

CAMBRIDGE TRACTS IN MATHEMATICS

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**181 Totally Positive Matrices**



# Totally Positive Matrices

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**CAMBRIDGE**  
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS  
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,  
São Paulo, Delhi, Dubai, Tokyo

Cambridge University Press  
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)  
Information on this title: [www.cambridge.org/9780521194082](http://www.cambridge.org/9780521194082)

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First published 2010

Printed in the United Kingdom at the University Press, Cambridge

*A catalogue record for this publication is available from the British Library*

ISBN 978-0-521-19408-2 Hardback

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This monograph is dedicated to the memory of  
I. J. Schoenberg, M. G. Krein, F. R. Gantmacher and S. Karlin,  
the four pioneers of the theory of total positivity.

*We work in the dark – we do what we can – we give what we have.  
Our doubt is our passion, and our passion is our task.*



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# Foreword

In this monograph was present the central properties of finite totally positive matrices. As such, the monograph has only six main chapters. We consider the basic properties of such matrices, determinantal criteria for when a matrix is totally positive, their variation diminishing properties, various examples of totally positive matrices, their eigenvalue/eigenvector properties, and factorizations of such matrices. Numerous topics are excluded from this exposition. Total positivity is a theory of considerable consequence, and the most glaring omissions of this monograph are undoubtedly its various applications to diverse areas. Aside from the many applications mentioned in Gantmacher, Krein [1950] and Karlin [1968], applications can be found in approximation theory (see Schumaker [1981], Pinkus [1985c]), combinatorics (see Brenti [1989], [1995], [1996]), graph theory (see Fomin, Zelevinsky [2000], Berenstein, Fomin, Zelevinsky [1996]), Lie group theory (see Lusztig [1994]), majorization (see Marshall, Olkin [1979]), noncommutative harmonic analysis (see Gross, Richards [1995]), shape preservation (see Goodman [1995]), computing using totally positive matrices (see de Boor, Pinkus [1977], Koev [2005], Demmel, Koev [2005], Koev [2007]), refinement equations and subdivision (see Cavaretta, Dahmen, Micchelli [1991], Micchelli, Pinkus [1991]), and infinite totally positive banded matrices (see Cavaretta, Dahmen, Micchelli, Smith [1981], de Boor [1982], Smith [1983], Dahmen, Micchelli, [1986]). See also the many references in these papers and also the many references to these papers. There has been no attempt to make this monograph all-encompassing, and we apologize to all who feel that their contributions to the theory have been slighted as a consequence.

The theory of totally positive matrices is an odd bird in the matrix theory aviary. Much of the motivation for its study has come from problems

in analysis, and the main initiators and contributors to the theory were analysts. I. J. Schoenberg was interested in the problem of estimating the number of real zeros of a polynomial, and this led him to his work on variation diminishing transformations (in the early 1930s) and Pólya frequency sequences, functions, and kernels (late 1940s and early 1950s). These, together with his work on splines (1960s and 1970s), are central topics in the theory of total positivity. M. G. Krein was led to the theory of total positivity via ordinary differential equations whose Green's functions are totally positive (mid 1930s). S. Karlin came to the theory of total positivity (in the 1950s and 1960s) by way of statistics, reliability theory, and mathematical economics. The two major texts on the subject *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*, by F. R. Gantmacher and M. G. Krein (see Gantmacher, Krein [1950]), and *Total Positivity. Volume 1*, by S. Karlin (see Karlin [1968]), are a blend of analysis and matrix theory (and in the latter case the emphasis is most certainly on analysis). (Their companion volumes *The Markov Moment Problem and Extremal Problems*, by M. G. Krein and A. A. Nudel'man (see Krein, Nudel'man [1977]) and *Tchebycheff Systems: with Applications in Analysis and Statistics*, by S. Karlin and W. J. Studden (see Karlin, Studden [1966]), are totally devoted to topics of analysis.) Thankfully we have the short monograph of T. Ando that eventually appeared as Ando [1987] (it was written a few years earlier) and was devoted to totally positive matrices. The present monograph is an attempt to update and expand upon Ando's monograph. A considerable amount of research has been devoted to this area in the past twenty years, and such an update is certainly warranted.

It was Schoenberg, in Schoenberg [1930], who coined the term *total positiv* (in German). Krein and Gantmacher (see Gantmacher, Krein [1935]), unaware of Schoenberg's earlier paper, used the term *complètement non négative* and *complètement positive* (French) for totally positive and strictly totally positive, respectively. As such, many authors use the term *totally nonnegative* and *totally positive* for totally positive and strictly totally positive, respectively, which, aside from the lack of consistency and order, all too often leads to confusion. We follow the Schoenberg/Karlin/Ando terminology.

It is a pleasure to acknowledge the help of Carl de Boor and David Tranah. All errors, omissions and other transgressions are the author's responsibility.

I would like to close this short foreword with a personal note. My first mathematical paper (jointly written with my doctoral supervisor Sam Karlin) was in the area of total positivity. It is said that as one gets old(er) one often returns to one's first love. I plead guilty on both counts.

Haifa, 2008.