MR2183481 (2006h:26008) 26A51 (41A05 44A60)

Pinkus, Allan (IL-TECH); **Wulbert, Dan** (1-UCSD)

Extending n-convex functions. (English summary)

Studia Math. **171** (2005), *no. 2,* 125–152.

Given points $x_1 < x_2 < \cdots < x_{n+1}$ on the real line $\mathbf{R}$, the Vandermonde determinant $V = V(x_1, \ldots, x_{n+1})$ of order $n+1$ has $x_1^{i-1}, \ldots, x_{n+1}^{i-1}$ in its ith row $(1 \le i \le n+1)$. If f is defined at the points x_j $(1 \le j \le n+1)$, denote by $U = U(x_1, \ldots, x_{n+1}; f)$ the determinant obtained from V replacing the last row by $f(x_1), \ldots, f(x_{n+1})$. A function f defined on a set $E \subset \mathbf{R}$ is n-convex if for any choice of points $x_i \in E$ the $(n+1)$st divided difference $[x_1, \ldots, x_{n+1}; f] = U/V$ is positive. Let E consist of m points $x_1, \ldots, x_m$ and let f be n-convex on E $(n \le m)$. Given a point $\xi \in (x_k, x_{k+1})$ the authors, following T. Popoviciu [Bull. Math. Soc. Roum. Sci. **36** (1934), 75–108; Zbl 0010.01601], construct two numbers $L(\xi)$ and $U(\xi)$ such that f has an n-convex extension to $E \cup \{\xi\}$ if and only if $L(\xi) \le U(\xi)$; if the condition is satisfied, then setting $f(\xi) = \alpha$ with $L(\xi) \le \alpha \le U(\xi)$ yields such an extension. An example is given of a 3-convex function f on a 6-point set E and of a point ξ not in E such that f has no 3-convex extension to $E \cup \{\xi\}$; and also of a 3-convex function on an 8-point set $E = \{x_1, \ldots, x_8\}$ which has a 3-convex extension to $E \cup \{\xi\}$ for any $\xi \in [x_1, x_8]$ but has no 3-convex extension to the whole interval $[x_1, x_8]$. Given $x_1 < \cdots < x_m$ the polynomials $B_j(t) = [x_j, \ldots, x_{j+n}; (\cdot - t)_+^{n-1}]$, $1 \le j \le m-n$, introduced by Popoviciu, are the B-splines of degree $n-1$ with knots $x_j, \ldots, x_{j+n}$. If f is n-convex on $x_1 < \cdots < x_m$ and $c_j = [x_j, \ldots, x_{j+n}; f]$, then f has an n-convex extension to $(a,b) \supset \{x_i\}$ if and only if there exists a bounded Borel measure $\mu \ge 0$ such that $c_j = \frac{1}{(n-1)!} \int_{x_1}^{x_m} B_j(t)\,d\mu(t)$. As μ varies in the set of all bounded positive Borel measures, the closed, convex cone $\mathcal{P}$ of the c_j in $\mathbf{R}^{m-n}$ is called the moment space induced by the B_j. If $n \ge 3$, $m \ge 6$, then $\mathcal{P}$ has infinitely many extreme rays of the form $\lambda(B_1(\xi), \ldots, B_{m-n}(\xi))$, $\xi \in (x_3, x_{m-2})$, $\lambda \ge 0$. The splines B_j form a weak Chebyshev system. It follows from a theorem of C. A. Micchelli and A. Pinkus [SIAM J. Math. Anal. **8** (1977), no. 2, 206–230; MR0435669 (55 #8627)] that if μ is a positive measure on $[x_1, x_{n+2r}]$ relative to $(B_j)_{1 \le j \le 2r}$ then there exists a lower principal representation, i.e., $\xi_1 < \cdots < \xi_r$ in (x_1, x_{n+2r}) and $\lambda_1, \ldots, \lambda_r > 0$ such that $\int_{x_1}^{x_{n+2r}} B(t)\,d\mu(t) = \sum_{k=1}^r \lambda_k B(\xi_k)$ for $B \in \mathcal{B} = \operatorname{span}\{B_1, \ldots, B_{2r}\}$. The authors prove that $x_{2k} < \xi_k < x_{n+2k-1}$ $(1 \le k \le r)$, the ξ_k and the λ_k are unique, and a Markov-Kreĭn-type inequality: if $\nu \ge 0$ satisfies $\int_{x_1}^{x_{n+2r}} B(t)\,d\nu(t) = \int_{x_1}^{x_{n+2r}} B(t)\,d\mu(t)$ for $B \in \mathcal{B}$, then $\int_{x_1}^{x_{n+2r}} g(t)\,d\nu(t) \ge \sum_{k=1}^r \lambda_k g(\xi_k)$ for g in the convexity cone of $\mathcal{B}$. The authors also investigate the extension to all of $\mathbf{R}$ of an n-convex function defined on a set of m points, and upper and lower envelopes to such functions.

Reviewed by *J. Horváth*

References

1. R. P. Boas and D. V. Widder, *Functions with positive differences,* Duke Math. J. **7** (1940), 496–503. MR0003436 (2,219e)

2. C. de Boor, *A remark concerning perfect splines*, Bull. Amer. Math. Soc. **80** (1974), 724–727. MR0338618 (49 #3382)
3. C. de Boor and A. Pinkus, *The B-spline recurrence relations of Chakalov and of Popoviciu*, J. Approx. Theory **124** (2003), 115–123. MR2010784 (2005d:41001)
4. A. L. Brown, *Best approximation by continuous n-convex functions*, ibid. **57** (1989), 69–76. MR0990804 (90d:41050)
5. P. S. Bullen, *A criterion for n-convexity*, Pacific J. Math. **36** (1971), 81–98. MR0274681 (43 #443)
6. H. G. Burchard, *Interpolation and approximation by generalized convex functions*, Ph.D. thesis, Purdue Univ., 1968. MR2617822
7. —, *Extremal positive splines with applications to interpolation and approximation by generalized convex functions*, Bull. Amer. Math. Soc. **79** (1973), 959–963. MR0338620 (49 #3384)
8. —, *Extremal positive splines with applications*, in: Approximation Theory, Academic Press, New York, 1973, 291–294. MR0338619 (49 #3383)
9. L. Galvani, *Sulle funzioni convesse di una o due variabili definite in un aggregato qualunque*, Rend. Circ. Mat. Palermo **41** (1916), 103–134.
10. E. Hopf, *Über die Zusammenhänge zwischen gewissen höheren Differenzenquotienten reeller Funktionen einer reellen Variablen und deren Differenzierbarkeitseigen-schaften*, dissertation, Univ. Berlin, 1926.
11. S. Kakeya, *On some integral equations III*, Tôhoku Math. J. **8** (1915), 14–23.
12. S. Karlin, *Total Positivity, Volume I*, Stanford Univ. Press, Stanford, 1968. MR0230102 (37 #5667)
13. S. Karlin and W. J. Studden, *Tchebycheff Systems: with Applications in Analysis and Statistics*, Interscience, New York, 1966. MR0204922 (34 #4757)
14. M. G. Krein, *The ideas of P. L. Chebyshev and A. A. Markov in the theory of limiting values of integrals and their further development*, Amer. Math. Soc. Transl. **12** (1951), 1–122. MR0113106 (22 #3947a)
15. M. G. Krein and A. A. Nudel'man, *The Markov Moment Problem and Extremal Problems. Ideas and problems of P. L. Chebyshev and A. A. Markov and their further development*, Transl. Math. Monogr. **50,** Amer. Math. Soc., Providence, RI, 1977. MR0458081 (56 #16284)
16. C. A. Micchelli and A. Pinkus, *Moment theory for weak Chebyshev systems with applications to monosplines, quadrature formulae and best one-sided L^1 approximation by spline functions with fixed knots*, SIAM J. Math. Anal. **8** (1977), 206–230. MR0435669 (55 #8627)
17. T. Popoviciu, *Sur quelques propriétés des fonctions d'une ou de deux variables réelles*, thèse, Faculté des Sciences de Paris, 1933. See also Mathematica (Cluj) **8** (1934), 1–85.
18. —, *Sur le prolongement des fonctions convexes d'ordre supérieur*, Bull. Math. Soc. Roumaine Sci. **36** (1934), 75–108.
19. —, *Les Fonctions Convexes*, Hermann, Paris, 1944.
20. I. J. Schoenberg and A. Whitney, *On Pólya frequency functions. III. The positivity of translation determinants with an application to the interpolation problem by spline curves*, Trans. Amer. Math. Soc. **74** (1953), 246–259. MR0053177 (14,732g)