GAUSSIAN QUADRATURE FORMULAE WITH MULTIPLE NODES

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1. FORMULATION AND STATEMENT OF MAIN RESULTS

An approximate evaluation of an integral by a suitable finite linear combination of point functionals (values of the integrand and some of its derivatives) is called a quadrature formula.

Generally one seeks a quadrature formula $Q(f)$ for which the error function

$$(1.1)\qquad R(f) = \int_a^b f(x)\phi(x)\,dx - \sum_{i=1}^{k} a_i f(t_i) = \int_a^b f(x)\phi(x)\,dx - Q(f)$$

is appropriately small for a sufficiently large class of functions. One traditional way to implement this desideratum is to choose $Q(f)$ ensuring that the quadrature formula evaluates the integral exactly $(R(f)=0)$ for a specified set of functions. A common determination has $Q(f)$ implying $R(x^i)=0$, $i=0,1,\ldots,k-1$, or more generally, $R(u_i)=0$, $i=1,\ldots,k$, where $\{u_i(x)\}_{i=1}^{k}$ is a Chebyshev (T)-system. These requirements can be fulfilled with any prescribed real nodes $\{t_i\}_{i=1}^{k}$ since the calculation of the corresponding coefficients $\{a_i\}$ ensue by the fact that the determinant of the corresponding linear system is non-zero. In general, with arbitrary nodes, one can expect to do no better. However, on the assumption that $\phi(x)$ is sufficiently positive, it is a classical result that a judicious choice of the nodes $\{t_i^*\}_{i=1}^k$ and coefficients $\{a_i^*\}_{i=1}^k$ produces the phenomenon of double precision in the quadrature formula, viz. $R(x^i) = 0$, $i = 0,1,\ldots,2k-1$. This fact is also available in the context of a T-system $\{u_i(x)\}_{i=1}^{2k}$. Moreover, the quadrature formula with this property is unique and its nodes are located interior to the interval of integration having positive coefficients. These special nodes are commonly referred to as <u>Gaussian nodes</u>, and the corresponding quadrature expression as the <u>Gaussian quadrature formula</u>.

Discussions of the existence, uniqueness and construction of the Gaussian quadrature formula abound in the literature on numerical integration. We sketch here three alternative approaches.

1. The following is due to Jacobi [1]: For each set of distinct real $\{t_i\}_{i=1}^k$, we may uniquely determine the coefficient array $\{a_i\}_{i=1}^k$ maintaining

$$R(f) = \int_a^b f(x)\phi(x)\,dx - \sum_{i=1}^k a_i f(t_i) = 0$$

for all $f \in \mathcal{P}_{k-1}$ = the collection of polynomials of degree $\leq k-1$. Define $p(x) = \prod_{i=1}^k (x-t_i)$. It is elementary to check that the double precision property $R(x^i) = 0$, $i = 0,1,\ldots,2k-1$, is equivalent to

$$\int_a^b p(x)q(x)\phi(x)\,dx = 0, \quad \text{for all } q \in \mathcal{P}_{k-1}. \tag{1.2}$$

Therefore the Gaussian nodes $\{t_i^*\}_{i=1}^k$ are the zeros of $p^*(x)$, where $p^*(x)$ is the unique solution of

$$\min \int_a^b [p(x)]^2 \phi(x)\, dx ,$$

and the minimum is taken over all monic polynomials $p(x) \in \mathcal{P}_k$. From (1.2) it follows that the points $\{t_i^*\}_{i=1}^k$ are distinct in (a,b) .

2. A.A. Markov [9] develops the Gaussian quadrature formula as follows: With each $\{t_i\}_{i=1}^k$, there certainly exist unique $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$ such that

$$R(f) = \int_a^b f(x)\phi(x)dx - \sum_{i=1}^k a_i f(t_i) - \sum_{i=1}^k b_i f'(t_i)$$

satisfies $R(f) = 0$ for $f \in \mathcal{P}_{2k-1}$. It can then be argued that $b_i = 0$, $i = 1,\dots,k$ iff the nodes $\{t_i\}_{i=1}^k$ coincide with the Gaussian nodes.

3. A third derivation of the Gaussian quadrature formula is via the geometry of moment spaces. The Gaussian quadrature formula corresponds to the unique lower principal representation for the measure $\phi(x)dx$ (see [8] or [7, p.136]).

The first and third methods of proof extend to any T-system $\{u_i(x)\}_{i=1}^{2k}$ defined in $[a,b]$, mutatis mutandis. Method 2 unnecessarily imposes more restrictions on the class of functions in terms of differentiability.

Turán [12] generalized the quadrature endowments of the Gaussian nodes to the following setting: Let k and ℓ be given positive integers, ℓ odd. Then there exist a unique set of nodes and coefficients, such that

$$(1.3) \quad \int_a^b f(x)\phi(x)dx = \sum_{i=1}^k \sum_{j=0}^{\ell-1} a_{ij}^* f^{(j)}(t_i^*) \text{ , for all } f \in \mathcal{P}_{(\ell+1)k-1} .$$

Observe that for any specification of k nodes, we can readily achieve the equation (1.3) for all f in $\mathcal{P}_{\ell k-1}$ by solving an appropriate system of linear equations. Where ℓ is even, this is the maximal extent of exactness that can be secured (see Proposition 1 later).

Turán's analysis extends the concept of method 1. For completeness we sketch the analysis. Consider

$$\min \int_a^b [p(x)]^{\ell+1} \phi(x)dx$$

where the valuation is extended over all monic polynomials p of degree k. Note that $\ell+1$ is even and positive. The minimum is uniquely attained by some $p^*(x) \in \mathcal{P}_k$. A standard variational argument leads to the orthogonality relations

$$\text{(1.3a)} \qquad \int_a^b [p^*(x)]^\ell q(x)\phi(x)dx = 0$$

for all $q \in \mathcal{P}_{k-1}$. It follows from (1.3a) that $p^*(x) = \prod_{i=1}^{k} (x-t_i^*)$ involves distinct points $\{t_i^*\}_{i=1}^k$ in (a,b). Next we ascertain the unique quadrature formula with nodes $\{t_i^*\}_{i=1}^k$ involving coefficients $\{a_{ij}^*\}$ having the remainder functional

$$R(f) = \int_a^b f(x)\phi(x)dx - \sum_{i=1}^{k} \sum_{j=0}^{\ell-1} a_{ij}^* f^{(j)}(t_i^*)$$

satisfying $R(f) = 0$ for all $f \in \mathcal{P}_{\ell k-1}$. Now, any $f \in \mathcal{P}_{(\ell+1)k-1}$ may be expressed in the form $f(x) = [p^*(x)]^\ell q(x) + q_1(x)$, where $q(x) \in \mathcal{P}_{k-1}$, and $q_1(x) \in \mathcal{P}_{\ell k-1}$. Therefore, $R(f) = R([p^*]^\ell q)$. However, $R([p^*]^\ell q) = \int_a^b [p^*(x)]^\ell q(x)\phi(x)dx = 0$, by (1.3a), and the existence of a quadrature formula with the desired precision is established.

Uniqueness of the above quadrature formula can be demonstrated as follows. Assume the existence of a quadrature formula of the form (1.3) with nodes $\{s_i^*\}_{i=1}^k$, that is exact for $\mathcal{P}_{(\ell+1)k-1}$. Let $\tilde{p}(x) = \prod_{i=1}^{k} (x-s_i^*)$. Then,

$$\int_a^b [\tilde{p}(x)]^\ell \, q(x)\phi(x)\,dx = 0$$

for all $q \in \mathcal{P}_{k-1}$. Now $p^*(x)-\tilde{p}(x) \in \mathcal{P}_{k-1}$, and therefore

$$\int_a^b [p^*(x)]^\ell \, (p^*(x)-\tilde{p}(x)) \, \phi(x)\,dx = 0 \quad \text{and}$$

$$\int_a^b [\tilde{p}(x)]^\ell \, (p^*(x)-\tilde{p}(x)) \, \phi(x)\,dx = 0 . \quad \text{Thus,}$$

$$\int_a^b [(p^*(x))^\ell-(\tilde{p}(x))^\ell] \, (p^*(x)-\tilde{p}(x)) \, \phi(x)\,dx = 0 .$$ Since ℓ is odd, this is patently impossible unless $p^*(x) \equiv \tilde{p}(x)$.

Observe that the foregoing proof of existence and uniqueness relies decisively on the divisibility characteristics inherent to the power functions.

Popoviciu generalized the Turán result as follows:

Theorem A [11]. *For any prescribed positive odd integers* $\{\mu_i\}_{i=1}^k$, *there exist distinct* $\{t_i^*\}_{i=1}^k$ *in* (a,b), *and real numbers* $\{a_{ij}^*\}_{i=1}^{k} {}_{j=0}^{\mu_i-1}$ *such that*

$$(1.4) \qquad R(f) = \int_a^b f(x)\phi(x)\,dx - \sum_{i=1}^{k} \sum_{j=0}^{\mu_i-1} a_{ij}^* \, f^{(j)}(t_i^*)$$

vanishes for $f \in \mathcal{P}_{n-1}$, *where* $n = \sum_{i=1}^{k} \mu_i + k$.

As is asserted in Proposition 1 below, one can do no better.

Theorem A subsumes Turán's result which occurs for $\mu_i = \ell$,

$i = 1,\ldots,k$. However, in the general Popoviciu setting uniqueness is lacking.

Popoviciu studied the following minimum problem. Consider

$$\min \int_a^b \Big[\prod_{i=1}^k (x-t_i)^{\mu_i+1} \Big] \phi(x)\,dx \,, \tag{1.5}$$

where the minimum is evaluated with respect to all real $\{t_i\}_{i=1}^k$. Since the class of admissible polynomials in (1.5) is non-convex, the Turán procedure is not directly applicable. However, if there exists a minimum achieved by $\prod_{i=1}^k (x-t_i^*)^{\mu_i+1}$, where the $\{t_i^*\}_{i=1}^k$ are distinct, then a standard variational argument gives

$$\int_a^b \Big[\prod_{i=1}^k (x-t_i^*)^{\mu_i} \Big] q(x)\phi(x)\,dx = 0 \tag{1.6}$$

for all $q \in \mathcal{P}_{k-1}$. It is easy to deduce on the basis of (1.6) that the $\{t_i^*\}_{i=1}^k$ are in (a,b) , and the assertion of (1.4) follows. An analysis of the second variation proves that the minimum actually involves distinct nodes. Again as previously noted, the method decisively utilizes properties of the power functions. Because of the lack of uniqueness, Popoviciu stated that a resolution of (1.4) may not necessarily correspond to a solution of an appropriate minimum problem as in our previous examples. This statement is not quite correct as we shall shortly explain.

In this paper, we extend the findings of Turán and Popoviciu to apply to Extended Complete Chebyshev (ECT) systems and in the process establish uniqueness subject to an ordering requirement. The proofs are more elaborate since the factoring features of the polynomials are not available for general ECT systems. The principal theorem of this work is:

Theorem 1. Let $\{u_m(x)\}_{m=1}^{n}$ *be an ECT-system on* $[a,b]$. *Assume* $\{\mu_i\}_{i=1}^{k}$ *are prescribed positive odd integers such that* $n = \sum_{i=1}^{k} \mu_i + k$. *Then there exist unique* $\{t_i^*\}_{i=1}^{k}$, $a<t_1^*<t_2^*<\ldots<t_k^*<b$, *and unique real numbers* $\{a_{ij}^*\}_{i=1}^{k}{}_{j=0}^{\mu_i-1}$ *satisfying*

$$(1.7)\quad \int_a^b u_m(x)\phi(x)\,dx = \sum_{i=1}^{k} \sum_{j=0}^{\mu_i-1} a_{ij}^* \, u_m^{(j)}(t_i^*) \ , \quad m=1,\ldots,n \ .$$

Thus uniqueness is present provided the increasing order of the $\{t_i^*\}$ is stipulated. If $\mu_i = \ell$, $i = 1,\ldots,k$, then there is one and only one solution. Generally, the number of distinct formulae with the property (1.4) is $\frac{k!}{m_1!\ldots m_s!}$, for some s $(1 \leqslant s \leqslant k)$, where m_i is the number of nodes of multiplicity $\mu_j = i$, each node counted exactly once, $\sum_{i=1}^{s} m_i = k$.

In the next paper, we prove that each set of relations of the form (1.7) derives from a minimum problem of the kind (1.5), with restricted domain. By solving the minimum problem, the quadrature relations (1.7) may be achieved, but we know of no proof of uniqueness deduced via the extremal analysis even for the power functions, except for the Turán special case $\mu_i = \ell$, $i = 1,\ldots,k$.

Our proof of Theorem 1 relies upon a continuity analysis coupled with the implicit function theorem in several variables. This method was partially developed in [6], [5] and [10].

We have heretofore reviewed some of the natural background to Theorem 1. Our interest in this subject was mostly motivated by the work of the first author on *extended* *monosplines* ([2] and [3]).

Let $K(x,s)$ be a real-valued Extended Totally Positive (ETP) kernel on $T \times T$ where T is an interval of the real line, and assume $[a,b] \subseteq T$. Then,

Definition 1. An extended monospline is a function of the form

$$(1.8) \quad g(x) = \int_a^b K(x,s)\phi(s)ds - \sum_{i=1}^{k} \sum_{j=0}^{\mu_i - 1} a_{ij} \frac{\partial^j K(x,t_i)}{\partial s^j}$$

where t_i *are distinct in* T, *and* $\phi(s)$ *is a positive function.*

The following is an immediate consequence of Theorem 1.

Theorem 2. Let $K(x,s)$ *be as above,* μ_i, $i = 1,\ldots,k$, *prescribed odd positive integers. Let* $x_1 \leq \ldots \leq x_n$, *all* $x_i \in T$, *be given with* $n = \sum_{i=1}^{k} \mu_i + k$. *The number of coincident* x *'s indicates the multiplicity of that* x. *Then there exists a unique extended monospline* $g(x)$ *of the form* (1.8), (i.e., *real* a_{ij} *and* $t_1 < \ldots < t_k$, *with* $t_i \in (a,b)$) *such that* $g(x)$ *vanishes precisely on the set* $\{x_i\}_{i=1}^{n}$.

Proof. Let $u_m(s) = K(x_m,s)$, $m = 1,\ldots,n$, where with repeated x values we take the usual successive derivatives of $K(x,s)$ with respect to x. Theorem 2 then emanates from Theorem 1 since the ETP property of $K(x,s)$ entails that $\{u_m(s)\}_{m=1}^{n}$ is an ECT-system on T. Q.E.D.

Proposition 1 below implies that where $g(x)$ admits the representation of the form (1.8), having the prescribed zeros $X = \{x_i\}_1^n$, then $g(x)$ vanishes exactly on X. We state the result with respect to an ECT-system $\{u_m(x)\}_{m=1}^{n}$, although a version exists with respect to an ETP kernel (see

[2, Theorem 2]).

Proposition 1. ([2], [3], [10]). Assume $\{u_m(x)\}_{m=1}^n$ is an ECT-system on $(c,d) \supset [a,b]$, and

$$\int_a^b u_m(x)\phi(x)\,dx = \sum_{i=1}^{k} \sum_{j=0}^{\mu_i-1} a_{ij}\, u_m^{(j)}(t_i) \ , \quad m = 1,\ldots,n \ ,$$

where $t_i \neq t_j$, $i \neq j$; $t_i \in (c,d)$; $a_{i,\mu_i-1} \neq 0$, $i = 1,\ldots,k$; and $\phi(x)$ is a positive weight function. Then,

$$n \leq \sum_{i=1}^{k} \mu_i + k - D \ , \tag{1.9}$$

where D is the number of nodes among $\{t_1,\ldots,t_k\}$ for which either μ_i is even, or $t_i \notin (a,b)$.

Theorem 1 is, in effect, a converse of Proposition 1.

2. PROOF OF THEOREM 1 AND RAMIFICATIONS

It is easier to prove the more general result of Theorem 3.

Theorem 3. Let p,q and k be prescribed non-negative integers, and $\{\nu_i\}_{i=1}^p$, $\{\delta_i\}_{i=1}^q$ and $\{\mu_i\}_{i=1}^k$ sets of positive integers, where the elements of $\{\mu_i\}_{i=1}^k$ are all odd. Write $\sum_{i=1}^{p} \nu_i + \sum_{i=1}^{q} \delta_i + \sum_{i=1}^{k} \mu_i + k = n$, and assume $\{u_m(x)\}_{m=1}^n$ is an ECT-system with respect to the open interval $(c,d) \supset [a,b]$. Then for given $\{\eta_i\}_{i=1}^p$, $c < \eta_1 < \ldots < \eta_p \leq a$, and $\{\xi_i\}_{i=1}^q$, $b \leq \xi_1 < \ldots < \xi_q < d$, there exist unique real coefficients $\{a_{ij}^*\}_{i=1}^{p}{}_{j=0}^{\nu_i-1}$, $\{b_{ij}^*\}_{i=1}^{k}{}_{j=0}^{\mu_i-1}$, $\{c_{ij}^*\}_{i=1}^{q}{}_{j=0}^{\delta_i-1}$

and unique nodes $\{t_i^*\}_{i=1}^k$, $a < t_1^* < \ldots < t_k^* < b$, *establishing the multiple node quadrature formula*

$$\int_a^b u_m(x)\phi(x)\,dx = \sum_{i=1}^{p}\sum_{j=0}^{\nu_i-1} a_{ij}^* u_m^{(j)}(\eta_i) + \tag{2.1}$$
$$+ \sum_{i=1}^{k}\sum_{j=0}^{\mu_i-1} b_{ij}^* u_m^{(j)}(t_i^*) + \sum_{i=1}^{q}\sum_{j=0}^{\delta_i-1} c_{ij}^* u_m^{(j)}(\xi_i) ,$$

for $m = 1,\ldots,n$.

In this theorem, it is essential that $\phi(x)$ be non-negative and sufficiently positive. For ease of notation, we stipulate $\phi(x) \equiv 1$. In Section 3, we indicate more precisely the scope of permissible $\phi(x)$.

The proof of Theorem 3 proceeds by induction on k , the number of nodes interior to $[a,b]$.

Induction Hypothesis: *We assume the validity of Theorem 3 when the number of nodes interior to* $[a,b]$ *does not exceed* $k-1$.

Note that the induction hypothesis involves only k , and is independent of the position and number of nodes exterior to (a,b) . The proof of Theorem 3 is facilitated by a series of lemmas. The first lemma is required to commence the induction.

Lemma 1. *For* $k = 0$, *Theorem* 3 *obtains*.

Proof. For given $\{\eta_i\}_{i=1}^p$ and $\{\xi_i\}_{i=1}^q$, obeying $c < \eta_1 < \ldots < \eta_p \leq a < b \leq \xi_1 < \ldots < \xi_q < d$, and associated positive set of integers $\{\nu_i\}_{i=1}^p$, $\{\delta_i\}_{i=1}^q$, we need to ascertain the existence of unique coefficient arrays $\{a_{ij}^*\}_{i=1}^{p}{}_{j=0}^{\nu_i-1}$ and $\{c_{ij}^*\}_{i=1}^{q}{}_{j=0}^{\delta_i-1}$ verifying the equations

$$\int_a^b u_m(x)\,dx = \sum_{i=1}^{p} \sum_{j=0}^{\nu_i-1} a^*_{ij}\, u_m^{(j)}(\eta_i) + \sum_{i=1}^{q} \sum_{j=0}^{\delta_i-1} c^*_{ij}\, u_m^{(j)}(\xi_i) \ ,$$

$$m = 1,\dots,n \ ,$$

where $n = \sum_{i=1}^{p} \nu_i + \sum_{i=1}^{q} \delta_i$.

Since the number of equations (linear in this case) and unknowns agree, we need to check that the determinant for the matrix C of the system is non-zero. The matrix C is composed from the rows $\{u_1^{(j)}(\eta_i),\dots,u_n^{(j)}(\eta_i)\}$, $j = 0,1,\dots,\nu_i-1$; $i = 1,\dots,p$, followed by the rows $\{u_1^{(j)}(\xi_i),\dots,u_n^{(j)}(\xi_i)\}$, $j = 0,1,\dots,\delta_i-1$; $i = 1,\dots,q$.

Since $\{u_m(x)\}_{m=1}^{n}$ is an ECT-system, this matrix, by definition, has a positive determinant. Q.E.D.

Our next task is to advance the induction. Let p,q,k, $\{\nu_i\}_{i=1}^{p}$, $\{\delta_i\}_{i=1}^{q}$, $\{\mu_i\}_{i=1}^{k}$, $\{\eta_i\}_{i=1}^{p}$ and $\{\xi_i\}_{i=1}^{q}$ be given satisfying the assumptions of the theorem. Let

$$\underline{e} = (a_{10},\dots,a_{p,\nu_p-1},b_{10},\dots,b_{k,\mu_k-1},c_{10},\dots,c_{q,\delta_q-1})$$

denote an $\sum_{i=1}^{p} \nu_i + \sum_{i=1}^{q} \delta_i + \sum_{i=1}^{k} \mu_i = n-k$ dimensional real vector, and designate by e_ℓ the ℓth component of $\underline{e}$, $\ell = 1,\dots,n-k$. Let $\underline{t} = (t_1,\dots,t_k)$ denote a k-tuple, with components arranged in increasing order $c < t_1 < \dots < t_k < d$. Define,

$$(2.2)\qquad \begin{aligned} h_m(\underline{e},\underline{t}) &= \int_a^b u_m(x)\,dx - \sum_{i=1}^{p} \sum_{j=0}^{\nu_i-1} a_{ij}\, u_m^{(j)}(\eta_i) - \\ &- \sum_{i=1}^{k} \sum_{j=0}^{\mu_i-1} b_{ij}\, u_m^{(j)}(t_i) - \sum_{i=1}^{q} \sum_{j=0}^{\delta_i-1} c_{ij}\, u_m^{(j)}(\xi_i)\ , \end{aligned}$$

$$\text{for}\quad m = 1,\dots,n\ ,$$

where the components $\{\eta_i\}_{i=1}^{p}$, $\{\xi_i\}_{i=1}^{q}$ and $\{t_i\}_{i=1}^{k}$ are all distinct. It is convenient to label the mapping H , implicit to (2.2), of a subset of Euclidean n space E^n into E^n by

$$(2.3a)\qquad H(\underline{e},\underline{t}) = (h_1(\underline{e},\underline{t}),\dots,h_n(\underline{e},\underline{t}))\ .$$

We also assign the name K to the contracted mapping induced by H with the last component on the right of (2.3a) deleted, viz.,

$$(2.3b)\qquad K(\underline{e},\underline{t}) = (h_1(\underline{e},\underline{t}),\dots,h_{n-1}(\underline{e},\underline{t}))\ .$$

Fix $t_1(t) = t$, the node with associated multiplicity μ_1 , to be in $(c,a]$, and distinct from all η_i , $i = 1,\dots,p$. We will implement a continuity argument with t serving as a parameter. Invoking the induction hypothesis we obtain unique $\underline{e}(t)$ and $\underline{t}(t)$, where $a < t_2(t) < \dots < t_k(t) < b$, displaying the dependence on the parameter t., satisfying

$$(2.4)\qquad h_m(\underline{e}(t),\underline{t}(t)) = 0\ ,\quad m = 1,\dots,n-1\ ,$$

$$\text{i.e., } K(\underline{e}(t),\ \underline{t}(t)) = \underline{0}\ .$$

The assumption that t not be coincident to some η_i , $i = 1,\dots,p$, is unnecessary. We can allow $t = \eta_i$ where the node at η_i becomes of multiplicity $\mu_1+\nu_i$. However, to avoid non-essential technical details and notational inconveniences we will assume henceforth that $\eta_p < a$ and $\xi_1 > b$.

<u>Lemma 2.</u> <u>For each</u> $t \in (\eta_p, a]$, <u>there exists a neighborhood</u> $\mathcal{N}(t)$ <u>of</u> t <u>such that the relations</u> (2.4) <u>hold, and each component of</u> $\underline{e}(t)$ <u>and</u> $\underline{t}(t)$ <u>is continuously differentiable for</u> t <u>in</u> $\mathcal{N}(t)$.

<u>Proof.</u> Each $h_m(\underline{e},\underline{t})$, $m=1,2,\ldots,n-1$, is obviously continuously differentiable with respect to each of the components of $\underline{e}$ and $\underline{t}$. With each $t_1(t) = t \in (\eta_p, a]$, we have already determined the existence of unique $\underline{e}(t)$ and $\underline{t}(t)$ yielding $h_m(\underline{e}(t),\underline{t}(t)) = 0$, for $m = 1,\ldots,n-1$. The result of Lemma 2 is obtained by an appeal to the implicit function theorem once we verify that the associated Jacobian has non-zero determinant.

The Jacobian matrix J is the matrix $\left\|\frac{\partial h_m(\underline{e},\underline{t})}{\partial e_\ell}\right\|_{m=1}^{n-1}\,{}_{\ell=1}^{n-k}$ adjoined to the matrix $\left\|\frac{\partial h_m(\underline{e},\underline{t})}{\partial t_i}\right\|_{m=1}^{n-1}\,{}_{i=2}^{k}$ evaluated at the point $(\underline{e}(t),\underline{t}(t))$.

Some straightforward manipulations of the last $k-1$ columns leads to the evaluation

$$\det J = \varepsilon \cdot \prod_{i=2}^{k} b_{i,\mu_i-1}(t)$$

$$\times U\begin{pmatrix} 1, & 2, & & \cdot & & \cdot & & \cdot & & \cdot & & ,n-1 \\ \underbrace{\eta_1,\ldots,\eta_1}_{\nu_1}, & \ldots, & \underbrace{\eta_p,\ldots,\eta_p}_{\nu_p}, & \underbrace{t_1(t),\ldots,t_1(t)}_{\mu_1}, & \underbrace{t_2(t),\ldots,t_2(t)}_{\mu_2+1}, & \ldots, & \underbrace{t_k(t),\ldots,t_k(t)}_{\mu_k+1}, & \underbrace{\xi_1,\ldots,\xi_1}_{\delta_1}, & \ldots, & \underbrace{\xi_q,\ldots,\xi_q}_{\delta_q} \end{pmatrix}$$

where† $\varepsilon = +1$ or -1 .

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† See footnote on following page.

Because $\{u_m(x)\}_{m=1}^{n-1}$ is an ECT-system on (c,d), it follows that $\det J \neq 0$ iff $\prod_{i=2}^{k} b_{i,\mu_i-1}(t) \neq 0$. Where $b_{i,\mu_i-1}(t) = 0$ for some $i = 2,\ldots,k$, then the relations $h_m(\underline{e}(t),\underline{t}(t)) = 0$, $m = 1,\ldots,n-1$, and the result of Proposition 1 are incompatible since μ_i is odd.

Q.E.D.

Thus for each $t \in (\eta_p,a]$, there exists a neighborhood $\mathcal{N}(t)$ of t for which Lemma 2 holds.

A reading of Lemma 2 reveals that we only used the information of $t \in (\eta_p,a]$ to assure the existence of $\underline{e}(t)$ and $\underline{t}(t)$ fulfilling $K(\underline{e}(t),\underline{t}(t)) = \underline{0}$ (see (2.3b)) and the inequalities $t < t_2(t) < \ldots < t_k(t)$. Actually, for any t (not necessarily confined to $(\eta_p,a]$) such that $t < t_2(t) < \ldots < t_k(t)$ and $K(\underline{e}(t),\underline{t}(t)) = \underline{0}$, we can construct a neighborhood $\mathcal{N}(t)$ to which the statements of

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† We use the notation $U\begin{pmatrix}1,\ldots,\ell\\ x_1,\ldots,x_\ell\end{pmatrix} = \begin{vmatrix} u_1(x_1) & \ldots & u_1(x_\ell)\\ \vdots & & \vdots\\ u_\ell(x_1) & \ldots & u_\ell(x_\ell)\end{vmatrix}$,

for $x_1 < \ldots < x_\ell$ while $U\begin{pmatrix}1, & \cdot & \cdot & \cdot & ,\ell\\ \underbrace{x_1,\ldots,x_1}_{\alpha_1}, & \underbrace{x_2,\ldots,x_2}_{\alpha_2}, & \ldots, & \underbrace{x_r,\ldots,x_r}_{\alpha_r}\end{pmatrix}$,

$\ell = \sum_{i=1}^{r} \alpha_i$, shall denote the determinant of the matrix with columns

$\{u_1^{(j)}(x_i),\ldots,u_\ell^{(j)}(x_i)\}$, $j = 0,1,\ldots,\alpha_i-1$; $i = 1,\ldots,r$.