on (a,b). These functionals, called gauges, are positively defined and absolutely homogeneous, but they not verify the triangle inequality. In this paper the authors restrict their attention to the case $q = 1$ (in this case the above defined functionals are denoted simply by $|||\cdot|||$ and $|||\cdot|||_*$), although all the results (excepting Theorem 4.1) hold true for the case $q \in [1,\infty)$. Concerning the relation of these gauges with the uniform topology on C[0,1] the authors prove that if f_m and f are in C[0,1] and f_m converges uniformly to f then $|||f||| \leq \liminf |||f_m||| \leq \limsup |||f_m||| \leq |||f|||_*$. The authors consider the problem of best approximation of continuous functions by polynomials of degree n with respect to the distance induced by these gauges. Although these gauges are not norms, one regains most of the geometric flavour of best uniform approximation in C[0,1]. Denote by π_n the set of all polynomials of degree at most n and for $f \in C[0,1]$ let $d(f,\pi_n)$ and $d^*(f,\pi_n)$ be the distance from f to π_n with respect to $|||\cdot|||$ and $|||\cdot|||_*$, respectively. The infimum $d(f,\pi_n)$ is attained for all $f \in C[0,1]$, but the infimum $d^*(f,\pi_n)$ may be unattained. The main result of the paper is the following Chebyshev type characterization of polynomials of best approximation: Theorem 3.1. For every $f \in C[0,1]$ there exists a unique $p \in \pi_n$ such that $|||f-p||| = d(f,\pi_n)$ and this p is characterized by the following condition: there exist $n+2$ nonempty open intervals $I_1 < I_2 < \ldots < I_{n+2}$ and $\sigma \in \{-1,1\}$ fixed such that (a) $(-1)^k\sigma(f-p) \geq 0$ on I_k and (b) $(-1)^k\sigma\int_{I_k}(f-p)(x)\cdot dx \geq |||f-p|||$, for all $k = 1,\ldots,n+2$. A similar result holds for $|||\cdot|||_*$. This condition is simpler to check than the well known conditions of characterization of best L^q-approximation, namely,

$$\int_0^1 |f(x)-p(x)|^{q-1} sgn(f(x)-p(x)) x^k dx = 0,$$

for $k = 0,1,\ldots,n$. *S.Cobzaş*

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