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Strong uniqueness. (English summary)
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Let X be a Banach space, V a non-empty subset of X and let $x \in X$. An element $v_0 \in V$ is called a *strongly unique best approximation* to x if there exists $r > 0$ such that for any $v \in V$,

$$\|x - v\| \geq \|x - v_0\| + r\|v - v_0\|.$$

This notion was introduced by Newman and Shapiro in 1963. They proved that this property holds for $X = C[a, b]$ and V any Haar subspace of $C[a, b]$. Hence, in particular, it is satisfied by subspaces of all algebraic polynomials of degree $\leq n$ for any natural number n. It is clear that this property is stronger than the uniqueness of the best approximant.

Strong uniqueness applied to the case of Haar subspaces made it possible to simplify the proof of the convergence of the classical Remez algorithm as well as the proof of the Freud theorem concerning the Lipschitz continuity of the operator of best approximation. Strong uniqueness has been much studied, mainly during the 1970s, 1980s and 1990s, and over 100 research papers devoted to the subject have been published.

In this very interesting survey paper the authors present main results in this beautiful and deep theory.

Chapter I is devoted to classical strong uniqueness. Here mainly strong uniqueness in the uniform norm and in the L_1-norm is considered. Chapter II deals with so-called non-classical strong uniqueness. By this we mean that there exists a nonnegative, strictly increasing function φ on $\mathbb{R}_+$, and a constant $r > 0$ such that for any $v \in V$,

$$\|x - v\| \geq \|x - v_0\| + r\varphi(\|v - v_0\|).$$

Chapter III is devoted to various applications of strong uniqueness. The paper also contains a large number of references, which are very helpful for everyone interested in this theory.

Reviewed by *Grzegorz Lewicki*