Total positivity and the exact $n$-width of certain sets in $L^1$.


The authors consider the set \( K_r = \{ \sum_{j=1}^{r} \alpha_j k_j(t) + \int_0^1 K(t,s) ds \mid (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r, \|h\|_1 \leq 1 \} \) under the conditions that the set of functions \( \{k_1(t), \ldots, k_r(t)\} \) is a Čebyšev system on \((0,1)\) and that, for every \( 0 < x_1 < \cdots < x_m < 1 \), \( \{k_1(t), \ldots, k_r(t), K(t,x_1), \ldots, K(t,x_m)\} \) is a weak Čebyšev system.

They compute for \( K_r \) the exact values of the $n$-width in the sense of Kolmogorov, \( d_n(K_r, X) \), and the $n$-width in the sense of Gelfand, \( d_n(K_r, Y) \), where \( X = L^1[0,1] \) and \( Y = C[0,1] \).

The authors prove that \( d_n(K_r, X) = d^n(K_r, Y) = \infty \) for \( n < r \) and \( d_n(K_r, X) = d^n(K_r, Y) = \|g_{n,r}\|_{\infty} \) for \( n \geq r \), where \( g_{n,r}(s) = \int_0^1 h_\xi(t) \cdot K(t,s) dt \) equioscillates \( n-r+1 \) times on \([0,1]\), that is, \( g_{n,r}(\eta_i) = (-1)^i \sigma \|g_{n,r}\|_{\infty} \), \( i = 1, \ldots, n-r+1 \), for some points \( 0 \leq \eta_1 < \cdots < \eta_{n-r+1} \leq 1 \) and \( \sigma = +1 \) or \(-1\). The function \( h_\xi(t) = (-1)^i \), \( \xi_i \leq t < \xi_{i+1}, j = 0, 1, \ldots, n \), where \( \xi_0 = 0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} = 1 \) are points of \([0,1]\), \( \xi = (\xi_1, \ldots, \xi_n) \), and, furthermore, \( (h_\xi, k_i) = \int_0^1 h_\xi(t) k_i(t) dt = 0, i = 1, \ldots, r \).

The subspace \( X_{n,0}^0(\{X_{n,0}^0, \{X_{n,0}^0, \ldots, K(t,\tau_i), \ldots, K(t,\tau_{r-n})\}, \text{where} \tau_1, \ldots, \tau_{r-n} \text{are zeros of} g_{n,r}(t) \text{in} (0,1) \} \text{is an optimal subspace for the} d_n(K_r, X) \). Furthermore, when \( n \geq r \), \( d_n(K_r, X) = \sup_{f \in K_r} \|f - Sf\| \), where \( S \) is the linear mapping from \( C[0,1] \) onto \( X_{n,0}^0 \), defined by the interpolation conditions \( (Sf)(\xi_i) = f(\xi_i), i = 1, \ldots, n, f \in C[0,1] \). For the Gelfand width \( d^n(K_r, Y) \), with \( n \geq r \), the subspace \( L_{n,0}^0 = \{f: f \in C[0,1], f(\xi_i) = 0, i = 1, \ldots, n\} \) of codimension \( n \) is an optimal subspace for \( K_r \).

In the final section of their paper the authors describe briefly a matrix analogue of their results.

I. F. Sarygin (Moscow)