On a best estimator for the class $\mathcal{M}^r$ using only function values.


Let $\mathcal{M}^r = \mathcal{M}^r [0, 1]$ be the class of $r$-fold integrals $f$ of functions $\lambda f$ whose total variation $\|\lambda f\|$ on $[0, 1]$ is bounded. For fixed $x = (x_1, \ldots, x_n)$, $0 < x_1 < \cdots < x_n < 1$, and mappings $S: \mathcal{M}^r \rightarrow \mathcal{M}^r$ which recover the functions $f \in \mathcal{M}^r$ only from their values at $x_1, \ldots, x_n$, the error in the $L^1$-norm $\|\cdot\|_1$ on $[0, 1]$ is defined by $E(x; S) := \sup\{\|f - Sf\|_1/\|\lambda f\|: f \in \mathcal{M}^r\}$. The purpose of the paper is to find and describe an optimal mapping $S_0$ such that $E(x) := \inf_S E(x; S) = E(x; S_0)$.

It turns out that for any fixed $x = (x_1, \ldots, x_n)$ there exist $n - r$ knots $\xi_1, \ldots, \xi_{n-r}$ in $(0, 1)$ such that the interpolation at the points $x_1, \ldots, x_n$ by the spline subspace of degree $r - 1$ with knots $\xi_1, \ldots, \xi_{n-r}$ provides an optimal estimator $S_0$ in the above sense. Using results on $n$-widths in $L^1[0, 1]$ the authors are also able to determine a choice of $x$ which minimizes $E(x)$.

The authors prove these results in a somewhat more general setting than described above. Analogous problems for the supremum norm have been settled by C. Micchelli, T. Rivlin and S. Winograd [Numer. Math. 26 (1976), no. 2, 191–200].

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