

## Positive and Minimal Projections in Function Spaces

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### 1. INTRODUCTION

If  $L$  is a linear lattice and  $E$  is a linear subspace of  $L$ , it is natural to ask whether there is a positive projection of  $L$  onto  $E$  (a projection  $P$  is *positive*, or *monotone*, if  $x \geq 0$  implies  $Px \geq 0$ ). This is always the case, for example, when  $L$  is  $L^p(\mu)$  and  $E$  is a closed linear sublattice [4, Chap. 3]. However, much less is known about the situation when  $L$  is the function space  $C(X)$  ( $X$  compact, Hausdorff) with supremum norm, though for certain subspaces Korovkin's theorem implies that there is no positive projection.

In Section 2, we give necessary and sufficient conditions for there to be a positive projection of a normed linear lattice  $L$  onto an  $n$ -dimensional subspace  $L_n$ . As a corollary, we see that if  $M$  is a closed sublattice of a Banach lattice  $L$  and there is a positive projection of  $M$  onto  $L_n$ , then there is a positive projection of  $L$  onto  $L_n$ . In particular, every finite-dimensional sublattice of  $L$  admits a positive projection. When  $L$  is  $C(X)$ , our characterization reduces to the following:  $L_n$  admits a positive projection if and only if there exist positive functions  $b_1, \dots, b_n$  in  $L_n$  and points  $x_1, \dots, x_n$  of  $X$  such that  $b_i(x_j) = \delta_{ij}$ .

In Section 3, we study the companion problem for finite-codimensional subspaces of  $C(X)$ . We prove, in fact, that if  $X$  has no isolated points, then such subspaces never admit positive projections.

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In Section 4, we are concerned with projections of  $C(X \times Y)$  onto certain natural subspaces. Here we consider minimal as well as positive projections. More precisely, let  $M$  be the subspace consisting of all functions of the form  $\phi(x, y) = f(x) + g(y)$ . Also, for fixed  $x^* \in X, y^* \in Y$ , let  $C_0(X \times Y)$  be the set of functions in  $C(X \times Y)$  that vanish at  $(x^*, y^*)$ , and let  $M_0 = M \cap C_0(X \times Y)$ . Then there exists no positive projection of  $C(X \times Y)$  onto  $M$ , and exactly one positive projection  $P^*$  of  $C_0(X \times Y)$  onto  $M_0$ . Furthermore, if  $X$  and  $Y$  are infinite, then for any projection  $P$  of  $C(X \times Y)$  onto  $M$ ,  $\|P\| \geq 3$ , while for any projection  $P$  of  $C_0(X \times Y)$  onto  $M_0$ ,  $\|P\| \geq \|P^*\| = 2$ . These results easily generalize to the product of  $k$  spaces  $X^i$ . Also, the method of proof establishes exact estimates (in the first case) for the norms when some or all of the spaces  $X^i$  are finite.

## 2. FINITE-DIMENSIONAL SUBSPACES

Our first result applies to general normed linear lattices. A *linear lattice* (or *Riesz space*) is a linear space (over the real field) with a lattice ordering  $\geq$  such that

$$\begin{aligned} x \geq 0, \quad y \geq 0 & \text{ implies } x + y \geq 0, \\ x \geq 0, \quad \lambda \in \mathbb{R}^+ & \text{ implies } \lambda x \geq 0. \end{aligned}$$

We use the usual notation:  $\sup\{x, y\} = x \vee y, \inf\{x, y\} = x \wedge y, |x| = x \vee (-x)$ . A *normed linear lattice* is a normed linear space equipped with a lattice ordering such that  $|x| \leq |y|$  implies  $\|x\| \leq \|y\|$ . If the space is also complete with respect to the norm, it is called a *Banach lattice*.

Let  $L_n$  denote an  $n$ -dimensional linear subspace of a normed linear lattice  $L$ . Suppose that there is a positive projection  $P$  of  $L$  onto  $L_n$ . It is then elementary (and well-known) that the ordering of  $L_n$  is a lattice ordering: in fact, we have

$$\sup_{L_n}(x, y) = P(x \vee y)$$

for  $x, y \in L_n$ . (This does not mean that  $L_n$  is a sublattice of  $L$  since  $x \vee y$  need not belong to  $L_n$ ). The positive cone in  $L_n$  is closed, hence Archimedean. By a standard result on finite-dimensional linear lattices [4, p. 70],  $L_n$  has a basis  $\{b_1, \dots, b_n\}$  such that  $\sum_1^n \lambda_i b_i \geq 0$  if and only if  $\lambda_i \geq 0$  for all  $i$ . We deduce the following characterization of finite-dimensional subspaces that admit positive projections:

**THEOREM 1.** *Let  $L_n$  be an  $n$ -dimensional subspace of a normed linear lattice  $L$ . Then the following statements are equivalent:*

- (i) *there is a positive projection of  $L$  onto  $L_n$ ,*  
 (ii) *there exist positive elements  $b_i$  of  $L_n$  and positive linear functionals  $f_i$  on  $L$  ( $i = 1, \dots, n$ ) such that  $f_i(b_j) = \delta_{ij}$ .*

*Proof.* If (ii) holds, then a positive projection  $P$  is simply given by  $Px = \sum_1^n f_i(x) b_i$ .

Conversely, suppose that there is a positive projection  $P$ . Then  $L_n$  has a basis  $\{b_1, \dots, b_n\}$  as above. For  $x = \sum_1^n \lambda_i b_i \in L_n$ , let  $g_i(x) = \lambda_i$ . The  $g_i$  are positive linear functionals defined on  $L_n$ , and  $g_i(b_j) = \delta_{ij}$ . The required functionals on  $L$  are given by  $f_i = g_i \circ P$ . ■

**COROLLARY.** *Let  $L$  be a Banach lattice,  $M$  a closed linear sublattice. Let  $L_n$  be an  $n$ -dimensional subspace of  $M$ . If there is a positive projection of  $M$  onto  $L_n$ , then there is a positive projection of  $L$  onto  $L_n$ .*

*Proof.* Theorem 1 gives us positive linear functionals  $f_i$  defined on  $M$ . It is well-known that every positive functional on a Banach lattice is continuous ([4, p. 84]; if there were positive elements  $x_k$  with  $\|x_k\| \leq 2^{-k}$  and  $f(x_k) \geq k$ , then no definition would be possible for  $f(\sum x_k)$ ). Further, every continuous positive functional defined on  $M$  has a positive extension defined on  $L$  [4, p. 86]. ■

In particular, every finite-dimensional linear sublattice admits a positive projection. As we shall see, this is far from being the case for infinite-dimensional sublattices of  $C(X)$ , though it is true in  $L^p(\mu)$ ,  $1 \leq p < \infty$  [4, p. 212].

*Remark.* It is sufficient in Theorem 1 if  $L$ , instead of having a lattice ordering, has an Archimedean ordering satisfying the Riesz decomposition property, that is, if  $x_1, x_2 \geq 0$  and  $0 \leq y \leq x_1 + x_2$ , then  $y = y_1 + y_2$ , where  $0 \leq y_i \leq x_i$ ,  $i = 1, 2$ . Finite-dimensional spaces with this property are order-isomorphic to  $\mathbb{R}^n$  with the usual order.

The next result shows that in the case  $L = C(X)$ , we can take the functionals in Theorem 1 to be point-evaluations.

**THEOREM 2.** *Let  $X$  be a compact, Hausdorff space and let  $L_n$  be an  $n$ -dimensional subspace of  $C(X)$ . Then the following statements are equivalent:*

- (i) *there is a positive projection of  $C(X)$  (or of a closed linear sublattice of  $C(X)$ ) onto  $L_n$ ,*  
 (ii) *there exist non-negative functions  $b_1, \dots, b_n$  in  $L_n$  and points  $x_1, \dots, x_n$  in  $X$  such that  $b_i(x_j) = \delta_{ij}$ .*

*Proof.* Let the functionals  $f_i$  be as in Theorem 1, and let  $S(f_i)$  denote the support of  $f_i$ . For  $j \geq 2$ , we have  $f_1(b_j) = 0$ , hence  $b_j(x) = 0$  for all  $x$  in  $S(f_1)$ .

Since  $f_1(b_1) = 1$ , there exists  $x_1$  in  $S(f_1)$  with  $b_1(x_1) > 0$ . Replace  $b_1$  by a positive scalar multiple to obtain  $b_1(x_1) = 1$ . The points  $x_2, \dots, x_n$  are found similarly. ■

*Remarks.* (1) In the same way, one sees that if for a certain set of non-negative functions  $b_1, \dots, b_n$  in  $L_n$ , there exist *unique* points  $x_i$  such that  $b_i(x_i) > 0$  and  $b_i(x_j) = 0$  for  $j \neq i$ , then there is a unique positive projection onto  $L_n$ .

(2) For any positive linear mapping  $T$  of  $C(X)$  into itself, we have  $\|T\| = \|Te\|$ , where  $e$  is the function with constant value 1. Hence if  $P$  is a positive projection onto a subspace containing  $e$ , then  $\|P\| = 1$  (and if  $e$  is in the subspace,  $\|P\| = 1$  implies  $P$  is positive). For the positive projection  $Pu = \sum_1^n u(x_i) b_i$  given by Theorem 2, we have  $\|P\| = \|b_1 + \dots + b_n\|$ . If the subspace does not contain  $e$ , this may well be greater than 1, even when a non-positive projection of norm 1 exists. To obtain a simple example, let  $X$  be a 3-point set, so that  $C(X)$  is  $\mathbb{R}^3$ . Let  $L_2$  be the subspace consisting of elements  $(x, y, z)$  satisfying  $x = y + 2z$ . Then there is an unique positive projection given by  $P(x, y, z) = (y + 2z, y, z)$ . (This corresponds to  $b_1 = (1, 1, 0)$ ,  $b_2 = (2, 0, 1)$ .) Clearly,  $\|P\| = 3$ . However, there is a non-positive projection with norm 1, namely,  $Q(x, y, z) = (x, y, (x - y)/2)$ .

On the other hand, if the subspace does contain  $e$ , then the problem of finding the minimal norm projection is equivalent, in a certain sense, to that of finding the "least negative" projection. By this we mean the following. It is easy to show that for  $f \geq 0$ ,

$$\frac{1}{\|f\|} \min(Pf)(x) \geq \frac{1}{2} (1 - \|P\|),$$

and equality is attained when we take the infimum over all  $f \geq 0$ .

It was shown by Morris and Cheney [2, Theorem 9] that if  $n \geq 3$  and  $L_n$  is an  $n$ -dimensional Chebyshev subspace of  $C[a, b]$  containing the constant functions, then every projection onto  $L_n$  has norm greater than 1. Consequently there is no positive projection onto  $L_n$ . Using our Theorem 2, we can prove the following stronger statement.

**COROLLARY.** *Let  $L_n$  be an  $n$ -dimensional subspace of  $C[a, b]$ . Assume that  $L_n$  contains an  $m$ -dimensional Chebyshev subspace  $X_m$ , where  $m \geq 3$ . Then there is no positive projection onto  $L_n$ .*

*Proof.* Since  $X_m$  is a Chebyshev subspace, for each  $x_0$  in  $(a, b)$  there exists an  $f_0$  in  $X_m$  such that  $f_0 \geq 0$  on  $[a, b]$  and  $f_0(x) = 0$  only for  $x = x_0$ .

Assume that there is a positive projection of  $C[a, b]$  onto  $L_n$ . Let  $b_1, \dots, b_n$  and  $x_1, \dots, x_n$  be as in the statement of Theorem 2. Choose  $x_0$  in  $(a, b) \setminus \{x_1, \dots, x_n\}$ , and let  $f_0$  be as above. Express  $f_0$  in the form  $\sum_1^n \lambda_i b_i$ .

Since  $f_0(x_i) > 0$ , we have  $\lambda_i > 0$  for  $i = 1, \dots, n$ . However,  $f_0(x_0) = 0$ , so  $b_i(x_0) = 0$  for each  $i$ . This holds for all  $x_0$  as above, which implies that each  $b_i$  is identically zero, a contradiction. ■

Actually, an even stronger statement is true. We say that a subspace  $E$  of  $C(X)$  has the "Korovkin property" if the identity is the only positive operator of  $C(X)$  into itself that agrees with the identity on  $E$ . (This differs slightly from the usual definition, which refers to a sequence of positive operators.) The Korovkin property implies, of course, that there is no positive projection onto  $E$ .

In the situation of the above corollary,  $X_m$  (and hence  $L_n$ ) has the Korovkin property. For  $f$  in  $C[a, b]$  and each  $x$ , set

$$\begin{aligned}\bar{f}(x) &= \inf\{g(x): g \in X_m, g \geq f\}, \\ \underline{f}(x) &= \sup\{g(x): g \in X_m, g \leq f\}.\end{aligned}$$

It is easily proved that if  $X_m$  is an  $m$ -dimensional Chebyshev subspace, with  $m \geq 3$ , then  $\bar{f}(x) = \underline{f}(x) = f(x)$  for each  $x$  in  $(a, b)$ . Let  $T$  be any positive operator of  $C[a, b]$  into itself which is the identity on  $X_m$ , and take any  $f$  in  $C[a, b]$ . Then it follows from the above equalities that  $(Tf)(x) = f(x)$  for all  $x$  in  $(a, b)$ , and hence that  $Tf = f$  (cf. Berens and Lorentz [1] and Šaškin [3]).

While the Korovkin property implies the non-existence of a positive projection, the converse is certainly not true. For example, if  $E$  has the Korovkin property, then so has any subspace containing  $E$ . No such inclusion property holds for the existence or non-existence of positive projections. Two specific examples will be given after Proposition 3.

When  $L_n$  is considered as a subspace of  $L^p[a, b]$ , a statement similar to the above corollary holds even for  $m = 2$ . In fact, we have as a corollary of Theorem 1:

**COROLLARY.** *Let  $L_n$  be an  $n$ -dimensional subspace of  $C[a, b]$ . Assume that  $L_n$  contains an  $m$ -dimensional Chebyshev subspace  $X_m$ , where  $m \geq 2$ . Then there is no positive projection of  $L^p[a, b]$  onto  $L_n$  (where  $1 \leq p < \infty$ ).*

*Proof.* By the previous corollary, we need only consider the case  $m = 2$ . Since  $X_2$  is a Chebyshev subspace, there exist  $g$  in  $X_2$  such that  $g(a) = 0$  and  $g(x) > 0$  for  $a < x \leq b$ , and  $h$  in  $X_2$  such that  $h(a) > 0$ . Let  $b_1, \dots, b_n$  and  $f_1, \dots, f_n$  be as in the statement of Theorem 1. Then  $g$  can be expressed as  $\sum_1^n \mu_i b_i$ , with each  $\mu_i \geq 0$ . Since  $g - \epsilon h$  takes negative values for each  $\epsilon > 0$ , we must have  $\mu_k = 0$  for some  $k$ . It follows that  $f_k(g) = 0$ . But this is impossible, since  $f_k$  is a non-zero, non-negative element of  $L^p[a, b]$  and  $g$  is a non-negative, continuous function that vanishes only at  $a$ . ■

Let  $\pi_n$  denote the space of algebraic polynomials of degree  $\leq n$ . It follows from the elementary form of Korovkin's theorem (or from the above) that there is no positive projection of  $C[a, b]$  onto  $\pi_n$  for  $n \geq 2$ . We give here a simple direct proof of a slightly stronger statement, together with a variant which is not obtainable from Korovkin-type theorems. Let  $C_0[a, b]$  denote the set of functions in  $C[a, b]$  that are 0 at  $a$ , and let  $\pi_n^0 = \pi_n \cap C_0[a, b]$ .

PROPOSITION 3. (i) *If  $n \geq 2$ , then there is no positive projection of  $\pi_{n+1}$  onto  $\pi_n$ .*

(ii) *If  $n \geq 3$ , then there is no positive projection of  $\pi_{n+1}^0$  onto  $\pi_n^0$ .*

*Proof.* It is sufficient to consider  $[a, b] = [0, 1]$ . Write  $r_k(x) = x^k$ . For both (i) and (ii), suppose that there is a positive projection  $P$ , and let  $P(r_{n+1}) = u$ . Now  $0 \leq r_{n+1} \leq r_n$ , so  $0 \leq u \leq r_n$ . It is elementary that this, together with the fact that  $u$  is in  $\pi_n$ , implies that  $u = \alpha r_n$  for some  $\alpha$  in  $[0, 1]$ . Now  $x^{n+1} \geq nx^2 - (n-1)x$  for  $x$  in  $[0, 1]$ . Hence  $u(x) \geq nx^2 - (n-1)x$ . In particular,  $u(1) \geq 1$ , so  $\alpha \geq 1$ .

For (i), let

$$h_k(x) = k \left( x - \frac{1}{4} \right)^2 + \frac{1}{2.4^n}.$$

Then  $h_k$  is in  $\pi_2$ , and for a suitable  $k$  we have  $h_k \geq r_{n+1}$  ( $h_k$  is a "narrow" quadratic having twice the value of  $r_{n+1}$  at  $\frac{1}{4}$ ). Hence  $h_k \geq u$ . Evaluation at  $\frac{1}{4}$  gives  $\alpha \leq \frac{1}{2}$ , a contradiction.

For (ii), modify this slightly, as follows. Let

$$g_k(x) = k \left( x - \frac{1}{4} \right)^2 + \frac{1}{2.4^{n-1}}.$$

Choose  $k$  such that  $x^n \leq g_k(x)$ , so that  $x^{n+1} \leq xg_k(x)$ . The function  $xg_k(x)$  is in  $\pi_3^0$ , so we obtain  $\alpha x^n \leq xg_k(x)$ . This gives  $\alpha \leq \frac{1}{2}$ , as before. ■

*Remarks.* (1) There is a positive projection of  $C[0, 1]$  onto  $\pi_1$ , and in fact onto any two-dimensional subspace containing the constant functions. For if  $f$  is a non-constant function, then we can define  $f_1 = \alpha f + \beta$  such that  $0 \leq f_1 \leq 1$  and  $f_1$  attains the values 0, 1. Then  $f_1$  and  $1 - f_1$  satisfy the conditions of Theorem 2. The positive projection onto  $\pi_1$  is unique.

(2) The subspace  $\pi_n^0$  does not have the Korovkin property in  $C[0, 1]$ . This is shown by the mapping  $(Tf)(x) = f(x) + f(0)$ .

An additional example, containing the constant functions, is as follows. Let  $L_3$  denote the subspace of  $C[-1, 1]$  spanned by the functions  $1, x^2, x^4$ . The mapping  $(Tf)(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x)$  shows that  $L_3$  does not have the

Korovkin property. By an application of Theorem 2 (and zero counting), or by the method of Proposition 3, it is easily seen that there is no positive projection onto  $L_3$ .

EXAMPLE. The subspace  $\pi_2^0$ , consisting of polynomials of the form  $ax + bx^2$ , displays several interesting features. Note first that  $ax + bx^2 \geq 0$  on  $[0, 1]$  if and only if  $a, a + b \geq 0$ . Consequently,  $\pi_2^0$  has a basis  $\{b_1, b_2\}$  such that  $\lambda_1 b_1 + \lambda_2 b_2 \geq 0$  if and only if  $\lambda_1 \geq 0$  and  $\lambda_2 \geq 0$ : let  $b_1(x) = x - x^2$ ,  $b_2(x) = x^2$ . We prove that there is no positive projection of  $C[0, 1]$  onto  $\pi_2^0$ , showing that the existence of such a basis is not in itself sufficient. Let  $g_1, g_2$  be the positive functionals on  $\pi_2^0$  such that  $g_i(b_j) = \delta_{ij}$ . We show that there is no positive extension of  $g_1$  defined on  $C[0, 1]$  (or even on  $\pi_2$ ); this implies the non-existence of a positive projection. Suppose, in fact, that  $f_1$  were such an extension, and let  $\alpha > 0$ . Since

$$0 \leq (x - \alpha)^2 = x^2 - 2\alpha x + \alpha^2,$$

we have

$$0 \leq -2\alpha + \alpha^2 f_1(e),$$

so  $f_1(e) \geq 2/\alpha$  for all  $\alpha > 0$ , which is impossible.

Since  $C_0[0, 1]$  is a sublattice of  $C[0, 1]$ , it follows from the corollary of Theorem 1 that there is no positive projection of  $C_0[0, 1]$  onto  $\pi_2^0$ . However, it is not hard to show that for each  $n \geq 3$ , there is a unique positive projection  $P_n$  of  $\pi_n^0$  onto  $\pi_2^0$ : if  $f(x) = a_1 x + \dots + a_n x^n$ , then  $(P_n f)(x) = a_1 x + (a_2 + \dots + a_n) x^2$ . By considering  $f_n(x) = 1 - (1 - x)^n$ , one sees that  $\|P_n\| \rightarrow \infty$  as  $n \rightarrow \infty$ .

### 3. FINITE-CODIMENSIONAL SUBSPACES OF $C(X)$

In this section, we prove:

**THEOREM 4.** *Let  $X$  be a compact, Hausdorff space with no isolated points. Then there is no positive projection of  $C(X)$  onto any proper, finite-codimensional subspace.*

The proof will be achieved by a series of lemmas. Suppose that  $P$  is such a projection, and let  $E$  be its (finite-dimensional) kernel. We shall work with  $E$  rather than the range of  $P$ . Note first that if  $f \in E$  and  $0 \leq g \leq f$ , then  $Pg = 0$ , so  $g \in E$  (that is,  $E$  is "order-convex"). We deduce:

**LEMMA 1.**  *$E$  contains no non-zero, non-negative function.*

*Proof.* Suppose that  $E$  contains a non-zero, non-negative function  $f$ . We may assume that  $f \geq 1$  on some open set  $G$ . Since  $X$  has no isolated points,  $G$  is infinite. Let  $(x_n)$  be a sequence of distinct points in  $G$ . For each  $n$ , there is a function  $f_n$  in  $C(X)$  such that  $0 \leq f_n \leq 1$ ,  $f_n(x_n) = 1$ ,  $f_n(x_i) = 0$  for  $i < n$  and  $f_n(x) = 0$  for all  $x$  in  $X \setminus G$ . Then  $0 \leq f_n \leq f$ , so  $f_n \in E$ , and it is clear that the sequence  $(f_n)$  is linearly independent, contradicting the fact that  $E$  is finite-dimensional. ■

LEMMA 2. Let  $E_1 = \{f \in E : \|f\| = 1\}$ , and for  $f$  in  $C(X)$ , let  $i(f) = \inf\{f(x) : x \in X\}$ . Then there exists  $c > 0$  such that  $i(f) \leq -c$  for all  $f \in E_1$ .

*Proof.* This is clear, since  $i$  is continuous,  $E_1$  is compact and, by Lemma 1,  $i(f) < 0$  for  $f \in E_1$ . ■

LEMMA 3. There exist  $x_1, \dots, x_k$  in  $X$  such that if  $f \in E$  and  $f(x_i) \geq 0$  for each  $i$ , then  $f = 0$ .

*Proof.* Let  $c$  be as in Lemma 2. Since  $E_1$  is a compact subset of  $C(X)$ , it is equicontinuous. Therefore for each  $x$  in  $X$ , there is a neighbourhood  $U(x)$  such that if  $y \in U(x)$ , then  $|f(y) - f(x)| \leq c/2$  for all  $f$  in  $E_1$ . The space  $X$  can be covered by a finite choice of such neighbourhoods, say  $U(x_1), \dots, U(x_k)$ . If  $f$  is in  $E_1$ , then  $f(x) \leq -c$  for some  $x$ , so  $f(x_i) \leq -c/2$  for some  $i$ . ■

LEMMA 4. Write  $Q = I - P$ . If  $f$  is in  $C(X)$  and  $f(x_i) = 0$  for all  $i$ , then  $Qf = 0$ .

*Proof.* It is sufficient to prove this for non-negative  $f$  (then consider  $f^+$  and  $f^-$ ). Suppose that  $f \geq 0$  and  $f(x_i) = 0$  for each  $i$ . Then  $Pf = f - Qf \geq 0$ , so  $(Qf)(x_i) \leq 0$  for all  $i$ . Since  $Qf$  is in  $E$ , Lemma 3 gives  $Qf = 0$ . ■

*Proof of Theorem 4.* Choose some  $f \geq 0$  with  $Qf \neq 0$ . By Lemma 1,  $Qf$  has both positive and negative values. Hence  $(Qf)(y) > 0$  for some  $y$  different from  $x_1, \dots, x_k$ . Let  $h$  be a non-negative function taking the value 1 at each  $x_i$  and 0 at  $y$ . Let  $g = fh$ . Then  $g(x_i) = f(x_i)$  for all  $i$ , so by Lemma 4,  $Qg = Qf$ . In particular,  $(Qg)(y) > 0$ . But  $g(y) = 0$ , so  $(Pg)(y) < 0$ . This contradicts the positivity of  $P$  since  $g \geq 0$ . ■

*Remarks.* (1) Let  $A$  be a closed, proper subset of  $X$  and let  $C(X, A)$  denote the set of functions in  $C(X)$  that vanish on  $A$ . The same reasoning shows that there is no positive projection of  $C(X, A)$  onto any proper, finite-codimensional subspace of itself. It is well-known that every closed, order-convex linear sublattice (i.e., closed *lattice ideal*) of  $C(X)$  is of the form  $C(X, A)$ .

(2) Let  $C(X, x_0)$  denote the set of functions in  $C(X)$  that vanish at  $x_0$ . If  $x_0$  is not an isolated point (whether or not other isolated points exist), then there is no positive projection of  $C(X)$  onto  $C(X, x_0)$ . This follows easily from the fact that any projection onto  $C(X, x_0)$  has the form  $Pf = f - f(x_0)g$ , where  $g$  is a function with  $g(x_0) = 1$ .

#### 4. CERTAIN SUBSPACES OF $C(X \times Y)$

We start with an elementary result.

**PROPOSITION 5.** *Let  $X$  be a compact, Hausdorff space and  $Y$  any topological space. Let  $L_n$  be an  $n$ -dimensional subspace of  $C(X)$  spanned by  $f_1, \dots, f_n$ . Let  $B^i$  ( $i = 1, \dots, n$ ) be subspaces of  $C(Y)$ , each containing the constant functions. Let  $A_n$  be the set of functions of the form  $\sum_1^n f_i(x)g_i(y)$ , where  $g_i \in B^i$  for each  $i$ . If there is no positive projection of  $C(X)$  onto  $L_n$ , then there is no positive projection of  $C(X \times Y)$  onto  $A_n$ .*

*Proof.* Choose any  $y^* \in Y$ , and let  $(Q\phi)(x) = \phi(x, y^*)$  for  $\phi$  in  $C(X \times Y)$ . If  $P$  were a positive projection of  $C(X \times Y)$  onto  $A_n$ , then  $QP|_{C(X)}$  would be a positive projection of  $C(X)$  onto  $L_n$ . ■

A natural application of this is the extension of Proposition 3 to polynomials in two or more variables. For  $x, y \in I = [0, 1]$ , set

$$\tilde{\pi}_n^2 = \left\{ \sum a_{ij}x^i y^j : i + j \leq n \right\},$$

$$\pi_{n,m}^2 = \left\{ \sum a_{ij}x^i y^j : i \leq n, j \leq m \right\}.$$

Then for  $n \geq 2$ , there is no positive projection of  $C(I^2)$  onto  $\tilde{\pi}_n^2$  or  $\pi_{n,m}^2$ . (In fact, examination of the proofs of Propositions 3 and 5 shows easily that there is no positive projection of  $\tilde{\pi}_{n+1}^2$  onto  $\tilde{\pi}_n^2$ , or of  $\pi_{n+1,m}^2$  onto  $\pi_{n,m}^2$ .) There is a unique positive projection of  $C(I^2)$  onto  $\pi_{1,1}^2$ , given by interpolation at the corners of  $I^2$ . It follows from Theorem 2 that there is no positive projection of  $C(I^2)$  onto  $\tilde{\pi}_1^2$ . The method of proof of Theorem 6 actually shows that there is no positive projection of  $\pi_{1,1}^2$  onto  $\tilde{\pi}_1^2$ .

Let  $X, Y$  be compact, Hausdorff spaces neither containing only one point. Let  $M$  be the subspace of  $C(X \times Y)$  consisting of functions of the form  $\phi(x, y) = f(x) + g(y)$ . Also, for a chosen  $x^* \in X, y^* \in Y$ , let  $C_0(X \times Y)$  denote the set of functions in  $C(X \times Y)$  which vanish at  $(x^*, y^*)$ , and let  $M_0 = M \cap C_0(X \times Y)$ . We shall consider both positive and minimal projections of  $C(X \times Y)$  onto  $M$ , and of  $C_0(X \times Y)$  onto  $M_0$ . Our first result totally characterizes the positive projections.

**THEOREM 6.** *There is no positive projection of  $C(X \times Y)$  onto  $M$ . There is exactly one positive projection  $P^*$  of  $C_0(X \times Y)$  onto  $M_0$ , given by  $(P^*\phi)(x, y) = \phi(x, y^*) + \phi(x^*, y)$ .*

*Proof.* Suppose that  $P$  is a positive projection of  $C(X \times Y)$  onto  $M$ . Let  $f \in C(X)$ ,  $g \in C(Y)$  be arbitrary functions such that  $0 \leq f, g \leq 1$ , and for which there exist points  $x_0, x_1 \in X$  and  $y_0, y_1 \in Y$  satisfying  $f(x_0) = g(y_0) = 0$ ,  $f(x_1) = g(y_1) = 1$ . Set  $u(x, y) = f(x)g(y)$ , and let  $(Pu)(x, y) = h(x) + k(y)$ , with  $h(x_0) = 0$ . Then  $(Pu)(x_0, y) = k(y) \geq 0$  for all  $y$  in  $Y$ . Now,  $f(x) \geq u(x, y)$ , so  $f(x) \geq h(x) + k(y)$  for all  $x, y$ . Taking  $x = x_0$ , we see that  $k = 0$ . It follows by similar reasoning that  $h = 0$ . By construction,  $|1 - f(x)||1 - g(y)| \geq 0$ . Application of  $P$  gives  $1 - f(x) - g(y) \geq 0$  for all  $x, y$ . Set  $x = x_1$  and  $y = y_1$  to obtain a contradiction.

Obviously  $P^*$  is a positive projection of  $C_0(X \times Y)$  onto  $M_0$ . Let  $P$  be any positive projection of  $C_0(X \times Y)$  onto  $M_0$ . For  $\phi \in C_0(X \times Y)$ , define  $\psi(x, y) = \phi(x, y) - \phi(x, y^*) - \phi(x^*, y)$ . The result follows if we prove that  $P\psi = 0$ . Now,  $\psi$  enjoys the property that  $\psi(x, y^*) = \psi(x^*, y) = 0$  for all  $x, y$ . Set  $(P\psi)(x, y) = h(x) + k(y)$ , where  $h(x^*) = k(y^*) = 0$ . Define  $\psi^+(x) = \max\{\psi(x, y) : y \in Y\}$ , and  $\psi^-(x) = \min\{\psi(x, y) : y \in Y\}$ . Thus  $\psi^+(x) \geq \psi(x, y) \geq \psi^-(x)$  for all  $x, y$ . Since  $\psi^+, \psi^- \in M_0$ , it follows that  $0 = \psi^+(x^*) \geq h(x^*) + k(y) \geq \psi^-(x^*) = 0$  for all  $y \in Y$ . Thus  $k = 0$ . Similarly  $h = 0$ . This proves the theorem. ■

*Remark.* The first part of the proof also shows that there is no positive projection of  $C(X \times Y)$  onto  $M_0$ . This is interesting in the light of the results of Sections 2 and 3.

For any  $(x^*, y^*) \in X \times Y$  the map defined by  $(P\phi)(x, y) = \phi(x, y^*) + \phi(x^*, y) - \phi(x^*, y^*)$  is a projection of  $C(X \times Y)$  onto  $M$  of norm 3. We prove that this is minimal if both  $X$  and  $Y$  contain an infinite number of points.

**THEOREM 7.** *Let  $X$  and  $Y$  be infinite, compact, Hausdorff spaces. Let  $P$  be any projection of  $C(X \times Y)$  onto  $M$ . Then  $\|P\| \geq 3$ .*

*Proof.* Let  $\{x_i\}_{i=1}^n$  and  $\{y_i\}_{i=1}^n$  be any two sets of  $n$  distinct points in  $X$  and  $Y$ , respectively. Let  $f_i \in C(X)$ ,  $i = 1, \dots, n$ , satisfy

$$\begin{aligned} f_i(x_j) &= \delta_{ij}, & i, j &= 1, \dots, n, \\ f_i(x) &\geq 0, & i &= 1, \dots, n, \quad x \in X, \\ \sum_{i=1}^n f_i(x) &= 1, & x &\in X. \end{aligned}$$

Similarly define  $g_i \in C(Y)$ ,  $i = 1, \dots, n$ , with respect to the points  $\{y_i\}_{i=1}^n$ .

Set  $\phi^{rs}(x, y) = f_r(x) g_s(y)$ ,  $r, s = 1, \dots, n$ . It easily follows that

$$\phi^{rs}(x_i, y_j) = \delta_{ri} \delta_{sj}, \quad r, s, i, j = 1, \dots, n, \quad (1)$$

$$\sum_{r=1}^n \phi^{rs}(x, y) = g_s(y), \quad s = 1, \dots, n, \quad (2)$$

$$\sum_{s=1}^n \phi^{rs}(x, y) = f_r(x), \quad r = 1, \dots, n. \quad (3)$$

Now, set  $(P\phi^{rs})(x_i, y_j) = a_{ij}^{rs}$ .

From (1), (2), and (3), we obtain

$$\sum_{r=1}^n a_{ij}^{rs} = \delta_{sj}, \quad s, i, j = 1, \dots, n, \quad (4)$$

$$\sum_{s=1}^n a_{ij}^{rs} = \delta_{ri}, \quad r, i, j = 1, \dots, n. \quad (5)$$

Furthermore, since  $P$  is a projection onto  $M$ , and functions in  $M$  satisfy  $\phi(z_0, w_0) + \phi(z_1, w_1) = \phi(z_0, w_1) + \phi(z_1, w_0)$ , we have

$$a_{ij}^{rs} = a_{i1}^{rs} + a_{1j}^{rs} - a_{11}^{rs}, \quad r, s, i, j = 1, \dots, n. \quad (6)$$

By (6),

$$\sum_{i,j=1}^n a_{ij}^{ij} = \sum_{i,j=1}^n [a_{i1}^{ij} + a_{1j}^{ij} - a_{11}^{ij}],$$

and applying (4) and (5) we obtain

$$\sum_{i,j=1}^n a_{ij}^{ij} = 2n - 1.$$

We show that  $a_{ij}^{ij} \geq (3 - \|P\|)/2$  for each  $i, j$ . It then follows that

$$2n - 1 \geq (3 - \|P\|) n^2/2$$

or  $\|P\| \geq 3 - (4n - 2)/n^2$ . This is true for all  $n$ , so  $\|P\| \geq 3$ .

Consider  $i, j = 1$  (a similar proof holds for each choice of  $i, j$ ). Since  $0 \leq (1 - f_1(x))(1 - g_1(y)) \leq 1$ , we have

$$|1 - 2f_1(x) - 2g_1(y) + 2f_1(x)g_1(y)| \leq 1$$

for all  $x \in X, y \in Y$ . Applying  $P$  and evaluating at  $x = x_1, y = y_1$ , gives

$$|1 - 2f_1(x_1) - 2g_1(y_1) + 2(P\phi^{11})(x_1, y_1)| \leq \|P\|$$

from which  $|-3 + 2a_{11}^{11}| \leq \|P\|$ . Thus  $a_{11}^{11} \geq (3 - \|P\|)/2$ . ■

*Remark.* If  $X$  contains exactly  $n$  points and  $Y$  contains exactly  $m$  points, then the above argument shows that for any projection  $P$  of  $C(X \times Y)$  onto  $M$ ,  $\|P\| \geq 3 - (2n + 2m - 2)/nm$ . This lower bound is in fact attained by the choice

$$\begin{aligned} a_{ij}^{rs} &= -1/nm, & r \neq i, \quad s \neq j, \\ &= (n-1)/nm, & r = i, \quad s \neq j, \\ &= (m-1)/nm, & r \neq i, \quad s = j, \\ &= (n+m-1)/nm, & r = i, \quad s = j, \end{aligned}$$

where the  $\{a_{ij}^{rs}\}_{r,i=1}^n \{s,j=1}^m$  are understood to be defined as in the proof of Theorem 7.

The projection  $P^*$  of  $C_0(X \times Y)$  onto  $M_0$  as given in Theorem 6 is of norm 2. Our next result, which is a variant of Theorem 7, shows that this is minimal.

**THEOREM 8.** *Let  $X$  and  $Y$  be infinite, compact, Hausdorff spaces. Let  $P$  be any projection of  $C_0(X \times Y)$  onto  $M_0$ . Then  $\|P\| \geq 2$ .*

*Proof.* As in the proof of Theorem 7, let  $\{x_i\}_{i=1}^n$  and  $\{y_i\}_{i=1}^n$  be any two sets of  $n$  distinct points ( $n \geq 2$ ) in  $X$  and  $Y$ , respectively, with  $x_1 = x^*$  and  $y_1 = y^*$ . Let  $\{f_i\}_{i=1}^n$  and  $\{g_i\}_{i=1}^n$  be as in the proof of Theorem 7, and set  $\phi^{rs}(x, y) = f_r(x)g_s(y)$  for  $r, s = 1, \dots, n; (r, s) \neq (1, 1)$ . For notational convenience, set  $\phi^{11} = 0$ . Thus  $\phi^{rs} \in C_0(X \times Y)$  for all  $r, s = 1, \dots, n$ . Now

$$\phi^{rs}(x_i, y_j) = \delta_{ri}\delta_{sj}, \quad r, s, i, j = 1, \dots, n, \quad (r, s) \neq (1, 1), \quad (7)$$

$$\sum_{r=1}^n \phi^{rs}(x, y) = g_s(y), \quad s = 2, \dots, n, \quad (8)$$

$$\sum_{s=1}^n \phi^{rs}(x, y) = f_r(x), \quad r = 2, \dots, n. \quad (9)$$

Set  $(P\phi^{rs})(x_i, y_j) = a_{ij}^{rs}$ . Since  $P$  maps  $C_0(X \times Y)$  onto  $M_0$ ,  $a_{11}^{rs} = 0$ ,  $r, s = 1, \dots, n$ . From the definition of  $a_{ij}^{rs}$ , we have

$$a_{ij}^{rs} = a_{i1}^{rs} + a_{1j}^{rs}, \quad a_{ij}^{11} = 0, \quad a_{11}^{rs} = 0, \quad r, s, i, j = 1, \dots, n, \quad (10)$$

and from (7), (8) and (9),

$$\sum_{r=1}^n a_{ij}^{rs} = \delta_{sj}, \quad s = 2, \dots, n, \quad (11)$$

$$\sum_{s=1}^n a_{ij}^{rs} = \delta_{ri}, \quad r = 2, \dots, n. \quad (12)$$

By (10), (11) and (12),

$$\begin{aligned} \sum_{i,j=1}^n a_{ij}^{ij} &= \sum_{i,j=1}^n [a_{i1}^{ij} + a_{1j}^{ij}] \\ &= \sum_{i=2}^n \sum_{j=1}^n a_{i1}^{ij} + \sum_{j=2}^n \sum_{i=1}^n a_{1j}^{ij} \\ &= 2(n-1). \end{aligned}$$

We show that  $a_{ij}^{ij} \geq 2 - \|P\|$ , for  $i, j = 1, \dots, n$ ,  $(i, j) \neq (1, 1)$ . It then follows that

$$2(n-1) = \sum_{i,j=1}^n a_{ij}^{ij} \geq (n^2 - 1)(2 - \|P\|)$$

or  $\|P\| \geq 2 - 2/(n+1)$ . This is true for all  $n$ , so  $\|P\| \geq 2$ .

We divide the proof of this claim into two cases. First assume that  $i, j > 1$ . For convenience, consider  $i = j = n$ . Since  $0 \leq f_n, g_n \leq 1$ ,

$$0 \leq f_n(x) + g_n(y) - f_n(x)g_n(y) \leq 1.$$

Because  $f_n + g_n - f_n g_n \in C_0(X \times Y)$ , we can apply  $P$  and evaluate at  $x = x_n$ ,  $y = y_n$  to obtain

$$|2 - a_{nn}^{nn}| \leq \|P\|.$$

So  $a_{nn}^{nn} \geq 2 - \|P\|$ .

Now assume that either  $i$  or  $j$ , but not both, is equal to 1. For convenience set  $i = 1, j = n$ . The function  $g_n - (1 - f_1)(1 - g_n) \in C_0(X \times Y)$ , and since  $0 \leq f_1, g_n \leq 1$ ,

$$|g_n(y) - (1 - f_1(x))(1 - g_n(y))| \leq 1.$$

Applying  $P$  and evaluating at  $x = x_1, y = y_n$  gives

$$|2 - a_{1n}^{1n}| \leq \|P\|.$$

So  $a_{1n}^{1n} \geq 2 - \|P\|$ . This completes the proof. ■

*Remark.* Theorems 7 and 8 generalize as follows. Let  $X^1, X^2, \dots, X^k$  be infinite, compact, Hausdorff spaces. There exists no positive projection of  $C(X^1 \times X^2 \times \dots \times X^k)$  onto  $M = C(X^1) + C(X^2) + \dots + C(X^k)$ , and any projection of  $C(X^1 \times X^2 \times \dots \times X^k)$  onto  $M$  is of norm at least  $2k - 1$ . Let  $x_i^* \in X^i$ ,  $i = 1, \dots, k$ . Set  $C_0(X^1 \times \dots \times X^k) = \{\phi: \phi \in C(X^1 \times \dots \times X^k), \phi(x_1^*, \dots, x_k^*) = 0\}$  and  $M_0 = M \cap C_0(X^1 \times \dots \times X^k)$ . There is a unique positive projection  $P^*$  of  $C_0(X^1 \times \dots \times X^k)$  onto  $M_0$ , given by  $(P^*\phi)(x_1, \dots, x_k) = \phi(x_1, x_2^*, \dots, x_k^*) + \dots + \phi(x_1^*, x_2, \dots, x_k)$ . For every projection  $P$  of  $C_0(X^1 \times \dots \times X^k)$  onto  $M_0$ ,  $\|P\| \geq \|P^*\| = k$ .

If  $X^i$  contains exactly  $m_i$  points,  $i = 1, \dots, k$ , then every projection  $P$  of  $C(X^1 \times \dots \times X^k)$  onto  $M$  satisfies

$$\|P\| \geq (2k - 1) - (2k - 2) \left[ \sum_i 1/m_i - \sum_{i \neq j} 1/m_i m_j + \dots + (-1)^{k-1} / m_1 m_2 \dots m_k \right],$$

and this lower bound is attained (see the remark after Theorem 7).

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*Note added in proof. Remark.* Many results relating to positive projections are to be found in Donner [5]. In particular, it follows as a special case of his Theorem 4.7 that a finite-dimensional subspace  $L_n$  of a Banach lattice  $L$  admits a positive projection if (and only if) it (i) is a lattice in the induced ordering and (ii) any subset of  $L_n$  that has an upper bound in  $L$  has an upper bound in  $L_n$ . Our Theorem 1 provides a simple proof of this. One need only establish that any positive linear functional on  $L_n$  has a positive extension defined on  $L$ . This is an immediate consequence of the Hahn-Banach theorem and the fact that there is a  $K$  such that  $\|(x^+)_{L_n}\| \leq K \|(x^+)_{L}\|$  for all  $x \in L_n$ , where  $(x^+)_{E} = \sup_E(x, 0)$ . To prove this, assume instead that there are elements  $x_n$  with  $\|(x_n^+)_{L}\| \leq 2^{-n}$  and  $\|(x_n^+)_{L_n}\| > n$ . Let  $y = \sum_1^\infty (x_n^+)_{L}$  and  $A = \{x \in L_n : x \leq y\}$ . Then  $A$  contains 0 and all  $x_n$ , which leads to a contradiction of (ii).

REFERENCES

1. H. BERENS AND G. G. LORENTZ, Theorems of Korovkin type for positive linear operators on Banach lattices, in "Approximation Theory" (G. G. Lorentz, Ed.), pp. 1-30, Academic Press, New York, 1973.
2. P. D. MORRIS AND E. W. CHENEY, On the existence and characterization of minimal projections, *J. Reine Angew. Math.* **270** (1974), 61-76.
3. YU. A. ŠAŠKIN, Korovkin systems in spaces of continuous functions, *Amer. Math. Soc. Transl.* (2) **54** (1966), 125-144 (*Izv. Akad. Nauk SSSR, Ser. Mat.* **26** (1962), 495-512).
4. H. H. SCHAEFER, "Banach Lattices and Positive Operators," Springer, Berlin, 1974.
5. K. DONNER, "Extensions of Positive Operators and Korovkin Theorems," Springer Lecture Notes 904, Springer-Verlag, Berlin/New York, 1982.