Mathematische Zeitschrift manuscript No.
(will be inserted by the editor)

Yehuda Pinchover

On uniqueness and nonuniqueness of the positive Cauchy problem for parabolic equations with unbounded coefficients

Received: January 1995 / Accepted: April 1995

1 Introduction

Let X be a domain in $\mathbb{R}^n$ (or more generally, a smooth manifold) and consider a second order parabolic operator

$$Lu = u_t + \rho(x)Pu \tag{1.1}$$

defined on $X_T = X \times [0,T)$. Here ρ is a positive function and P is a time-independent second order linear elliptic operator of the form

$$P(x,\partial_x) = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(x)\partial_i + c(x)\ , \tag{1.2}$$

where $\partial_i = \partial/\partial x_i$.

A nonnegative solution $u(x,t)$ of the equation $Lu = 0$ in $X_T = X \times [0,T)$ with a nonnegative initial data $u(x,0) = u_0(x)$ is called *a solution of the positive Cauchy problem*. Such a solution can be thought of as the temperature field at the time t in an n-dimensional body which occupies the domain X. The coefficients $a_{ij}(x)$ are the components of the matrix of the thermal conductivities while the function $1/\rho(x)$ is the heat capacity of a unit hypervolume. Hence nonnegative solutions of the Cauchy problem are natural objects to study. In particular, it is of interest to know whether the

---

Research supported in part by the ISRAEL SCIENCE FOUNDATION administered by THE ISRAEL ACADEMY OF SCIENCES AND HUMANITIES and the fund for the promotion of research at the Technion.

Y. Pincover
Department of Mathematics
Technion-Israel Institute of Technology, 32000 Haifa, Israel
Tel.: +972-4-8294086
E-mail: pincho@tx.technion.ac.il

initial condition u_0 determines the solution u or whether some boundary conditions at the "boundary" of X are needed.

The aim of this paper is to study uniqueness and nonuniqueness of the positive Cauchy problem. There are basically two methods for proving the uniqueness of the positive Cauchy problem (UP), one is by showing that nonnegative solutions satisfy a growth condition which guarantees the uniqueness of the general Cauchy problem (see for example [1,2,6,23,24]). In the second method one shows that nonnegative solutions of the equation $Lu = 0$ satisfy a (restricted) uniform Harnack inequality (UHI, see definition 26) and this property implies UP (see [16,18,20,27]).

In Section 3 and Section 4 we prove that the uniqueness of the positive Cauchy problem holds for two classes of parabolic operators in $\mathbb{R}^n$ and in certain subdomains of $\mathbb{R}^n$ by showing that in these cases UHI is satisfied. It turns out that in both cases neither boundary condition nor growth condition are needed. Moreover, we show in Section 7 that although by proving that UHI is satisfied we obtain new and in some cases almost sharp results concerning UP, nevertheless, there are cases where UP holds but UHI does not hold true. So, UHI is a stronger property than UP.

Sections 5 and 6 are devoted to nonuniqueness results (for previous nonuniqueness results see [8,9,20–23,25]). In Section 5 we formulate a general sufficient condition for the nonuniqueness of the positive Cauchy problem depending only on properties of the Green function and the positive solutions of the *elliptic* equation $Pu = 0$ in X (Inequality (2.19), see also (5.1)). It turns out that a positive solution u of the stationary equation $Pu = 0$ in X which satisfies this condition admits a smaller, time-dependent nonnegative solution of the equation $Lu = 0$ in X_T with the same initial condition u.

In Section 6 we show how this general sufficient condition can be applied in the case of radially symmetric Schrödinger semigroups.

2 Preliminaries

Let X be a noncompact connected countable at infinity smooth manifold of dimension n and let $Lu = u_t + Pu$ be a second order parabolic linear operator on $X_T = X \times [0,T)$. Here P is a time-independent elliptic operator which in any coordinate system $(U; x_1, \ldots, x_n)$ has the form (1.2).

We assume that for every $x \in X$ the real quadratic form

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ , \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \tag{2.1}$$

is positive definite. The matrix $[a^{ij}]$, the inverse of $[a_{ij}]$, defines a Riemannian metric g on the manifold X

$$g(\partial/\partial x_i, \partial/\partial x_j) = a^{ij} \ . \tag{2.2}$$

Let $|\cdot|$ and dx be the distance and volume element induced by the Riemannian metric (2.2). We denote by $\mathrm{dist}(x, A)$ the distance of the point x from the set A and by $d(x) = \mathrm{dist}(x, \partial\Omega)$ the distance of x from $\partial\Omega$, the boundary of Ω.

We also denote by $B(x,r) \subset \mathbb{R}^n$ the open ball of radius r centered at x, and let $S(r) = \partial B(0,r)$ the sphere of radius r. The *parabolic box* $Q(x_0,t_0,R,\theta)$ is the set

$$Q(x_0,t_0,R,\theta) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid x \in B(x_0,R),\ t \in (t_0,t_0+\theta R^2)\}\ . \quad (2.3)$$

We assume that the coefficients of the elliptic operator P are real and Hölder continuous. Let $u_0 \in C(X)$ be a nonnegative function. A function u is a *solution of the positive Cauchy problem* in $X_T = X \times [0,T)$ with the initial data u_0 if u is a nonnegative continuous function in X_T, $u(x,0) = u_0(x)$ and $Lu = 0$ in $X \times (0,T)$ in the classical sense.

We say that the *uniqueness of the positive Cauchy problem* for the operator L in X_T holds (and it will be refered to as UP) when any two solutions of the positive Cauchy problem satisfying the same initial condition are identically equal in X_T. Note that no global growth condition or boundary condition are imposed. If nonuniqueness of the positive Cauchy problem takes place we shall denote it by NUP.

It is known (see for example, [6,23]) that UP holds if and only if any solution of the positive Cauchy problem in X_T with initial data $u_0 = 0$ is the zero solution. Hence, it is enough to consider the positive Cauchy problem with zero initial data.

A (nonnegative) solution u of the positive Cauchy problem in X_T with initial data $u_0 = 0$ can be extended to a nonnegative solution of the equation $Lu = 0$ in $X \times (-\infty,T)$ by defining $u(x,t) = 0$ if $t < 0$. Therefore (see [16]), UP holds if any nonzero nonnegative solution of the equation $Lu = 0$ in $X \times (-\infty,T)$ is strictly positive.

We denote the cone of all nonnegative solutions of the equation $Lu = 0$ in $X \times (-\infty,0)$ by $\mathcal{C}_L(X)$ and the cone of all positive solutions of the elliptic equation $Pu = 0$ in X by $\mathcal{C}_P(X)$.

Definition 21 *Let M denote a parabolic operator L or an elliptic operator P defined on X. A solution $u \in \mathcal{C}_M(X)$ is said to be* **minimal** *if for any $v \in \mathcal{C}_M(X)$ with $v \le u$ there exists $c \ge 0$ such that $v = cu$ (in other words, u belongs to an extreme ray of $\mathcal{C}_M(X)$).*

By the Choquet theorem all the nonzero functions of the cone $\mathcal{C}_L(X)$ are strictly positive if and only if each nontrivial minimal solution in $\mathcal{C}_L(X)$ is strictly positive and again, this property implies UP.

The structure of the cone $\mathcal{C}_L(X)$ and its minimal solutions were investigated by many mathematicians in various cases [4,12,16,18,20,27, and the references therein]. The following principle holds true in many of the investigated cases.

Definition 22 *We say that the* **separation principle** *holds in $\mathcal{C}_L(X)$ (and it will be refered to as SP) provided that $\mathcal{C}_L(X) \neq \emptyset$ and a function u is a nontrivial minimal solution in $\mathcal{C}_L(X)$ if and only if there exist a real number $\lambda \le \lambda_0(P,X)$ and a minimal positive solution v_λ of the equation $(P-\lambda)v = 0$ in X such that*

$$u(x,t) = \mathrm{e}^{-\lambda t} v_\lambda(x) \quad \text{on } X \times (-\infty,0). \quad (2.4)$$

Remark 23 *It turns out that* $\lambda_0(P,X)$ *is the generalized principal eigenvalue of the operator* P *in* X *which is given by*

$$\lambda_0(P,X) = \sup\{\lambda \in \mathbb{R} \mid \mathcal{C}_{P-\lambda}(X) \neq \emptyset\} \ . \tag{2.5}$$

Since each element in $\mathcal{C}_P(X)$ is a strictly positive function it follows that SP is a sufficient condition for UP. Moreover, SP implies also an integral representation theorem for solutions in $\mathcal{C}_L(X)$ (see [18,27]). For other applications of SP see [12,20,30–32].

Nonnegative solutions of parabolic equations satisfy the following (local) parabolic Harnack inequality due to Krylov and Safanov [17].

Theorem 24 *Let* u *be a nonnegative solution of the parabolic equation* $Lu = 0$ *in the parabolic box* $Q = Q(x_0, t_0, R, \theta) \subset \mathbb{R}^n$*, and assume that* $\theta > 1$ *and* $R \leq 2$*. Suppose that*

$$\sum_{i=1}^{n} |b_i(x)| + |c(x)| \leq \mu \ , \quad x \in Q \tag{2.6}$$

and

$$\mu^{-1} \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \mu \sum_{i=1}^{n} \xi_i^2 \ , \tag{2.7}$$

for all $x \in Q$ *and* $\xi = (\xi_1 \ , \ldots , \xi_n) \in \mathbb{R}^n$*. Then*

$$u(x_0, t_0 + R^2) \leq Cu(x, t_0 + \theta R^2) \tag{2.8}$$

for every $|x - x_0| \leq R/2$*, where* $C = C(\mu, \theta, n)$.

In the sequel we shall need the following simple refinement of the Krylov-Safanov Harnack inequality.

Lemma 25 *Let* u *be a nonnegative solution of the parabolic equation* $Lu = 0$ *in the parabolic box* $Q_r = Q(x_0, t_0, r, \theta)$ *, and assume that* $\theta > 1$ *and* $r \leq 2$ *. Suppose that*

$$\sum_{i=1}^{n} r|b_i(x)| + r^2|c(x)| \leq \mu \ , \quad x \in Q \tag{2.9}$$

and (2.7) is satisfied. Then

$$u(x_0, t_0 + r^2) \leq Cu(x, t_0 + \theta r^2) \tag{2.10}$$

for every $|x - x_0| \leq r/2$ *, where* $C = C(\mu, \theta, n)$.

Proof. Let us introduce the parabolic scaling

$$x - x_0 = r\eta \ ; \quad t - t_0 = r^2\tau \tag{2.11}$$

and consider the function $v(\eta, \tau) = u(x_0 + r\eta, t_0 + r^2\tau) = u(x,t)$. Then v is a nonnegative solution of the parabolic equation

$$\begin{aligned} L_r v = v_\tau - \textstyle\sum_{i,j=1}^{n} a_{ij}(x_0 + r\eta)\partial_{\eta_i}\partial_{\eta_j} v + \\ r \textstyle\sum_{i=1}^{n} b_i(x_0 + r\eta)\partial_{\eta_i} v + r^2 c(x_0 + r\eta) v = 0 \end{aligned} \tag{2.12}$$

in $Q_1 = Q(0,0,1,\theta)$. The coefficients of the operator L_r satisfy conditions (2.6) and (2.7) in Q_1 and therefore, (2.10) follows by applying the local Harnack inequality (Theorem 24) to the operator L_r and the solution v in $Q = Q_1$. □

The following notion plays a key role in proving SP.

Definition 26 *We say that the* **uniform (restricted) parabolic Harnack inequality** *holds in* $\mathcal{C}_L(X)$ *(and it will be refered to as* UHI*) if* $\mathcal{C}_L(X) \neq \emptyset$ *and for any* $\epsilon > 0$ *there exists a positive constant* $C > 0$ *such that*

$$u(x, t-\epsilon) \le Cu(x,t) \tag{2.13}$$

for all $(x,t) \in X \times (-\infty, 0)$ *and all* $u \in \mathcal{C}_L(X)$.

It is well known (see [16,18,20,27, and the references therein]) that UHI implies SP (and therefore also UP). Moreover, Murata [20] proved that UHI is equivalent to the SP.

We summarize the above results in the following lemma

Lemma 27 *Let* $Lu = u_t + Pu$ *be a parabolic operator defined on* X , *where* P *is an elliptic operator of the form (1.2). Consider the following statements:*

(i) The uniform (restricted) Harnack inequality holds in $\mathcal{C}_L(X)$ *(UHI).*
(ii) The separation principle holds true in $\mathcal{C}_L(X)$ *(SP).*
(iii) Any nontrivial minimal solution in $\mathcal{C}_L(X)$ *is strictly positive.*
(iv) Any nontrivial solution in $\mathcal{C}_L(X)$ *is strictly positive.*
(v) A nonnegative solution of the equation $Lu = 0$ *in* X_T *with a zero initial data is identically zero.*
(vi) The uniqueness of the positive Cauchy problem holds in X_T *(UP).*

Then

$$(i) \Longleftrightarrow (ii) \Longrightarrow (iii) \Longleftrightarrow (iv) \Longrightarrow (v) \Longleftrightarrow (vi)\ . \tag{2.14}$$

For the second part of our paper dealing with nonuniqueness results we need some additional notions and results.

Let $\Omega \subset X$ be a domain and let $\{\Omega_k\}_{k=1}^{\infty}$ be a sequence of smooth relatively compact domains such that $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\cup_{k=1}^{\infty} \Omega_k = \Omega$. Assume that $\mathcal{C}_P(\Omega) \neq \emptyset$ then for every $k \ge 1$ the Dirichlet Green function $G_P^{\Omega_k}(x,y)$ exists and is positive. By the generalized maximum principle $\{G_P^{\Omega_k}(x,y)\}_{k=1}^{\infty}$ is an increasing sequence which, by the elliptic Harnack inequality, converges uniformly in every compact subdomain either to $G_P^{\Omega}(x,y)$, the positive *minimal Green function* of P in Ω and P is said to be *subcritical operator* in Ω , or to infinity and in this case P is *critical* in Ω. The operator P is said to be *supercritical* if $\mathcal{C}_P(\Omega) = \emptyset$. Similarly, one defines $k_L^{\Omega}(x,y,t)$ the *(minimal) heat kernel* of the parabolic operator L in Ω .

Assume that the elliptic operator P is subcritical in X and denote by $k(x,y,t)$ the positive minimal *heat kernel* of the operator L in X . It is well known that

$$G_P^X(x,y) = \int_0^{\infty} k(x,y,t)dt\ . \tag{2.15}$$

Let $u \in \mathcal{C}_P(X)$. It follows from the generalized maximum principle that either

$$\int_X k(x,y,t)u(y)dy = u(x) \text{ for some (and hence for all) } x \in X \tag{2.16}$$

or

$$\int_X k(x,y,t)u(y)dy < u(x) \text{ for some (and hence for all) } x \in X\,. \tag{2.17}$$

Moreover, the function $v(x,t) = u(x) - \int_X k(x,y,t)u(y)dy$ is a nonnegative solution of the equation $Lu = 0$ in X_T with $v(x,0) \equiv 0$. Therefore, in order to show that UP does not hold for the operator L on X it is sufficient to show that v is a nontrivial nonnegative (and hence positive) solution for some $u \in \mathcal{C}_P(X)$.

Remark 28 *A positive solution $u \in \mathcal{C}_P(X)$ which satisfies (2.16) is called a* positive invariant solution. *In 1982 D. Stroock has conjectured that if $\lambda_0 = 0$ then P admits a positive invariant solution. Recently, it turned out that this conjecture is false [29]. Note that if $P1 = 0$ and Equation (2.16) holds for $u = 1$ one says that L* conserves probability*.*

It is well known [5] that Inequality (2.17) holds true if and only if there exists $\lambda < 0$ such that

$$-\lambda \int_X G^X_{P-\lambda}(x,y)u(y)dy < u(x) \tag{2.18}$$

for some x (and hence for all) x in X . Furthermore, as in [24], it follows that (2.18) is satisfied if

$$\int_X G^X_P(x,y)u(y)dy < \infty \tag{2.19}$$

for some x and hence for all x in X . Inequalities (2.18) and (2.19) are closely related to the notions of the equivalence of two Green functions and of small perturbations.

Definition 29 *Let $P_i,\ i = 1,\ 2$ be two subcritical operators in $\Omega \subset X$. We say that the Green functions $G^{\Omega}_{P_1}(x,y)$ and $G^{\Omega}_{P_2}(x,y)$ are* **equivalent** *($G^{\Omega}_{P_1}(x,y) \sim G^{\Omega}_{P_2}(x,y)$) if there exists $C > 0$ such that*

$$C^{-1}G^{\Omega}_{P_2}(x,y) \le G^{\Omega}_{P_1}(x,y) \le CG^{\Omega}_{P_2}(x,y) \tag{2.20}$$

for all $x,\ y \in \Omega,\ x \ne y$.

Definition 210 *Let P be a subcritical operator in Ω and let $V \in C^{\alpha}(\Omega)$. We say that V is a* **small perturbation** *of P in Ω if*

$$\lim_{k\to\infty} \left\{ \sup_{x,y\in\Omega\setminus\Omega_k} \int_{\Omega\setminus\Omega_k} \frac{G^{\Omega}_P(x,z)|V(z)|G^{\Omega}_P(z,y)}{G^{\Omega}_P(x,y)}\, dz \right\} = 0\,. \tag{2.21}$$

It is known [28] that if the operators P and $P+V$ are subcritical in Ω and V is a small perturbation of P in Ω then $G_P^{\Omega}(x,y) \sim G_{P+V}^{\Omega}(x,y)$. This follows from pointwise estimates of the iterated Green kernels corresponding to the Neumann series expansion of the Green function $G_{P+\epsilon V}^{\Omega}(x,y)$ in terms of $G_P^{\Omega}(x,y)$ and V (see [28]). In particular, if $V \equiv 1$ is a small perturbation of the subcritical operator P in X then $G_P^X(x,y) \sim G_{P+1}^X(x,y)$. It is also known [26] that if $G_P^X(x,y) \sim G_{P+1}^X(x,y)$ then (2.19) holds for each $u \in \mathcal{C}_P(X)$.

We summarize the above results in the following lemma

Lemma 211 *Let $Lu = u_t + Pu$ be a parabolic operator defined on X , where P is a subcritical elliptic operator in X of the form (1.2). Consider the following statements:*

(i) $V \equiv 1$ *is a small perturbation of the operator P in X .*
(ii) $G_P(x,y) \sim G_{P+1}^X(x,y)$.
(iii) There exists $u \in \mathcal{C}_P(X)$ such that (2.19) is satisfied.
(iv) There exists $u \in \mathcal{C}_P(X)$ such that (2.18) is satisfied.
(v) There exists $u \in \mathcal{C}_P(X)$ such that (2.17) is satisfied.
(vi) UP does not hold true on X_T .

Then

$$(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv) \Longleftrightarrow (v) \Longrightarrow (iv). \tag{2.22}$$

Remark 212 *Throughout the paper, we usually do not indicate explicitly the quantities that a given constant C depends on. Moreover, in a given context, the same letter C may be used to denote different constants depending on the same set of arguments.*

3 Separation principle in $\mathbb{R}^n$

In this section we deal with a parabolic equation in $\mathbb{R}^n$. We have

Theorem 31 *Let L be a parabolic operator with real Hölder continuous coefficients defined on $X = \mathbb{R}^n$, $n \geq 1$. Let ρ be a Hölder continuous function on X such that*

$$C^{-1}(1+|x|) \leq \rho(x) \leq C(1+|x|) \quad \textit{for all } x \in \mathbb{R}^n \ . \tag{3.1}$$

Suppose that L has the form

$$\begin{gathered} Lu = u_t + Pu = \\ u_t - \rho(x)^{\gamma} \textstyle\sum_{i,\, j=1}^n a_{ij}(x)\partial_i\partial_j u + \sum_{i=1}^n b_i(x)\partial_i u + c(x)u \ . \end{gathered} \tag{3.2}$$

Assume that $\gamma \leq 2$ and there exists a positive constant C such that for all $x \in X$ and for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$,

$$C^{-1}\sum_{i=1}^n \xi_i^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq C\sum_{i=1}^n \xi_i^2 \, , \tag{3.3}$$

and

$$\rho(x)^{-\gamma/2}\sum_{i=1}^{n}|b_i(x)| + |c(x)| \leq C \ . \tag{3.4}$$

Then the separation principle (SP) holds true in $\mathcal{C}_L(\mathbb{R}^n)$. In particular, the uniqueness of the positive Cauchy problem (UP) holds true for L in $\mathbb{R}^n$.

Proof. It follows from Lemma 27 that it is enough to prove that UHI holds true on $\mathbb{R}^n$. It is clear that (3.4) implies that $\mathcal{C}_{P-\lambda}(X) \neq \emptyset$ for $\lambda < 0$ small enough and therefore, $\mathcal{C}_L(X) \neq \emptyset$. Consider $u \in \mathcal{C}_L(X)$ and let $\epsilon > 0$ be fixed. Since the local (Krylov-Safanov) Harnack inequality holds true in any compact subset of $\mathbb{R}^n$ it is sufficient to prove (2.13) for every $x \in \mathbb{R}^n$ such that $|x| \geq 2$. Let us fix (x_0,t_0), $|x_0| = R \geq 2$, $t_0 < -(1+\epsilon)$. Consider the parabolic annulus

$$A_R = \left\{(x,t) \mid R - \frac{R^{\gamma/2}}{2} < |x| < R + \frac{R^{\gamma/2}}{2} \ ; \ 0 \leq t - t_0 \leq 1+\epsilon\right\} \ . \tag{3.5}$$

Let us introduce the dilation

$$x = R\eta \ ; \quad t - t_0 = R^{(2-\gamma)}\tau \ ; \quad x_0 = R\eta_0 \tag{3.6}$$

and consider the function $v(\eta,\tau) = u(R\eta, t_0 + R^{(2-\gamma)}\tau) = u(x,t)$. The annulus A_R is mapped onto the annulus

$$A_{1,R} = \left\{(\eta,\tau) \mid \ |\,|\eta| - 1\,| < \frac{R^{(\gamma/2-1)}}{2} \ ; 0 \leq \tau \leq (1+\epsilon)R^{\gamma-2}\right\} . \tag{3.7}$$

The parabolic box $Q = Q(\eta_0, 0, R^{(\gamma/2-1)}/2, 1+\epsilon)$ is contained in $A_{1,R}$. The nonnegative function v is a solution of the equation $L_R v = 0$ in Q , where

$$L_R v = v_\tau - R^{-\gamma}\rho(R\eta)^\gamma \textstyle\sum_{i,j=1}^n a_{ij}(R\eta)\partial_{\eta_i}\partial_{\eta_j} v + \\ R^{1-\gamma}\textstyle\sum_{i=1}^n b_i(R\eta)\partial_{\eta_i} v + R^{2-\gamma}c(R\eta)v \ . \tag{3.8}$$

In this box the coefficients of L_R satisfy (2.7) and (2.9) with $r = R^{(\gamma/2-1)}/2$. Hence, Lemma 25 implies that there exists a constant C which may depend on ϵ, n and the constants in (3.3) and (3.4) but not on (x_0,t_0) and u such that

$$u(x_0, t_0+1) = v(\eta_0, R^{\gamma-2}) \leq Cv(\eta_0, R^{\gamma-2}(1+\epsilon)) = Cu(x_0, t_0+1+\epsilon) \ .$$

Since such an inequality holds true for any $\epsilon > 0$ it means that nonnegative solutions of the operator L satisfy UHI in $\mathbb{R}^n$. □

Remark 32 *In [2] the authors prove UP for a time-dependent parabolic operator L in $\mathbb{R}^n$. It is assumed there that L has the form (3.2), $0 \leq \gamma \leq 2$ and the coefficients of P and of P^*, the formal adjoint of P, satisfy (3.3) and*

$$\rho(x)^{-1}\sum_{i=1}^{n}|b_i(x)| + \rho(x)^{\gamma-2}|c(x)| \leq C \ . \tag{3.9}$$

So, there is a gap between the lower order terms growth condition (3.4) which guarantees SP and (3.9) which (under some probably unnecessary additional assumptions) guarantees UP. It seems that in this gap SP does not hold, see examples 71, 72, and 73. See also [1, 10, 11, 13–15, 18, 21, 33, and the references therein] for related uniqueness results.

As can be checked from the proof of Theorem 31 the density function ρ need not be two sided bounded by the distance function. We have the following extension of Theorem 31

Theorem 33 *Let L be a parabolic operator with real Hölder continuous coefficients defined on $X = \mathbb{R}^n, n \geq 1$. Let ρ be a Hölder continuous function on X such that*

(i)

$$0 < \rho(x) \leq C(1 + |x|)^2 \quad \textit{for all } x \in \mathbb{R}^n \ , \tag{3.10}$$

(ii)

$$C^{-1} \leq \frac{\rho(x)}{\rho(x_0)} \leq C \tag{3.11}$$

for all $x_o \in \mathbb{R}^n$, $|x_0| \geq 1$ and $x \in B(x_0, r(x_0))$, where $r(x_0) = \rho(x_0)^{1/2}/(2C|x_0|)$.

Suppose that L has the form

$$Lu = u_t + Pu = \\ u_t - \rho(x) \textstyle\sum_{i,j=1}^n a_{ij}(x)\partial_i\partial_j u + \sum_{i=1}^n b_i(x)\partial_i u + c(x)u \ , \tag{3.12}$$

the coefficients a_{ij} satisfy (3.3) and there exists a positive constant C_1 such that for all $x_o \in \mathbb{R}^n$, $|x_0| \geq 1$ and $x \in B(x_0, r(x_0))$

$$\rho(x_0)^{-1/2} \sum_{i=1}^n |b_i(x)| + |c(x)| \leq C_1 \ . \tag{3.13}$$

Then the separation principle (SP) holds true in $\mathcal{C}_L(\mathbb{R}^n)$. In particular, the uniqueness of the positive Cauchy problem (UP) holds true for L in $\mathbb{R}^n$.

Example 34 *The following functions satisfy conditions (3.10) and (3.11) of Theorem 33.*

(1)

$$\rho_1(x) = (1 + |x|^\gamma) \log(1 + |x|) \quad \textit{in} \quad \mathbb{R}^n, \gamma < 2 \, ,$$

(2)

$$\rho_2(x) = \begin{cases} 1 + |x|^{\alpha_+} & \textit{if } x > 1 \\ 1 + |x|^{\alpha_-} & \textit{if } x < -1 \\ 2 & \textit{if } |x| \leq 1 \ , \end{cases}$$

where $\alpha_\pm \leq 2$ and $x \in \mathbb{R}$.

4 Separation principle for domains satisfying the cone condition

In this section we discuss the separation principle for a certain class of parabolic operators defined on domains in $\mathbb{R}^n$ satisfying the cone condition. Note that in [3] this condition plays an important role in proving intrinsic ultracontractivity which implies NUP (see [20,21]).

Definition 41 *(Internal Cone Condition). An open set $\Omega \subset \mathbb{R}^n$ is said to satisfy the (A, δ, β)-***internal cone condition** *if there exist $A \subset S^{n-1}, \delta > 0$ and $\beta > 0$ such that the following holds: for each $x \in \Omega$ with $d(x) < \delta$ there is a point $y(x) \in \partial\Omega$ and a rotation $R_x \in SO(\mathbb{R}^n)$ so that x belongs to the open cone*

$$C(y(x); R_x(A); \delta) = \left\{ y \in \mathbb{R}^n \mid |y - y(x)| < \delta \ , \ \frac{y - y(x)}{|y - y(x)|} \in R_x(A) \right\} \ .$$

of vertex $y(x)$, base $R_x(A)$ and height δ and this cone is contained in Ω. Moreover,

$$dist \left(\frac{x - y(x)}{|x - y(x)|} \ , S^{n-1} \setminus R_x(A) \right) > \beta \ .$$

Theorem 42 *Let L be a parabolic operator with real Hölder continuous coefficients defined on a domain $X \subset \mathbb{R}^n$, $n \geq 1$, which satisfies the (A, δ, β)-internal cone condition. Let ρ be a Hölder continuous function on X such that*

$$C^{-1} \min(d(x), \delta) \leq \rho(x) \leq C \min(d(x), \delta) \tag{4.1}$$

for all $x \in X$. Suppose that L has the form (3.2) and that (3.3) and (3.4) are satisfied with $\gamma \geq 2$. Then the separation principle (SP) holds true in $\mathcal{C}_L(X)$. In particular, the uniqueness of the positive Cauchy problem (UP) holds true for L on X.

Proof. As in Theorem 31 it is clear that $\mathcal{C}_L(X) \neq \emptyset$ and let $u \in \mathcal{C}_L(X)$. It is enough to prove the UHI in a $\delta/2$-neighborhood of ∂X since outside this neighborhood the coefficients of the operator L are bounded and P is uniformly elliptic. Let $x_0 \in X, d(x_0) = R < \delta/2 < 1$. Without loss of generality, we may assume that $y(x_0) = 0$. Fix $\epsilon > 0$ and use the dilation (3.6) to dilate $C(0; R_{x_0}(A); \delta) \times [0, 1+\epsilon] \cap A_R$ onto $B_R = C(0; R_{x_0}(A); 2) \times [0, (1+\epsilon)R^{\gamma-2}] \cap A_{1,R}$, where A_R and $A_{1,R}$ are the annuli (3.5) and (3.7) respectively. There exists a constant $0 < C_\beta \leq 1$ which depends only on A, β and n such that the parabolic box $Q_\beta = Q(\eta_0, 0, C_\beta R^{(\gamma/2-1)}/2, 1+\epsilon)$ is contained in B_R. Consider the function $v(\eta, \tau) = u(R\eta, t_0 + R^{(2-\gamma)}\tau) = u(x,t)$. Again, the nonnegative function v is a solution of the equation $L_R v = 0$ in Q_β, where L_R is given by (3.8).

The rest of the proof is similar to the last part of the proof of Theorem 31. □

Remark 43 *(i) Theorem 42 can be extended to include more general density functions* ρ *(see Theorem 33).*

(ii) Note that unlike [25] we do not have any assumption on $\nabla\rho$ *and the domain* X *need not be* C^∞ *and bounded.*

(iii) It follows from the results in [25] that if X *is a smooth bounded domain (resp.* $X = \mathbb{R}^n$*), the operator* L *has the form* $Lu = u_t + \rho(x)^\gamma \Delta u$ *with* $\gamma < 2$ *(resp.* $\gamma > 2$*) and* ρ *satisfies (4.1) (resp. (3.1)) and also some additional conditions as in [25] then NUP holds. So, in terms of the exponent* γ *our results are sharp (see also Section 6 and [9, 25]).*

5 Nonuniqueness of the positive Cauchy problem

In this section we prove a nonuniqueness lemma for general parabolic operators of the form $Lu = u_t + \rho(x)Pu$. We shall use this result in the next section.

Lemma 51 *Consider the parabolic operator* $Lu = u_t + \rho(x)Pu$ *defined on* X *. Assume that* P *is subcritical in* X *and* ρ *is a positive Hölder continuous function. Then UP does not holds true if*

$$\int_X \frac{G_P^X(x,y)u(y)}{\rho(y)}dy < \infty\,. \tag{5.1}$$

for some $u \in \mathcal{C}_P(X)$ *and* $x \in X$ *. In particular, if* $1/\rho$ *is a small perturbation of* P *in* X *then NUP holds. Moreover, in this case the separated nonnegative solutions of the form (2.4) are not minimal solutions for all* $\lambda < \lambda_0(\rho(x)P, X)$ *, where* $\lambda_0(P, X)$ *is the generalized principal eigenvalue given by (2.5).*

Furthermore, if V *is a Hölder continuous small perturbation of* P *in* X *and the operator* $P + V$ *is subcritical in* X *then (5.1) implies that NUP holds also for the parabolic operator* $L + \rho(x)V(x)$ *.*

Proof. Consider the elliptic operator $P_\rho = \rho(x)P$ then the positive minimal Green function of P_ρ is given by

$$G_{P_\rho}^X(x,y) = \frac{G_P^X(x,y)}{\rho(y)}\,. \tag{5.2}$$

Moreover, $\mathcal{C}_{P_\rho}(X) = \mathcal{C}_P(X)$. Hence, the first assertion of the lemma follows directly from Lemma 211. Note also that by (5.2) the function 1 is a small perturbation of the operator P_ρ if and only if $1/\rho$ is a small perturbation of the operator P. Therefore, Lemma 211 implies that if $1/\rho$ is a small perturbation of the operator P in X then UP does not hold for the operator L on X_T .

Let $\lambda_2 < \lambda_1 < \lambda_0(\rho(x)P, X)$. A function u is a positive solution of the equation $(P_\rho - \lambda_1)u = 0$ if and only if $u > 0$ and satisfies the equation $(P - \lambda_1(\rho(x))^{-1})u = 0$. Since $1/\rho(x)$ is a small perturbation of the operator P in X it follows from [28] that $G_X^{P-\lambda_2(\rho(x))^{-1}}(x,y) \sim G_X^{P-\lambda_1(\rho(x))^{-1}}(x,y)$. Therefore, Theorem 2.3 of [26] implies that there exists a constant $C > 0$

such that for each minimal solution $v_{\lambda_1} \in \mathcal{C}_{P-\lambda_1(\rho(x))^{-1}}(X)$ there exists a minimal solution $v_{\lambda_2} \in \mathcal{C}_{P-\lambda_2(\rho(x))^{-1}}(X)$ such that

$$C^{-1}v_{\lambda_2}(x) \le v_{\lambda_1}(x) \le Cv_{\lambda_2}(x) \ , \quad x \in X \ . \tag{5.3}$$

Hence, $\exp(-\lambda_1 t)v_{\lambda_1}(x)$ is not a minimal solution of $\mathcal{C}_L(X)$. By the same reasons, if V is a small perturbation of P in X and (5.1) is satisfied for the operator P with some $u \in \mathcal{C}_P(X)$ then there exists $v \in \mathcal{C}_{P+V}(X)$ such that

$$\int_X \frac{G^X_{P+V}(x,y)v(y)}{\rho(y)}dy < \infty \ . \tag{5.4}$$

By the first part of the lemma this implies NUP for the operator $Lu = u_t + \rho(x)(P + V(x))u$. □

Remark 52 *According to Lemma 211 and Section 7 the separation principle SP may be thought as a "strong" uniqueness property (UP). Lemma 51 shows that if the function* $1/\rho$ *is a small perturbation of the operator* P *in* X *then we have a "strong" nonuniqueness property. Namely, NUP holds and "almost all" the separated solutions are not minimal solutions.*

6 NUP for radially symmetric Schrödinger semigroups

In this section we discuss some nonuniqueness results for a parabolic operator of the form $Lu = u_t + \rho(x)Pu$ on $\mathbb{R}^n$, $n \ge 2$, where P is a subcritical radially symmetric Schrödinger operator.

Consider a Schrödinger operator $P = -\Delta + V(|x|)$ on $X = \mathbb{R}^n$ with a radially symmetric potential V . We assume that $\mathcal{C}_P(\mathbb{R}^n) \ne \emptyset$. Denote by $g(r)$ the (normalized) radial positive solution of the equation $Pu = 0$ in $\mathbb{R}^n$ and let $f(r) = r^{(n-1)/2}g(r)$ and $w(r) = V(r) + (n-1)(n-3)/4r^2$. It is easy to check that f solves the Jacobi equation $f'' = w(r)f$. The operator P is subcritical in X if and only if $1/f \in L^2(1,\infty)$ (see [19]) and in this case the Green function at zero is radial and is given by

$$G^{\mathbb{R}^n}_P(0,y) = C_n g(|y|) \int_{|y|}^{\infty} f(t)^{-2}dt \ . \tag{6.1}$$

We need the following

Lemma 61 *(i) Suppose that* $w \ge 0$ *and* $n \ge 2$ *then*

$$f(r)f'(r) \int_r^{\infty} f(t)^{-2}dt \le 1 \ . \tag{6.2}$$

(ii) Let $n > 2$ *and suppose that* V_i , $i = 0$, 1 *are subcritical radial potentials such that* $V_0 \le V_1$ *and* $w_0(r) = V_0(r) + (n-1)(n-3)/4r^2 \ge 0$. *Let* g_i , f_i *be the corresponding solutions. Then*

$$\frac{f_1(r)}{f_1'(r)} \le \frac{f_0(r)}{f_0'(r)} \ , \quad r > 0 \ . \tag{6.3}$$

Proof. (i) Since $f(0)=0$ and $f''\geq 0$ it follows that $f'>0$. On the other hand,

$$\frac{1}{f^2(t)}\leq \frac{1}{f^2(t)}\left(1+\frac{f^2(t)w(t)}{(f'(t))^2}\right)=-\left(\frac{1}{f(t)f'(t)}\right)' . \tag{6.4}$$

Hence,

$$\int_r^\infty \frac{dt}{f^2(t)}\leq \frac{1}{f(r)f'(r)} , \tag{6.5}$$

which implies (6.2).

(ii) We may assume that V_i are smooth functions. The positive functions $F_i=f_i'/f_i$ solve the equations

$$F_i'+F_i^2=w_i , \quad i=0 , 1 \tag{6.6}$$

in $(0,\infty)$. Therefore, the function $F=F_1-F_0$ satisfies the differential equation

$$F'+(F_0+F_1)F=V_1-V_0 . \tag{6.7}$$

After integrating (6.7) we obtain

$$f_0(r)f_1(r)F(r)=\int_\epsilon^r (V_1(s)-V_0(s))f_0(s)f_1(s)ds+f_0(\epsilon)f_1(\epsilon)F(\epsilon) . \tag{6.8}$$

It follows from the asymptotic behavior of f_i near 0 that

$$\lim_{\epsilon\to 0} f_0(\epsilon)f_1(\epsilon)F(\epsilon)=0 \tag{6.9}$$

which together with (6.8) implies that $F\geq 0$ and therefore, (6.3) is proved.

□

Lemma 62 *Let ρ be a positive Hölder continuous function defined on $\mathbb{R}^n$ and let $1/\tilde\rho(r)=\int_{S(1)}1/\rho(r\xi)d\sigma(\xi)$, where $r=|x|$,$\xi\in S(1)$ and $r\xi$ is the polar coordinate representation of the point x. Consider the parabolic operator $Lu=u_t+\rho(x)Pu$, where P is a subcritical radially symmetric Schrödinger operator in $\mathbb{R}^n$. Then UP does not hold true for the operator L in $\mathbb{R}^n$ if*

$$C_n\int_1^\infty f^2(r)(\tilde\rho(r))^{-1}\int_r^\infty f(t)^{-2}dt\, dr<\infty . \tag{6.10}$$

Moreover, if $w\geq 0$ and $n>2$ then NUP holds if

$$\int_1^\infty \frac{f(r)}{f'(r)\tilde\rho(r)}dr<\infty . \tag{6.11}$$

Proof. We want to show that (5.1) holds true with $x = 0$ and $u(x) = g(|x|)$, where $g(|x|)$ is the positive radial solution of the equation $Pu = 0$ in $\mathbb{R}^n$. Recall that $f(r) = r^{(n-1)/2}g(r)$. Using polar coordinates, (6.1) and (6.10) we have

$$\int_{\mathbb{R}^n \setminus B(0,1)} \frac{G_P^{\mathbb{R}^n}(0,y)g(|y|)}{\rho(y)} dy =$$

$$C\int_1^\infty f^2(r)(\tilde{\rho}(r))^{-1}\int_r^\infty f(t)^{-2}dt\, dr < \infty \tag{6.12}$$

and therefore, Lemma 51 implies that NUP holds.

Using estimate (6.2) of Lemma 61 we see that (6.10) is satisfied if (6.11) holds true. □

Remark 63 (i) *Assume that* $V_0(|x|) \le V_1(|x|)$, $w_0(|x|) \ge 0$ *and* $0 < \rho_0(x) \le \rho_1(x)$, $x \in \mathbb{R}^n$, $n > 2$. *Consider the corresponding parabolic operators* L_i , $i = 0$, 1 . *It follows from Lemma 61 that if condition (6.11) holds for* L_0 *in* $\mathbb{R}^n$ *it is also true for* L_1 *and therefore, the positive Cauchy problem for* L_1 *does not have a unique solution in* $\mathbb{R}^n$.

(ii) *If the operator L satisfies the conditions of Lemma 62 then NUP holds true also for subcritical small perturbations of the operator P (see Lemma 51).*

Corollary 64 *Consider the parabolic operator* $Lu = u_t + \rho(x)Pu$, *where* $P = -\Delta + V(|x|)$ *is a subcritical radially symmetric Schrödinger operator in* $\mathbb{R}^n$. *Assume that* $V \ge 0$ *and that* $V^{-1/2} \in L^1(1,\infty)$. *Then the positive Cauchy problem does not have a unique solution if* $1/\rho$ *is a bounded function.*

Proof. It follows from [22] that under the above conditions $f/f' \in L^1(1,\infty)$ and therefore, Lemma 62 implies NUP. □

Note that under the same assumptions on the potential V (using a different approach) Murata [22] has shown nonuniqueness for $\rho = 1$. Moreover, under some additional assumptions on the potential V it was shown in [21] that $V^{-1/2} \in L^1(1,\infty)$ is also a necessary condition for NUP ($\rho = 1$).

Corollary 65 *Let P be a Schrödinger operator with a radially symmetric potential defined on* $\mathbb{R}^n$. *Assume that* $\dim \mathcal{C}_P(X) > 1$ *and let* $\rho(x)$ *be a positive Hölder continuous function such that* $\rho(x) \ge C(1+|x|^2)$. *Then UP does not hold true for the operator* $Lu = u_t + \rho(x)Pu$ *in* $\mathbb{R}^n$.

Proof. It is known [19] that $\dim \mathcal{C}_P(X) > 1$ if and only if

$$\int_1^\infty f^2(r)r^{-2}\int_r^\infty f(t)^{-2}dt\, dr < \infty\ . \tag{6.13}$$

Since $\rho(x) \ge C(1+|x|^2)$ it follows that (6.10) is satisfied and this implies NUP. □

Corollary 66 *Suppose that that* $V(x) = a|x|^\beta$, $|x| > 1$, a , $\beta \in \mathbb{R}$ *and assume that* $P = -\Delta + V$ *is subcritical in* $\mathbb{R}^n$.

(i) If $\beta > -2$ *and* $a > 0$ *then NUP holds if*

$$r^{-\beta/2}\tilde{\rho}(r)^{-1} \in L^1(1,\infty)\ . \tag{6.14}$$

(ii) If $\beta \leq -2$ and $n \geq 3$, then NUP holds if

$$|x|^{2-n}(\rho(x))^{-1} \in L^1(\mathbb{R}^n) \ . \tag{6.15}$$

Proof. (i) In this case the radial solution f is given by

$$f(r) = Cr^{1/2} I_{(n-2)/(\beta+2)}(cr^{(\beta/2+1)}) \ , \tag{6.16}$$

where I_ν is the modified Bessel function of order ν and $c = a^{1/2}(\beta/2+1)^{-1}$ (see [19]). It follows from the asymptotic behavior of the Bessel functions at infinity that $f(r)/f'(r) \sim r^{-\beta/2}$. Hence, condition (6.11) is satisfied if (6.14) holds.

(ii) Consider the parabolic operator $Lu = u_t + \rho(x)\Delta u$ in $\mathbb{R}^n \times [0,T)$, $n \geq 3$. Using either Lemma 51 with $u = 1$ or estimate (6.11) we deduce that if (6.15) is satisfied then the Cauchy problem does not have a unique solution in the class of nonnegative solutions and also in the class of bounded solutions. The function $|x|^\beta$, $\beta < -2$ is a small perturbation of the Laplacian in $\mathbb{R}^n$, $n \geq 3$ [19,26,28]. Therefore, NUP for $\beta < -2$ follows from Lemma 51.

Suppose now that $\beta = -2$. Then the product of the positive (radial) solution $u \in \mathcal{C}_P(\mathbb{R}^n)$ and the Green function at the origin equals $C|x|^{(2-n)}$. Hence, NUP follows from (6.15) and Lemma 51. □

Remark 67 *An indirect proof of the NUP for the parabolic operator $Lu = u_t + \rho(x)\Delta u$ in $\mathbb{R}^n \times [0,T)$, $n \geq 3$, with a density ρ satisfying (6.15) can be deduced from a recent paper of Eidus and Kamin [9]. In fact, it is shown there that for each bounded initial data there exists a solution for the Cauchy problem which decays at infinity in an appropriate integral sense. Moreover, for nonnegative initial data this solution is nonnegative. This clearly implies nonuniqueness of the Cauchy problem in the class of bounded solutions and also in the class of nonnegative solutions (NUP) since the bounded solution $u \equiv 1$ does not satisfy this decay condition. For related nonuniqueness results see [8,13,25]*

7 Examples

In this section we present some examples which demonstrate the sharpness and the drawback of our results. The first examples show that UP may hold true while the separation principle is violated.

Example 71 *Consider the parabolic operator $Lu = u_t - \Delta u - |x|^\beta u$, $0 < \beta \leq 2$ on $X = \mathbb{R}^n$, $n \geq 1$. Then $\mathcal{C}_{P-\lambda}(X) = \emptyset$ for every $\lambda \in \mathbb{R}$ and therefore, SP is not satisfied. On the other hand, it follows from [2,21] that UP holds true.*

The next example shows that UHI may not hold true when UP holds and $\mathcal{C}_L(X) \neq \emptyset$.

Example 72 *Consider the parabolic operator $Lu = u_t - \Delta u - x_1 u$ on $X = \mathbb{R}^n$, $n \geq 1$. Then the function*

$$u(x,t) = \exp(t^3/3 + x_1 t) \tag{7.1}$$

is a positive solution in $\mathcal{C}_L(\mathbb{R}^n)$ *which does not satisfy UHI. On the other hand, it follows from [2,21] that UP holds true.*

Example 73 *Consider the parabolic operator* $Lu = u_t - \Delta u - x_1 u_{x_1}$ *on* $X = \mathbb{R}^n$. *Then for every* $\lambda \in \mathbb{R}$ *the function*

$$u_\lambda(x,t) = \exp(\lambda^2 \mathrm{e}^{2t} + \sqrt{2}\,\lambda x_1 \mathrm{e}^t) \tag{7.2}$$

is a positive solution $\mathcal{C}_L(\mathbb{R}^n)$ *which does not satisfy UHI. In fact,* u_λ *is a (nonseparated) minimal solution in* $\mathcal{C}_L(X)$.

More generally, let B *be a* 2×2 *real matrix with eigenvalues* λ_i *such that* $\Re\lambda_i > 0$, $i = 1$, 2 . *Consider the two dimensional Orenstein-Uhlenbeck parabolic operator*

$$Lu = u_t - \Delta u - (Bx, \nabla u) \quad \text{on } X = \mathbb{R}^2 \; . \tag{7.3}$$

The minimal nonnegative solutions $M(x,t,y), y \in \mathbb{R}^2$ *of* $\mathcal{C}_L(X)$ *were computed explicitly in [4]. It is easy to check that* $M(x,t,y), y \in \mathbb{R}^2$ *are strictly positive solutions which do not satisfy SP. Hence, any nontrivial solution in* $\mathcal{C}_L(X)$ *is strictly positive and UP holds true. On the other hand,* $\mathcal{C}_P(X) \neq \emptyset$ *but the separation principle does not hold. Note that the fact that UP holds true for every Orenstein-Uhlenbeck process in* $\mathbb{R}^n$ *follows also from the more general results in [2].*

Example 74 *(see also [7] p. 294) Consider the operator*

$$Lu = u_t - a(x)(a(x)u_{xx} + a'(x)u_x) = u_t + a(x)Pu \quad x \in \mathbb{R} \; , \tag{7.4}$$

where a *is a positive smooth function and* $Pu = -(au')'$ *is a Sturm-Liouville operator. Two linearly independent solutions of the equation* $Pu = 0$ *in* $\mathbb{R}$ *are* $u = 1$ *and* $v(x) = \int_{x_0}^x a(t)^{-1}dt$. *By changing the independent variable* $x \mapsto v = v(x) = \int_0^x a(t)^{-1}dt$ *the operator* L *on* $\mathbb{R}$ *is transformed to the classical heat operator on an interval* $I \subseteq \mathbb{R}$. *In particular, UP holds true for the operator* L *on* $\mathbb{R}$ *if and only if it holds for the heat operator on* I . *Furthermore,* $I = \mathbb{R}$ *if and only if*

$$\int_{-\infty}^0 a(t)^{-1}dt = \int_0^\infty a(t)^{-1}dt = \infty \; . \tag{7.5}$$

Since UP holds for the heat operator on I *if and only if* $I = \mathbb{R}$ *it follows that UP holds for* L *on* $\mathbb{R}$ *if and only if (7.5) is satisfied.*

On the other hand, in order to prove nonuniqueness results using Lemma 51 we need that P *is a subcritical operator and this happens if and only if (7.5) is not satisfied. If we suppose further that* $1/a \in L^1(\mathbb{R})$ *then it is easy to show that all the conditions of Lemma 51 are satisfied and therefore, UP does not hold for* L *on* $\mathbb{R}$. *But condition (5.1) is not satisfied when* $1/a \in L^1(-\infty,0)$ *and* $1/a \notin L^1(0,\infty)$. *Hence, the global condition (5.1) may not show nonuniqueness which is caused by the behavior of the coefficients of the operator* L *near one part of the ideal boundary of* X .

Acknowledgements This paper was written while the author was visiting the Forschungsinstitut für Mathematik, ETH. The author wishes to thank Prof. J. Moser for his hospitality. The author also thanks S. Kamin, V. Lin and M. Murata for valuable discussions and the referee for his helpful remarks.

References

1. Aronson, D. G.: Uniqueness of positive weak solutions of second order parabolic equations, Ann. Polonici Mathematici **16**, 285–303 (1965).
2. Aronson, D. G., Besala, P.: Uniqueness of positive solutions of parabolic equations with unbounded coefficients, Colloquium Mathematicum **18**, 126–135 (1967).
3. Cipriani, F.: Intrinsic ultracontractivity of Dirichlet Laplacians in nonsmooth domains, Potential Analysis **3**, 203–218 (1994).
4. Cranston, M., Orey, S., Rösler, U.: The Martin boundary of two dimensional Orenstein-Uhlenbeck processes, in: Kingman, J. F. C. et al (eds.): Probability, statistics and analysis (London Mathematical Society Lecture Note Series No. 79, pp. 63–78) Cambridge: Cambridge Univ. Press 1983.
5. Davies, E. B.: Heat kernel and spectral theory, Cambridge: Cambridge Univ. Press, 1989.
6. Donnely, H.: Positive solutions of the heat and eigenvalue equations on Riemannian manifolds, in: Naveira, A. M. et al (eds.): Differential geometry, Peñiscola 1985 (Lecture Notes in Math. Vol. 1209, pp. 143–151) Berlin Heidelberg New York: Springer 1986.
7. Eidel'man, S. D.: Parabolic systems, Amsterdam: North-Holland 1969.
8. Eidus, D.: The Cauchy problem for the non-linear filtration equation in an inhomogeneous medium, J. Differential Equations **84** , 309–318 (1990).
9. Eidus, D., Kamin, S.: The filtration equation in a class of functions decreasing at infinity, Proc. Amer, Math. Soc. **120**, 825–830 (1994).
10. Friedman, A.: On the uniqueness of the Cauchy problem for parabolic equations, Amer. J. Math. **81**, 503–511 (1959).
11. Friedman, A.: Partial differential equation of parabolic type, Englewood Cliffs: Prentice Hall 1964.
12. Freire, A.: The Martin boundary of Riemannian products, J. Differential Geometry **33**, 215–232 (1991).
13. Il'in, A. M., Kalashnikov, A. S., Oleinik, O. A.: Linear equations of the second order of parabolic type, Russian Math. Surveys **17**, 1–144 (1972).
14. Kondrat'ev, V. A., Eidel'man, S. D.: Properties of linear evolutionary systems with elliptic space part, Math. USSR Sbornik **10**, 369–397 (1970).
15. Kondrat'ev, V. A., Eidel'man, S. D.: Positive solutions of linear partial differential equations, Trans. Moscow Math. Soc. **31**, 81–148 (1974).
16. Koranyi, A., Taylor, J. C.: Minimal solutions of the heat equation and uniqueness of the positive Cauchy problem on homogeneous spaces, Proc. Amer. Math. Soc. **94**, 273–278 (1985).
17. Krylov, N. V., Safanov, M. V.: A certain property of solutions of parabolic equations with measurable coefficients, Math. USSR Izv. **16**, 151–164 (1981).
18. Lin, V., Pinchover, Y.: Manifolds with group actions and elliptic operators, Memoirs Amer. Math. Soc., Vol. **112**, No. 540 (1994).
19. Murata, M.: Structure of positive solutions to $(-\Delta + V)u = 0$ in $\mathbb{R}^n$, Duke Math. J. **53**, 869–943 (1986).
20. Murata, M.: Uniform restricted parabolic Harnack inequality, separation principle, and ultracontractivity for parabolic equations, in: Komatso, H. (ed.): Functional analysis and related topics, 1991 (Lecture Notes in Math. Vol. 1540, pp. 277–288) Berlin Heidelberg New York: Springer 1993.
21. Murata, M.: Non-uniqueness of the positive Cauchy problem for parabolic equations, J. Differential Equations, to appear.
22. Murata, M.: Sufficient condition for nonuniqueness of the positive Cauchy problem for parabolic equations, in: Yajima, K. (ed.): Spectral and scattering theory and applications (Advanced Studies in Pure Math., Vol. 23, pp. 275–282) Tokyo: Kinokuniya 1994.
23. Murata, M.: Uniqueness and nonuniqueness of the positive Cauchy problem for the heat equation on Riemannian manifolds, Proc. Amer. Math. Soc., to appear.
24. Murata, M.: Nonuniqueness of the positive Dirichlet problem for parabolic equations in cylinders, preprint, 1994.

25. Pang, M. M. H.: L^1 Properties of two classes of singular second order elliptic operators, J. London Math. Soc. (2) **38**, 525–543 (1988).
26. Pinchover, Y.: On positive solutions of second-order elliptic equations, stability results, and classification, Duke Math. J. **57**, 955–980 (1988).
27. Pinchover, Y.: Representation theorems for positive solutions of parabolic equations, Proc. Amer. Math. Soc., **104**, 507–515 (1988).
28. Pinchover, Y.: Criticality and ground states for second-order elliptic equations, J. Differential Equations **80**, 237–250 (1989).
29. Pinchover, Y.: On nonexistence of any λ_0-invariant positive harmonic function, a counter example to Stroock's conjecture, preprint.
30. Taylor, J. C.: Minimal functions, martingales, and Brownian motion on a noncompact symmetric space, Proc. Amer. Math. Soc. **100**, 725–730 (1987).
31. Taylor, J. C.: Do minimal solutions of the heat equations characterize diffusions?, Probab. Th. Rel. Fields **83**, 321–330 (1989).
32. Taylor, J. C.: The product of minimal functions is minimal, Bull. London Math. Soc. **22**, 499–504 (1990), see also erratum in London Math. Soc. **24**, 379–380 (1992).
33. Zitomirskii, Ja. I.: Exact classes of uniqueness for the solution of the Cauchy problem for second-order equations, Soviet Math. Dokl. **7**, 1407–1411 (1966).