On positivity of solutions of degenerate boundary value problems for second-order elliptic equations

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Abstract

In this paper we study the **degenerate mixed boundary value problem:**

$$Pu = f \;\text{ in } \Omega, \;\; Bu = g \;\text{ on } \partial\Omega \setminus \Gamma,$$

where Ω is a domain in $\mathbb{R}^n$, P is a second order linear elliptic operator with real coefficients, $\Gamma \subseteq \partial\Omega$ is a relatively closed set, and B is an oblique boundary operator defined only on $\partial\Omega \setminus \Gamma$ which is assumed to be a smooth part of the boundary.

The aim of this research is to establish some basic results concerning positive solutions. In particular, we study the solvability of the above boundary value problem in the class of nonnegative functions, and properties of the generalized principal eigenvalue, the ground state, and the Green function associated with this problem. The notion of criticality and subcriticality for this problem is introduced, and a criticality theory for this problem is established. The analogs for the generalized Dirichlet boundary value problem, where $\Gamma = \partial\Omega$, were examined intensively by many authors.

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1 Introduction

The aim of this paper is to study some positivity properties of a degenerate mixed boundary value problem for a second order elliptic operator P in a general domain in $\Omega \subseteq \mathbb{R}^n$, where an oblique boundary operator B is defined only on a smooth and relatively open portion of the boundary. On the remaining part of the boundary which we call the **singular set** Γ, we do not explicitly impose any boundary condition. Nevertheless, since we look for positive solutions of minimal growth at Γ (and at infinity), the boundary condition on Γ should be interpreted as a zero Dirichlet boundary condition in some generalized sense. Therefore, we indeed deal with a mixed boundary value problem.

Let Ω be a domain in $\mathbb{R}^n$, and let P be a second order linear elliptic differential operator with real coefficients of the form

$$Pu = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u, \quad x \in \Omega. \tag{1.1}$$

Let $\Gamma \subseteq \partial\Omega$ be a relatively closed set, and suppose that $\partial\Omega \setminus \Gamma$ is a $C^{2,\alpha}$-portion of $\partial\Omega$. For $x \in \partial\Omega \setminus \Gamma$, let $\vec{n}(x)$ be the unit outward normal from the boundary, and $\vec{\nu}(x)$ be a unit vector pointing outward from the boundary. Let B be an oblique boundary operator of the form

$$Bu = \gamma(x)u + \beta(x)\frac{\partial u}{\partial \nu}, \quad x \in \partial\Omega \setminus \Gamma. \tag{1.2}$$

We always assume that

$$a_{ij}, b_i, c \in C^{\alpha}(\bar{\Omega} \setminus \Gamma), \quad 1 \leq i, j \leq n, \tag{1.3}$$

and that for all $x \in \bar{\Omega} \setminus \Gamma$ and $\xi \in \mathbb{R}^n \setminus \{0\}$

$$0 < \Lambda_0(x) \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda(x) \sum_{i=1}^{n} \xi_i^2 \,. \tag{1.4}$$

Furthermore, it is always assumed that

$$\begin{gathered}\bar{\Gamma} = \Gamma,\ \partial\Omega \setminus \Gamma \in C^{2,\alpha},\ \gamma, \beta, \vec{\nu} \in C^{1,\alpha}(\partial\Omega \setminus \Gamma), \quad \text{and} \\ \gamma \geq 0,\ \beta > 0 \text{ and } \vec{\nu} \cdot\ \vec{n} > 0 \text{ on } \partial\Omega \setminus \Gamma.\end{gathered} \tag{1.5}$$

Remark 1.1 Sometimes we use the equivalent formulation for the boundary operator B, namely,

$$Bu = \gamma(x)u + \vec{\beta}(x) \cdot \nabla u \qquad x \in \partial\Omega \setminus \Gamma, \tag{1.6}$$

where $\gamma \geq 0$, $\gamma, \vec{\beta} \in C^{1,\alpha}(\partial\Omega \setminus \Gamma)$, and $\vec{\beta} \cdot \vec{n} > 0$.

We investigate the **degenerate mixed boundary value problem**

$$Pu = f \text{ in } \Omega, \ \ Bu = g \text{ on } \partial\Omega \setminus \Gamma. \tag{1.7}$$

Note that we do not exclude the case where $\Gamma = \emptyset$, and the case where $\Gamma = \partial\Omega$. Recall also that the boundary condition on Γ should be interpreted as a zero Dirichlet boundary condition in some generalized sense. It turns out that in the regular case, where the closed set Γ is a smooth relatively open part of $\partial\Omega$, we actually impose zero Dirichlet boundary condition on Γ. It follows that classical boundary value problems like the Dirichlet, Neumann, Robin, Zaremba and even some cases of the Poincaré problem are covered by our setting. For related results concerning these boundary value problems, see [2, 6, 8, 11, 12, 13, 14, 15, 20, 30, and the references therein].

We study the principal eigenvalue, criticality theory, and general properties of the cone of positive solutions. The analogs for the **generalized Dirichlet problem** (where $\Gamma = \partial\Omega$, and Ω is an arbitrary domain) were examined intensively in [18, 21, 22, 23, 25, and the references therein]. Note that this case is also covered by our setting.

When one compares the present problem with the generalized Dirichlet boundary value problem, one sees that some fundamental properties which hold true for the generalized Dirichlet boundary value problem are not valid or at least are not obvious in our case (see Example 7.8 and 8.21). Consequently, the construction of the Green function for our case is much more complicated. Moreover, even if we impose on Γ the Dirichlet boundary condition, then already in the smooth bounded domain case, the problem is in general not elliptic. Another difficulty that arises is that the natural adjoint boundary operator does not satisfy the assumption (1.5). In this paper, we refrain from discussing the adjoint problem.

The following sets of positive solutions and supersolutions play an important role in our study:

Definition 1.2 We define the following families

$$\begin{aligned}\mathcal{H}_{P,B}(\Omega)=\mathcal{H}_P= \{u\in C^2(\Omega)\cap C^1(\bar\Omega\setminus\Gamma)|\ u>0 \ \text{ in } \Omega,\\ Pu=0 \ \text{ in } \Omega, \text{ and } Bu\geq 0 \text{ on } \partial\Omega\setminus\Gamma\},\end{aligned} \tag{1.8}$$

$$\begin{aligned}\mathcal{H}^0_{P,B}(\Omega)=\mathcal{H}^0_P= \{u\in C^2(\Omega)\cap C^1(\bar\Omega\setminus\Gamma)|\ u>0 \text{ in } \Omega,\\ Pu=0 \text{ in } \Omega, \text{ and } Bu=0 \text{ on } \partial\Omega\setminus\Gamma\},\end{aligned} \tag{1.9}$$

$$\begin{aligned}\mathcal{SH}_{P,B}(\Omega)=\mathcal{SH}_P= \{u\in C^2(\Omega)\cap C^1(\bar\Omega\setminus\Gamma)|\ u>0 \text{ in } \Omega,\\ Pu\geq 0 \text{ in } \Omega, \text{ and } Bu\geq 0 \text{ on } \partial\Omega\setminus\Gamma\},\end{aligned} \tag{1.10}$$

$$\begin{aligned}\mathcal{SH}^0_{P,B}(\Omega)=\mathcal{SH}^0_P= \{u\in C^2(\Omega)\cap C^1(\bar\Omega\setminus\Gamma)|\ u>0 \text{ in } \Omega,\\ Pu\geq 0 \text{ in } \Omega, \text{ and } Bu=0 \text{ on } \partial\Omega\setminus\Gamma\}.\end{aligned} \tag{1.11}$$

If $u \in \mathcal{SH}_{P,B}(\Omega)$, then u is said to be a **positive supersolution** of the operator (P, B) in Ω.

We consider the one-parameter family of operators

$$P_t u := Pu - tW(x)u \quad \text{ in } \Omega,$$

where $W \in C^\alpha(\Omega \setminus \Gamma)$ is a real function, and $t \in \mathbb{R}$. We also introduce the set

$$S = \{t \in \mathbb{R}|\ \mathcal{H}_{P_t}(\Omega) \neq \emptyset\}.$$

If u belongs to one of the families (1.8)–(1.11), then Hopf's lemma implies that $u > 0$ on $\bar\Omega \setminus \Gamma$. The starting point of our analysis is Theorem 2.1, where we extend slightly the generalized maximum principle [26]; it holds if $\mathcal{SH}_P \neq \emptyset$ and either $\Gamma \neq \emptyset$ or Ω is unbounded. Furthermore, Theorem 5.2 states that in this case $\mathcal{H}^0_P \neq \emptyset$. Therefore,

$$S = \{t \in \mathbb{R}|\ \mathcal{H}^0_{P_t} \neq \emptyset\} = \{t \in \mathbb{R}|\ \mathcal{SH}_{P_t} \neq \emptyset\}.$$

Moreover, as in the Dirichlet case, it follows that S is a closed interval, and if $S \neq \emptyset$, then S is bounded if and only if W changes its sign (see lemmas 6.3, and 6.5).

In [14], G. M. Lieberman used the Perron method to derive the solvability of the *regular* oblique boundary value problem in bounded domains under the assumption that $c \geq 0$. Using the same approach, we generalize this result, and prove the solvability of the *degenerate* problem, in the class of nonnegative functions, and in any domain, under the weaker assumption that $\mathcal{SH}_P \neq \emptyset$. The key ingredient of this approach is the property of local solvability, which is proved in Section 3. In Section 4, the Perron process is applied and minimal positive solutions are obtained for two basic degenerate problems (see theorems 4.1 and 4.4).

In Section 5, we define the generalized principal eigenvalue for the boundary value problem (1.7) as

$$\lambda_0 = \lambda_0(\Omega, P, W, B) := \sup S = \sup\{t \in \mathbb{R} |\ \mathcal{H}_{P_t}(\Omega) \neq \emptyset\},$$

and prove in Lemma 7.1 that in regular cases, λ_0 is the classical principal eigenvalue [2], namely, the only eigenvalue with a positive eigenfunction.

H. Berestycki, L. Nirenberg and S. R. S. Varadhan (in [4]), considered the Dirichlet boundary value problem in a bounded domain. They proved that the principal eigenvalue λ_0^D is an increasing, concave, Lipschitz continuous function of c (with respect to the L^∞-norm), and a decreasing function of the domain (see also [17, 25]). In Section 7 it is shown that λ_0 is a concave, Lipschitz continuous, increasing function of the coefficient c, and a continuous monotone function of the weight function W. In addition, λ_0 is a decreasing function of the domain in an appropriate manner. Note that in general, the monotonicity with respect to the domain in the standard sense does not hold true even for the regular oblique derivative problem (see Example 7.8).

Section 8 is devoted to the criticality theory. First, we define for our problem the notions of **positive solutions of minimal growth at infinity of** Ω, the **ground state**, and the **Green function** $G_\Omega^B(x, y)$. In Theorem 8.5, it is proved that the Perron-solution of a certain problem in a neighborhood of infinity in Ω is a positive solution of minimal growth.

We generalize the notion of criticality and subcriticality which was studied in [18, 21, 22, 23, 29] for the generalized Dirichlet problem. The operator (P, B) is **critical** in Ω if the problem admits a ground state with eigenvalue zero, that is, a positive solution in $\mathcal{H}^0_{P,B}(\Omega)$ of minimal growth. The operator (P, B) is **subcritical** if it has a positive solution, but does not possess a ground state, and (P, B) is **supercritical** if $\mathcal{SH}_{P,B}(\Omega) = \emptyset$.

We summarize the main results of Section 8 in the following two theorems. These results are well known for the generalized Dirichlet problem.

Theorem 1.3 *The following assertions are equivalent:*
a) The operator (P, B) *is critical in* Ω.
b) $\dim \mathcal{SH}_P(\Omega) = 1$.
c) $\mathcal{H}^0_P(\Omega) \neq \emptyset$, *and* (P, B) *does not admit a Green function in* Ω.
d) $\mathcal{SH}_P(\Omega) = \mathcal{H}^0_P(\Omega) \neq \emptyset$.
e) For any $W \gneqq 0$, *the operator* $P+W$ *is subcritical and the operator* $P-W$ *is supercritical in* Ω.

Theorem 1.4 *The following assertions are equivalent:*
a) The operator (P, B) *is subcritical in* Ω.
b) $\dim \mathcal{SH}_P(\Omega) > 1$.
c) The operator (P, B) *admits a Green function in* Ω.
d) $\mathcal{SH}_P(\Omega) \setminus \mathcal{H}^0_P(\Omega) \neq \emptyset$.
e) For any $W \in C^\alpha_0(\Omega)$ *there exists* $\varepsilon_0 > 0$ *such that the operator* $P - \varepsilon W$ *is subcritical in* Ω *for every* $|\varepsilon| \leq \varepsilon_0$.

Notation

$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n |\ x_n > 0\}, \quad \mathbb{R}^n_0 = \{x \in \mathbb{R}^n |\ x_n = 0\}.$

$B_\delta(x_0) = \{x \in \mathbb{R}^n |\ |x - x_0| < \delta\}, \quad B_\delta = B_\delta(0).$

$$|u|_{0;\Omega} = \sup_\Omega |u|, \qquad |u|_{k;\Omega} = ||u||_{C^k(\bar\Omega)} = \sum_{|\beta|\leq k} |D^\beta u|_{0;\Omega}.$$

$$[u]_{\alpha;\Omega} = \sup_{\substack{x \neq y \\ x,y\in\Omega}} \frac{|u(x) - u(y)|}{|x-y|^\alpha}, \text{ where } 0 < \alpha \leq 1,$$

$$|u|_{k,\alpha;\Omega} = ||u||_{C^{k,\alpha}(\bar\Omega)} = \sum_{|\beta|\leq k} |D^\beta u|_{0;\Omega} + \sum_{|\beta|=k} [D^\beta u]_{\alpha;\Omega}.$$

Weighted Hölder norms:

Let $\Sigma \subseteq \partial\Omega$, $0 \leq k \in \mathbb{Z}$, $0 < \alpha \leq 1$, $k + \alpha + b \geq 0$;

$\tilde d(x) = \mathrm{dist}(x, \partial\Omega \setminus \Sigma), \quad \Omega_\delta = \{x \in \Omega |\ \tilde d(x) > \delta\}.$

$$|u|^{(b)}_{k,\alpha;\Omega} = \sup_{\delta>0}\{\delta^{k+\alpha+b}|u|_{k,\alpha;\Omega_\delta}\}, \quad C^{k,\alpha,(b)}(\Omega) = \{u \mid |u|^{(b)}_{k,\alpha;\Omega} < \infty\},$$

$$|f|^{(b)}_{k,\alpha;\Sigma} = \inf\{|g|^{(b)}_{k,\alpha;\Omega} |\ g \in C^{k,\alpha,(b)}(\Omega), \lim_{x\to x_0\in\Sigma} g(x) = f(x_0)\}.$$

In a given context, the same letter C will be used to denote different constants depending on the same set of arguments.

2 Auxiliary results

Let P and B be operators of the forms (1.1) and (1.2) satisfying (1.3)-(1.5). The first theorem is a version of the generalized maximum principle [26]:

Theorem 2.1 (Generalized maximum principle) *Let Ω be a domain in $\mathbb{R}^n$. In case that Ω is bounded, assume further that $\Gamma \neq \emptyset$. Suppose that $\mathcal{SH}_P \neq \emptyset$. Assume that $v \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma)$ satisfies*

$$Pv \geq 0 \text{ in } \Omega,\ Bv \geq 0 \text{ on } \partial\Omega \setminus \Gamma,\ \liminf_{\substack{x\to\Gamma \\ x\in\Omega}} \frac{v}{u} \geq 0, \text{ and } \liminf_{\substack{x\to\infty \\ x\in\Omega}} \frac{v}{u} \geq 0, \tag{2.1}$$

for some $u \in \mathcal{SH}_P$. Then either $v > 0$ in $\bar{\Omega} \setminus \Gamma$, or $v = 0$ in $\bar{\Omega} \setminus \Gamma$.

Proof: By the definition of $\mathcal{SH}_P$, and Hopf's lemma, $u > 0$ in $\bar{\Omega} \setminus \Gamma$, and

$$Pu \geq 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad Bu \geq 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma. \tag{2.2}$$

Define $P^u w = \frac{P(uw)}{u}$ on Ω, that is,

$$P^u w = P_0^u w + c^u w = -\sum_{i,j=1}^{n} a_{ij}^u \partial_{ij} w + \sum_{i=1}^{n} b_i^u \partial_i w + c^u w.$$

Clearly, the coefficients, $a_{ij}^u = a_{ij}$, $b_i^u = -\frac{2}{u}\sum a_{ij}\partial_j u + b_i$ are in $C(\bar{\Omega} \setminus \Gamma)$, and $c^u = P^u(1) = \frac{Pu}{u} \geq 0$. We also define $B^u w = \frac{B(uw)}{u}$ on $\partial\Omega \setminus \Gamma$, namely, $B^u w = \gamma^u w + \beta \frac{\partial w}{\partial \nu}$, where $\gamma^u = \frac{Bu}{u}$. Thus, $\gamma^u \geq 0$ and $\beta > 0$ on $\partial\Omega \setminus \Gamma$. Now, take $w = \frac{v}{u}$, so, $P^u w = \frac{Pv}{u} \geq 0$ in Ω, $B^u w = \frac{Bv}{u} \geq 0$ on $\partial\Omega \setminus \Gamma$, $\liminf_{x\to\Gamma} w \geq 0$, and $\liminf_{x\to\infty} w \geq 0$.

If $w = k$, where k is a constant, then $\liminf_{x\to\Gamma} w \geq 0$, or $\liminf_{x\to\infty} w \geq 0$ implies that $k \geq 0$, and $v = ku$. Therefore, either $v = 0$ in Ω, or $v > 0$ in Ω.

Suppose that $w \neq const.$. If a point $x \in \Omega$ such that $w(x) < 0$ exists, then by the strong maximum principle for the operator P^u, either

$$\min\left\{\liminf_{x\to\partial\Gamma} w, \liminf_{x\to\infty} w\right\} < 0,$$

or there is a minimum point $x_0 \in \partial\Omega \setminus \Gamma$ such that $w(x_0) < 0$. The first case contradicts our assumption. In the second case, since $P_0^u w \geq -c^u w \geq 0$ in a neighborhood of x_0, it follows from Hopf's lemma for the operator P_0^u and the function w that $\frac{\partial w(x_0)}{\partial \nu} < 0$. Thus, $B^u w(x_0) < 0$, contradicting $B^u w \geq 0$.

Consequently, $w \geq 0$ in Ω, and, by the strong maximum principle, either $w > 0$ in Ω, or $w = 0$ in Ω. Hence, either $v > 0$ in Ω, or $v = 0$ in Ω. Moreover, if $v > 0$ in Ω, then, by Hopf's lemma, $v > 0$ in $\bar{\Omega} \setminus \Gamma$. □

Remark 2.2 When Ω is a bounded domain and $\Gamma = \emptyset$, the above generalized maximum principle holds true provided that $u \in \mathcal{SH} \setminus \mathcal{H}^0$. Indeed, if $w \neq$ *const.*, then the proof is identical, and when $w = const.$ the proof is trivial.

We extend slightly a lemma of J. Serrin [28].

Lemma 2.3 *Let Ω be a bounded domain in $\mathbb{R}^n$ of class C^2. Assume that P is a uniformly elliptic operator in $\bar{\Omega}$ of the form (1.1) with $C^\alpha(\bar{\Omega})$ coefficients, and B is a boundary operator of the form (1.2) satisfying (1.5). If there exists a function $u(x) \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma)$ such that*

$$u > 0, \quad Pu > 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad Bu \geq 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma, \tag{2.3}$$

then there exists a function $\tilde{u}(x) \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma)$ such that

$$\inf_{\bar{\Omega} \setminus \Gamma} \tilde{u} > 0, \ P\tilde{u} > 0 \text{ in } \Omega, \text{ and } B\tilde{u} > 0 \text{ on } \partial\Omega \setminus \Gamma. \tag{2.4}$$

Proof: We use the Serrin's construction. Let ρ be a positive constant, and define a function

$$h(t) = \begin{cases} 3\rho^2 - 2\rho t - t^2 & 0 \leq t \leq \frac{\rho}{2}, \\ \rho^2 & t \geq \rho, \end{cases}$$

such that $h \in C^\infty[0, \infty)$, $\rho^2 \leq h \leq 3\rho^2$, $|h'| \leq 4\rho$ and $|h''| \leq 8$. Set

$$\tilde{u}(x) = u(x) + \varepsilon h(d(x)),$$

where $d \colon \Omega \to \mathbb{R}$ is the distance function from $\partial\Omega$. The function $h(d(x))$ is a smooth function of x for a sufficiently small ρ. Therefore, $\tilde{u}(x) > \varepsilon\rho^2 > 0$ in $\bar{\Omega} \setminus \Gamma$, and $\tilde{u}(x) \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma)$. Moreover,

$$Ph = -\sum_{i,j=1}^{n} a_{ij}\partial_i(d)\partial_j(d)h'' + \sum_{i=1}^{n} \left(b_i\partial_i(d) - \sum_{j=1}^{n} a_{ij}\partial_i\partial_j(d) \right) h' + ch.$$

By our assumption, $\{a_{ij}(x)\}$ is a continuous and positive definite matrix in $\bar{\Omega}$. Therefore, there exist positive numbers $\Lambda_0 \le \Lambda$ such that

$$\Lambda_0 \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda \sum_{i=1}^{n} \xi_i^2 \,, \tag{2.5}$$

for all $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^n$. It follows that

$$\begin{array}{ll} P\tilde{u} \ge Pu + \varepsilon[2\Lambda_0 - A(3\rho^2 + 3\rho)] & \text{for } d < \rho/2, \\ P\tilde{u} \ge Pu - \varepsilon[8\Lambda + A(3\rho^2 + 4\rho)] & \text{for } \rho/2 \le d \le \rho, \\ P\tilde{u} \ge Pu - \varepsilon A|h| \ge Pu - \varepsilon A\rho^2 & \text{for } d > \rho, \end{array}$$

where A is a positive constant depending on the maximum of $\partial_i\partial_j(d)$ in $0 \le d \le \rho$ and on the bounds of the coefficients in Ω. Furthermore,

$$B\tilde{u} = Bu + \varepsilon[\gamma h + \beta h' \frac{\partial d}{\partial \nu}] \ge Bu + \varepsilon[3\gamma\rho^2 + 2\beta\rho\vec{\nu}\cdot\vec{n}] \text{ on } \partial\Omega \setminus \Gamma.$$

It follows that for ρ and ε small enough,

$$\inf_{\bar{\Omega}\setminus\Gamma} \tilde{u} > 0, \quad P\tilde{u} > 0 \text{ in } \Omega, \quad \text{and } B\tilde{u} > 0 \text{ on } \partial\Omega \setminus \Gamma. \qquad \square$$

Remark 2.4 Let us assume in addition to the hypotheses of Lemma 2.3 that $u \in C(\bar{\Omega})$. Then there exists a function $\tilde{u}(x) \in C^2(\Omega) \cap C^1(\bar{\Omega}\setminus\Gamma) \cap C(\bar{\Omega})$ that satisfies $\tilde{u} > 0$ in $\bar{\Omega}$, $P\tilde{u} > 0$ in Ω, and $B\tilde{u} > 0$ on $\partial\Omega \setminus \Gamma$.

3 Local solvability

In this section, we prove the **local solvability** property which is the key for the construction of the Perron-solution of our mixed boundary value problem.

Definition 3.1 (Local solvability) The boundary value problem

$$Pu = f \text{ in } \Omega, \quad Bu = g \text{ on } \partial\Omega \setminus \Gamma \tag{3.1}$$

is **locally solvable** if for each $y \in \bar{\Omega} \setminus \Gamma$, there is a relatively open subset $N = N(y)$ of $\bar{\Omega} \setminus \Gamma$ containing y such that for any $h \in C(\bar{N})$ there is a (unique) solution $v \in C^2(N) \cap C(\bar{N})$ of the mixed boundary value problem

$$Pv = f \text{ in } N \cap \Omega, \; Bv = g \text{ on } N \cap \partial\Omega, \text{ and } v = h \text{ on } \partial' N, \tag{3.2}$$

where $\partial' N = \partial N \cap \Omega$. We denote this function v by $(h)_y$ to emphasize its dependence on h and y.

To establish the local solvability of (3.1), we first prove the solvability of (3.2) for N and Ω of a special form. For $0 < R < 1$, set

$$\begin{aligned} &x_0 = (0, \ldots, 0, -R), \\ &D_R = \{x \in \mathbb{R}^n_+ \colon |x - x_0| < 1\}, \\ &\Sigma_R = \{x \in \mathbb{R}^n_0 \colon |x - x_0| < 1\}, \end{aligned} \tag{3.3}$$

First, we derive some a priori bounds on solutions of mixed boundary value problems in D_R. These a priori bounds and their proofs are similar to Lemma 3 in [14], but here we do not assume that $c \geq 0$. These bounds, in conjunction with the solvability of a particular boundary value and an approximation argument, will be used to obtain the local solvability.

Lemma 3.2 *Let P and B be operators of the forms (1.1) and (1.6) which are defined on D_R and Σ_R, respectively. Suppose that*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \quad \geq \quad |\xi|^2 \quad \text{ for all } x \in D_R,\ \xi \in \mathbb{R}^n, \tag{3.4}$$

$$\sum_{i=1}^{n} \beta_i(x) n_i(x) \quad \geq \quad 1 \quad \text{ for all } x \in \Sigma_R. \tag{3.5}$$

Let m and M be positive constants with $m < 1$ and $M > 1$, such that

$$|c|_{0;D_R} + \sum_{i=1}^{n} (|b_i|_{0;D_R} + |\beta_i|_{0;\Sigma_R}) \leq M, \tag{3.6}$$

and

$$1 > R > \max\left\{\frac{\sqrt{3}}{2}, \frac{M}{\sqrt{1 - m + M^2}}, \frac{2M}{\sqrt{1 + 4M^2}}, (\sqrt{m^2 + 1} - m)\right\}. \tag{3.7}$$

If $u \in C^2(D_R \cup \Sigma_R) \cap C(\bar{D}_R)$ is a solution of

$$Pu = f \text{ in } D_R, \quad Bu = g \text{ on } \Sigma_R, \quad u = 0 \text{ on } \partial D_R \setminus \Sigma_R, \tag{3.8}$$

then

$$\sup_{D_R} \tilde{d}^{-m}|u| \leq K(|f|^{(2-m)}_{0;D_R} + |g|^{(1-m)}_{0;\Sigma_R}),$$

where $\tilde{d}$ is the distance function to $\partial D_R \setminus \Sigma_R$, and K is a positive constant which depends only on m.

Proof: Set $y = (y_1, \ldots, y_n) = x - x_0$, $r = |y|$, $w_R(x) = (1-r^2)^m$. Note that $\tilde{d}(x) = \mathrm{dist}(x, \partial D_R \setminus \Sigma_R) = 1 - r$. One can check that

$$\begin{cases} Pw_R \geq \frac{m(1-m)}{4}\tilde{d}^{(m-2)} & \text{in } D_R, \\ Bw_R \geq \frac{mM}{(1+4M^2)^{1/2}}\tilde{d}^{(m-1)} & \text{on } \Sigma_R. \end{cases} \tag{3.9}$$

Set

$$v_\pm = \pm u - K(|f|_0^{(2-m)} + |g|_0^{(1-m)})w_R,$$

where $K = \max\{\frac{4}{m(1-m)}, \frac{\sqrt{5}}{m}\}$. From (3.9) we infer that

$$\begin{aligned} Pv_\pm \leq \pm f - \tilde{d}^{m-2}|f|_0^{(2-m)} &\leq 0, \quad \text{in } D_R, \\ Bv_\pm \leq \pm g - \tilde{d}^{m-1}|g|_0^{(1-m)} &\leq 0, \quad \text{on } \Sigma_R, \\ v_\pm = \pm u - K(|f|_0^{(2-m)} + |g|_0^{(1-m)})w_R &= 0 \quad \text{on } \partial D_R \backslash \Sigma_R. \end{aligned}$$

Note that for $R > R'$ which satisfies (3.7), $w_{R'} \in \mathcal{SH}_P(D_R)$, and since $w_{R'} > 0$ on $\partial D_R \setminus \Sigma_R$, we obtain

$$\liminf_{x \to \partial D_R \backslash \Sigma_R} \frac{v_\pm}{w_{R'}} = 0.$$

Hence, by the generalized maximum principle (Theorem 2.1), $v_\pm \leq 0$ in D_R. Since $w_R(x) = (1+r)^m \tilde{d}^m \leq 2\tilde{d}^m$, it follows that,

$$|u|\tilde{d}^{-m} \leq 2K(|f|_0^{(2-m)} + |g|_0^{(1-m)}). \qquad \square$$

Remark 3.3 In fact, we have shown that there exists $w \in \mathcal{SH}_{P,B}(D_R)$ which is strictly positive in $\bar{D}_R$ provided that $1 - R > 0$ is sufficiently small. We note that for the Dirichlet boundary value problem, if Ω is a domain which is contained in a "narrow" strip and P is a uniformly elliptic operator with bounded coefficients, then $\mathcal{SH}_P(\Omega) \neq \emptyset$ (see [4]).

The next step is to show that (3.8) is solvable in D_R under hypotheses similar to those of Lemma 3.2.

Lemma 3.4 *Let us assume in addition to the hypotheses of Lemma 3.2 that* $0 < \alpha < 1$ *and that*

$$|a_{ij}|_{\alpha;D_R} + |b_i|_{\alpha;D_R} + |c|_{\alpha;D_R} + |\beta_i|_{1,\alpha;\Sigma_R} + |\gamma|_{1,\alpha;\Sigma_R} \leq M_1. \tag{3.10}$$

Then for every $f \in C^{\alpha,(2-m)}(D_R)$ and $g \in C^{1,\alpha,(1-m)}(\Sigma_R)$, the boundary value problem (3.8) has a unique solution $u \in C^{2,\alpha,(-m)}(D_R)$. Moreover,

$$|u|^{(-m)}_{2,\alpha;D_R} \le c(\alpha, m, M, M_1, R, n)(|f|^{(2-m)}_{\alpha;D_R} + |g|^{(1-m)}_{1,\alpha;\Sigma_R}). \tag{3.11}$$

Proof: First, we show that any solution $u \in C^{2,\alpha,(-m)}(D_R)$ of (3.8) obeys (3.11). Let u be a $C^{2,\alpha,(-m)}$ solution of (3.8). By the monotonicity properties of the weighted Hölder norms [9, Lemma 2.1],

$$|u|^{(0)}_0 \le |u|^{(-m)}_{0,m} \le C|u|^{(-m)}_{2,\alpha},$$

so, $u \in C^2(D_R \cup \Sigma_R) \cap C(\bar{D}_R)$. It follows from the up to the boundary weighted Schauder estimate [14, Lemma 1] with $b = -m \ge -2 - \alpha$, that

$$|u|^{(-m)}_{2,\alpha;D_R} \le C_1(\sup_{D_R} |\tilde{d}^{-m} u| + |f|^{(2-m)}_{\alpha;D_R} + |g|^{(1-m)}_{1,\alpha;\Sigma_R}).$$

Now, with the aid of Lemma 3.2, we have

$$|u|^{(-m)}_{2,\alpha;D_R} \le C(|f|^{(2-m)}_{0;D_R} + |g|^{(1-m)}_{0;\Sigma_R} + |f|^{(2-m)}_{\alpha;D_R} + |g|^{(1-m)}_{1,\alpha;\Sigma_R}),$$

which implies that

$$|u|^{(-m)}_{2,\alpha;D_R} \le C(|f|^{(2-m)}_{\alpha;D_R} + |g|^{(1-m)}_{1,\alpha;\Sigma_R}).$$

As in [14], in order to obtain the solvability of (3.8), we apply the method of continuity. Consider the Banach space

$$\mathcal{B} = \{u \in C^{2,\alpha,(-m)}(D_R) |\ u = 0 \text{ on } \partial D_R \setminus \Sigma_R\},$$

the normed linear space

$$\mathcal{N} = C^{\alpha,(2-m)}(D_R) \times C^{1,\alpha,(1-m)}(\Sigma_R),$$

with the norm

$$|(f, g)|_{\mathcal{N}} = |f|^{(2-m)}_{\alpha;D_R} + |g|^{(1-m)}_{1,\alpha;\Sigma_R},$$

and the bounded linear operators T_0 and T_1 from $\mathcal{B}$ to $\mathcal{N}$ given by

$$T_0 u = (-\Delta u, -\partial_n u) \quad \text{and} \quad T_1 u = (Pu, Bu).$$

We need to prove that T_1 is surjective. By [14], T_0 is surjective. Define

$$T_\tau = (1-\tau)T_0 + \tau T_1, \quad T_\tau = (P_\tau u, B_\tau u), \quad \text{for } 0 < \tau < 1.$$

We infer that T_1 is surjective if for every $0 \leq \tau \leq 1$

$$|u|_{\mathcal{B}} \leq C|T_\tau u|_{\mathcal{N}}. \tag{3.12}$$

Clearly, P_τ and B_τ satisfy the assumptions of the present lemma (Lemma 3.4). Hence, by the first part of the proof, (3.11) is satisfied with a constant C which is independent on τ. But this means exactly (3.12). $\square$

Next, we prove the solvability of mixed boundary value problems in D_R with nonzero Dirichlet boundary values on $\partial D_R \setminus \Sigma_R$.

Lemma 3.5 *Suppose that the operators P and B satisfy all the assumptions of Lemma 3.4. Then the problem*

$$Pu = f \ \text{in}\ D_R,\ \ Bu = g\ \text{on}\ \Sigma_R,\ \ u = h\ \text{on}\ \partial D_R \setminus \Sigma_R \tag{3.13}$$

has a unique solution $u \in C(\bar{D}_R) \cap C^2(D_R \cup \Sigma_R)$ for every $f \in C^\alpha(D_R)$, $g \in C^{1,\alpha}(\Sigma_R)$ and $h \in C(\partial D_R \setminus \Sigma_R)$.

Proof: By remark 3.3, there exists $w \in \mathcal{SH}_P(D_R) \cap C(\bar{D}_R)$ which is strictly positive in $\bar{D}_R$.

Let $\{h_k\}$ be a $C^3(\bar{D}_R)$ sequence of function which converges to h uniformly on $\partial D_R \setminus \Sigma_R$. Let v_k be the solution of

$$Pv_k = f - Ph_k \ \text{in}\ D_R, \quad Bv_k = g - Bh_k \ \text{on}\ \Sigma_R, \quad v_k = 0 \ \text{on}\ \partial D_R \setminus \Sigma_R,$$

(this problem is solvable by Lemma 3.4). Set $u_k = v_k + h_k$. Since

$$\begin{array}{ll} P(u_k - u_l) = 0 & \text{in}\ D_R, \\ B(u_k - u_l) = 0 & \text{on}\ \Sigma_R, \\ u_k - u_l = h_k - h_l & \text{on}\ \partial D_R \setminus \Sigma_R, \end{array}$$

the maximum principle for to the operator $P^w u = w^{-1}P(wu)$ implies that

$$\sup_{D_R}\left|\frac{u_k - u_l}{w}\right| \leq \sup_{\partial D_R}\left|\frac{u_k - u_l}{w}\right|.$$

Suppose that the supremum of the right hand side is positive, and assume that either a positive maximum point, or a negative minimum point of $\frac{u_k - u_l}{w}$ is achieved on Σ_R. By Hopf's lemma for the operator $P_0^w := P^w - c^w$ and the function $\frac{u_k - u_l}{w}$, we have $B^w(\frac{u_k - u_l}{w}) \neq 0$, and this is a contradiction. Hence,

$$\left|\frac{u_k - u_l}{w}\right|_{0;D_R} \leq \left|\frac{h_k - h_l}{w}\right|_{0;\partial D_R \setminus \Sigma_R},$$

thus,

$$|u_k - u_l|_{0;D_R} \leq \frac{\max |w|}{\min |w|} |h_k - h_l|_{0;\partial D_R \setminus \Sigma_R},$$

and the sequence $\{u_k\}$ converges uniformly to a continuous limit function u. Therefore, $|u_k|_0 \leq C$, and Lemma 1 in [14] implies that

$$|u_k|^{(0)}_{2,\alpha} \leq C(\sup_{D_R} |\tilde{d}^0 u_k| + |f|^{(2)}_{\alpha} + |g|^{(1)}_{1,\alpha}).$$

So, the $C^{2,\alpha,(0)}$ norms of the u_k are uniformly bounded. It follows that $u \in C(\bar{D}_R) \cap C^2(D_R \cup \Sigma_R)$ is a solution of (3.13). □

Finally, we prove the local solvability of (3.1) for a general domain.

Lemma 3.6 *Let P and B be operators of the forms (1.1) and (1.2) satisfying (1.3)-(1.5), and let $f \in C^{\alpha}(\bar{\Omega} \setminus \Gamma)$ and $g \in C^{1,\alpha}(\partial\Omega \setminus \Gamma)$. Then (3.1) is locally solvable.*

Proof: For $y \in \Omega$, take $\delta > 0$ sufficiently small such that $N = N(y) = B_\delta(y) \subset\subset \Omega$, and such that there exists $u \in C^2(N)$ such that $u > 0$ in $\bar{N}$ and $Pu \geq 0$ in N. Note that the existence of a positive supersolution u in a small ball N follows from Remark 3.3.

Now, since $\partial N = \partial' N$, and (3.2) is the Dirichlet problem, its unique solvability for any $h \in C(\partial N)$ is well-known. We remark that the existence of a positive supersolution in the relatively compact subdomain N substitutes for the usual assumption $c \geq 0$.

Let $y \in \partial\Omega \setminus \Gamma$. Using Lemma 3.5, the proof of the local solvability for $y \in \partial\Omega \setminus \Gamma$ is achieved exactly as in [14, Lemma 6] by straightening the boundary, and taking a sufficiently small domain D_R of the form (3.3). □

4 The Perron process for the degenerate problem

The Perron process, usually reserved for the Dirichlet problem, has been used in [14] to prove the solvability of a *regular* mixed boundary value problem. We use here a modification of this process to establish the solvability of the *degenerate* problem (1.7) in the class of nonnegative functions. As mentioned, the main ingredient of the Perron method is the local solvability which was proved in Lemma 3.6.

Theorem 4.1 *Let P and B be operators of the forms (1.1) and (1.2) satisfying (1.3)-(1.5). Let $f \in C_0^{\alpha}(\Omega)$ and $g \in C_0^{1,\alpha}(\partial\Omega \setminus \Gamma)$ be nonnegative functions such that f and g are not both (identically) equal to zero. Assume further that there exists a positive function $v \in C^2(\Omega) \cup C^1(\bar{\Omega} \setminus \Gamma)$ satisfying*

$$\begin{cases} Pv \geq 0 & \text{in } \Omega, \\ Pv \geq \delta_0 > 0 & \text{in } \operatorname{supp}(f), \\ Bv > 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases} \tag{4.1}$$

Then there exists a unique minimal positive solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma)$ of the problem

$$Pu = f \text{ in } \Omega, \ Bu = g \text{ on } \partial\Omega \setminus \Gamma. \tag{4.2}$$

That is, $u > 0$ satisfies (4.2), and if $w \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma)$ is a positive solution of (4.2), then $u \leq w$.

Proof: The case where $\partial\Omega \setminus \Gamma = \emptyset$ is the generalized Dirichlet boundary value problem, and it is known that the existence of a positive supersolution which is not a solution, is a sufficient condition for the solvability of this boundary value problem (see for example [25, Theorem 4.3.8]).

Suppose that $\partial\Omega \setminus \Gamma \neq \emptyset$. By Lemma 3.6, the problem (4.2) is locally solvable. Let $\{N(y)\}_{y \in \bar{\Omega} \setminus \Gamma}$ be the corresponding system of neighborhoods. A **Perron-sub(super)solution** of (4.2) is a function $w \in C(\bar{\Omega} \setminus \Gamma)$, such that

$$\limsup_{\substack{x \to \Gamma \\ x \in \Omega}} \frac{w}{v} \leq 0, \quad \limsup_{\substack{x \to \infty \\ x \in \Omega}} \frac{w}{v} \leq 0, \qquad (\liminf_{\substack{x \to \Gamma \\ x \in \Omega}} \frac{w}{v} \geq 0, \quad \liminf_{\substack{x \to \infty \\ x \in \Omega}} \frac{w}{v} \geq 0,)$$

and for any $y \in \bar{\Omega} \setminus \Gamma$, and any $h \in C(\bar{N}(y))$, if $h \geq w$ ($h \leq w$) on $\partial' N(y)$, then $(h)_y \geq w$ ($(h)_y \leq w$) in $N(y)$ (see Definition 3.1). The set of all Perron-subsolutions (supersolutions) of (4.2) is denoted by S^- (S^+).

Let $w \in C(\bar{\Omega} \setminus \Gamma)$, define $\bar{w} = \bar{w}_y$ the **lift** of w with respect to y, as

$$\bar{w}(x) = \bar{w}_y(x) = \begin{cases} (w)_y(x) & \text{if } x \in N(y), \\ w(x) & \text{otherwise.} \end{cases}$$

We now prove the following properties (1)-(5):

(1) If w_1, w_2 are in S^-, then $\max\{w_1, w_2\} \in S^-$. This property follows immediately from the definition of a Perron-subsolution.

(2) Take $w \in S^-$, $y \in \bar{\Omega} \setminus \Gamma$ and $N = N(y) \subset \bar{\Omega} \setminus \Gamma$. Then $\bar{w} = \bar{w}_y \in S^-$. Clearly, $\bar{w}$ is continuous in $\bar{\Omega} \setminus \Gamma$, $\limsup_{x\to\Gamma} \frac{\bar{w}}{v} \le 0$, and $\limsup_{x\to\infty} \frac{\bar{w}}{v} \le 0$. Take $y_1 \in \bar{\Omega} \setminus \Gamma$ and $N_1 = N(y_1)$. Let h be a function in $C(\bar{N}_1)$, such that $\bar{w} \le h$ on $\partial' N_1$, we claim that $\bar{w} \le (h)_{y_1}$ in N_1.

Since $w \in S^-$, we see that $w \le (w)_y = \bar{w}$ in N, and as $\bar{w} = w$ in $\bar{\Omega} \setminus N$, it follows that $w \le \bar{w}$ in $\bar{\Omega} \setminus \Gamma$. Set $N_2 = N \cap N_1$ and $N_3 = N_1 \setminus N_2$, thus, $N_1 = N_2 \cup N_3$.

We have $w \le \bar{w} \le h$ on $\partial' N_1$ and $w \in S^-$, hence, $w \le (h)_{y_1}$ in N_1 and therefore, $\bar{w} = w \le (h)_{y_1}$ in N_3. In particular, $\bar{w} \le (h)_{y_1}$ on $\partial' N_2 \cap \bar{N}_3$. Furthermore, we have $\bar{w} \le h = (h)_{y_1}$ on $(\partial' N_2 \cap N) \subset \partial' N_1$. Therefore,

$$\begin{aligned} P\bar{w} &= P(h)_{y_1} = f && \text{in } N_2, \\ B\bar{w} &= B(h)_{y_1} = g && \text{on } N_2 \cap \partial\Omega, \\ \bar{w} &\le (h)_{y_1} && \text{on } \partial' N_2, \end{aligned}$$

and by the generalized maximum principle (Theorem 2.1), $\bar{w} \le (h)_{y_1}$ in N_2. Thus, $\bar{w} \le (h)_{y_1}$ in $N_1 = N_2 \cup N_3$.

(3a) Let N be a neighborhood of the local solvability. If $w^\pm \in C^2(N \cap \Omega) \cap C(\bar{N})$ satisfy

$$Pw^+ = Pw^- \text{ in } N \cap \Omega,\ Bw^+ = Bw^- \text{ on } N \cap (\partial\Omega \setminus \Gamma),\ w^+ \ge w^- \text{ on } \partial' N,$$

then either $w^+ = w^-$ in N, or else $w^+ > w^-$ in N. This property follows from the generalized maximum principle (Theorem 2.1).

(3b) If $w^\pm \in S^\pm$, then $w^+ \ge w^-$ in Ω. Set

$$m = \sup_{y\in\Omega} \left(\frac{w^-}{v}(y) - \frac{w^+}{v}(y) \right).$$

We need to prove that $m \le 0$. Suppose that $m > 0$. Define

$$\mathcal{S} = \{ y \in \bar{\Omega} \setminus \Gamma |\ \frac{w^-}{v}(y) - \frac{w^+}{v}(y) = m \}.$$

Let $\xi \in \Gamma \cup \{\infty\}$, then

$$\limsup_{y\to\xi} \left(\frac{w^-}{v}(y) - \frac{w^+}{v}(y) \right) \le \limsup_{y\to\xi} \frac{w^-}{v}(y) - \liminf_{y\to\xi} \frac{w^+}{v}(y) \le 0. \quad (4.3)$$

Hence, for some $R > 0$, the set $\mathcal{S}$ is contained in the compact set $\bar{\Omega}_R := \overline{\Omega \cap B_R}$, and $\mathcal{S} \cap \partial B_R = \emptyset$. Consequently, there exists a sequence of $\{y_k\} \in \Omega_R$ such that $\frac{w^-}{v}(y_k) - \frac{w^+}{v}(y_k) \to m$. Take a subsequence $\{y_{k_l}\}$ which converges to y_0. It follows from (4.3) that $y_0 \notin \Gamma$. Hence, $y_0 \in \bar{\Omega}_R \setminus (\Gamma \cup \partial B_R)$, and from the continuity of $w^\pm$ and v in $\bar{\Omega}_R \setminus (\Gamma \cup \partial B_R)$,

$$\left(\frac{w^-}{v}(y_0) - \frac{w^+}{v}(y_0)\right) = m.$$

Thus, $\mathcal{S} \subset \bar{\Omega}_R \setminus (\Gamma \cup \partial B_R)$ is a nonempty closed set.

We claim that $\mathcal{S} \cap (\partial\Omega \setminus \Gamma) \neq \emptyset$. Otherwise, there is a closest point y_1 in $\mathcal{S}$ to $\partial\Omega_R$. Since $y_1 \in \Omega$, then $N_1 = N(y_1) \subset \Omega$, and $\partial' N_1 = \partial N_1$. Let $\bar{w}^\pm$ be the lifts of $w^\pm$ in N_1. Define $P^v w = \frac{P(vw)}{v}$. Now, $c^v \geq 0$, and

$$\begin{aligned} &P^v\left(\tfrac{\bar{w}^-}{v} - \tfrac{\bar{w}^+}{v}\right) = \tfrac{1}{v}(P(\bar{w}^-) - P(\bar{w}^+)) = 0 &&\text{in } N_1, \\ &\tfrac{\bar{w}^-}{v} - \tfrac{\bar{w}^+}{v} = \tfrac{w^- - w^+}{v} \leq m &&\text{on } \partial N_1. \end{aligned}$$

According to the weak maximum principle,

$$\sup_{N_1} \frac{\bar{w}^- - \bar{w}^+}{v} \leq \sup_{\partial N_1} \left(\frac{w^- - w^+}{v}\right)_+ \leq m,$$

and the strong maximum principle implies that either $\frac{\bar{w}^-}{v} - \frac{\bar{w}^+}{v} < m$ in N_1, or $\frac{\bar{w}^-}{v} - \frac{\bar{w}^+}{v} = m$ in N_1. Since $w^\pm$ is a Perron-super(sub)solution, and $\bar{w}^\pm = w^\pm$ on ∂N_1, then

$$\left(\frac{\bar{w}^-}{v} - \frac{\bar{w}^+}{v}\right)(y_1) \geq \left(\frac{w^-}{v} - \frac{w^+}{v}\right)(y_1) = m.$$

It follows that $\frac{\bar{w}^-}{v} - \frac{\bar{w}^+}{v} = m$ in N_1 and hence,

$$\frac{w^-}{v} - \frac{w^+}{v} = \frac{\bar{w}^-}{v} - \frac{\bar{w}^+}{v} = m \text{ on } \partial N_1,$$

which contains points of $\mathcal{S}$ closer to $\partial\Omega_R$ than y_1, contradicting the definition of y_1.

Let $y_1 \in \mathcal{S} \cap \partial\Omega \setminus \Gamma$, and let $\bar{w}^\pm$ be the lifts of $w^\pm$ in $N_1 = N(y_1) \subset \bar{\Omega} \setminus \Gamma$. Define

$$P^v w = P_0^v w + c^v w = \frac{P(vw)}{v}, \text{ and } B^v w = \frac{Bv}{v} w + \beta \frac{\partial w}{\partial \nu}.$$

Now, c^v, β^v are nonnegative, and $\gamma^v > 0$. Since

$$\begin{aligned} &P^v(\tfrac{\bar{w}^-}{v} - \tfrac{\bar{w}^+}{v}) = 0 \qquad \text{in } N_1 \\ &B^v(\tfrac{\bar{w}^-}{v} - \tfrac{\bar{w}^+}{v}) = 0 \qquad \text{on } \partial N_1 \cap \partial\Omega, \\ &\tfrac{\bar{w}^-}{v} - \tfrac{\bar{w}^+}{v} \le m \qquad\quad \text{on } \partial' N_1, \end{aligned}$$

by the strong maximum principle and the Hopf lemma for the operator P_0^v,

$$\text{either} \quad \frac{\bar{w}^-}{v} - \frac{\bar{w}^+}{v} < m, \quad \text{or} \quad \frac{\bar{w}^-}{v} - \frac{\bar{w}^+}{v} = m \quad \text{ in } \bar{N}_1.$$

From

$$\left(\frac{\bar{w}^-}{v} - \frac{\bar{w}^+}{v}\right)(y_1) \ge \left(\frac{w^-}{v} - \frac{w^+}{v}\right)(y_1) = m,$$

it follows that $\frac{\bar{w}^-}{v} - \frac{\bar{w}^+}{v} = m$ in $\bar{N}_1$. But then

$$B^v(\frac{\bar{w}^- - \bar{w}^+}{v}) = \gamma^v m = \frac{Bv}{v} m > 0 \text{ on } \partial N_1 \cap \partial\Omega,$$

and this is a contradiction. Consequently, $m \le 0$, so, $w^- \le w^+$ in Ω.

(4) $S^\pm$ are not empty. Clearly, $0 \in S^-$, and $kv \in S^+$, for k satisfying $k > \max\{\frac{|g|_0}{\varepsilon_0}, \frac{|f|_0}{\delta_0}\}$, where $\varepsilon_0 = \min\{Bv(x)|\ x \in \operatorname{supp}(g)\}$.

(5) Let N be a relatively open subset of $\bar{\Omega} \setminus \Gamma$, and let $\{u_k\}$ be a bounded sequence of $C^2(N) \cap C(\bar{N})$ solutions of

$$Pu_k = f \text{ in } N \cap \Omega, \;\; Bu_k = g \text{ on } N \cap \partial\Omega.$$

Then there is a subsequence $\{u_{k_l}\}$ converging to a solution u of

$$Pu = f \text{ in } N \cap \Omega \text{ and } Bu = g \text{ on } N \cap \partial\Omega.$$

The desired property follows directly from the up to the boundary weighted Schauder estimate [14, Lemma 1] and the Arzelà-Ascoli theorem.

We now define

$$u = \sup\{w|\ w \in S^-\},$$

and prove that u is a $C^2(\bar{\Omega} \setminus \Gamma)$ solution of (4.2). By (4), $S^\pm$ are not empty. Take $w^\pm \in S^\pm$. Let K be a compact set in $\bar{\Omega} \setminus \Gamma$. According to (3b), for every $w \in S^-$ we have $w \leq w^+$ in Ω, and hence in $\bar{K}$. Therefore, $\{w|\ w \in S^-\}$ is a nonempty bounded from above set in every compact set K, and $u = \sup\{w|\ w \in S^-\}$ is well defined. Set $y \in \bar{\Omega} \setminus \Gamma$ and let $\{w_k\} \subset S^-$ be a sequence such that $w_k(y) \to u(y)$. By replacing w_k with $\tilde{w}_k = \max\{w_k, w_1\}$ and using property (1), we obtain a locally bounded sequence of Perron-subsolutions. Property (2) implies that the sequence $\{\bar{w}_k\}$ of the lifts of $\tilde{w}_k$ with respect to y is contained in S^-. Moreover, $\bar{w}_k$ are locally uniformly bounded in Ω, and $w_1 \leq \tilde{w}_k \leq \bar{w}_k \leq w^+$. In view of property (5), $\{\bar{w}_k\}$ has a subsequence $\{\bar{w}_{k_l}\}$ which converges in $N = N(y)$ to a function $\bar{w} \in C^2(N)$ that satisfies

$$P\bar{w} = f \text{ in } N \cap \Omega, \text{ and } B\bar{w} = g \text{ on } N \cap \partial\Omega.$$

We have $\bar{w}_{k_l} \leq u$ in N, thus, $\bar{w} \leq u$ in N. Note that $\bar{w}(y) = u(y)$ (as $w_k(y) \leq \bar{w}_k(y) \leq u(y)$, and $w_k(y) \to u(y)$). We claim that $\bar{w} = u$ in N. Suppose that for some $z \in N$, $\bar{w}(z) < u(z)$, then there exists $w_0 \in S^-$ such that $\bar{w}(z) < w_0(z) \leq u(z)$. By replacing $\tilde{w}_k$ with $\hat{w}_k = \max\{\bar{w}_k, w_0\}$, and taking the lifts and then a converging subsequence, we obtain a solution $\hat{w} \in C^2(N)$ of

$$P\hat{w} = f \text{ in } N \cap \Omega, \text{ and } B\hat{w} = g \text{ on } N \cap \partial\Omega.$$

From our construction of $\hat{w}$, $\bar{w} \leq \hat{w}$ in N, and in particular, on $\partial' N$, hence, (3a) implies that either $\bar{w} < \hat{w}$, or $\bar{w} = \hat{w}$ in N. Since $\hat{w}_k \in S^-$, we have $\hat{w}_k \leq u$ and therefore, $\bar{w}(y) = u(y) \geq \hat{w}(y)$. Consequently, $\bar{w} = \hat{w}$ in N, which contradicts $\bar{w}(z) < w_0(z) \leq \hat{w}(z)$. Thus, $\bar{w} = u$ in N. Since y is an arbitrary point of $\bar{\Omega} \setminus \Gamma$, it follows that u is a $C^2(\bar{\Omega} \setminus \Gamma)$ solution of

$$Pu = f \text{ in } \Omega, \text{ and } Bu = g \text{ on } \partial\Omega \setminus \Gamma.$$

Recall that $0 \in S^-$, so, $u \geq 0$. Since by our assumptions either $Bu = g \neq 0$ on $\partial\Omega \setminus \Gamma$, or $Pu = f \neq 0$ in Ω, we conclude that $u > 0$ in $\bar{\Omega} \setminus \Gamma$.

Now, let $w \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma)$ be a positive solution of (4.2). Clearly, $w \in S^+$, hence, $u \leq w$ and u is the minimal positive solution of (4.2). □

Definition 4.2 The solution which was obtained in Theorem 4.1, is said to be a **Perron-solution** of problem (4.2).

From the minimality of the Perron-solution we obtain

Corollary 4.3 *The Perron-solution of problem (4.2) depends neither on the positive supersolution* $v \in \mathcal{SH}_P(\Omega)$*, nor on the system of neighborhoods* $\{N(y)\}_{y\in\bar{\Omega}\setminus\Gamma}$.

We can use the Perron method to obtain a positive solution for the following exterior degenerate mixed boundary value problem:

Theorem 4.4 *Suppose that* P *and* B *are operators of the forms (1.1) and (1.2) satisfying (1.3)-(1.5). Let* K *be a nonempty compact set in* Ω *with a smooth boundary, and let* $g \in C(\partial K)$ *be a positive function. Assume also that* $\mathcal{SH}_P(\Omega) \neq \emptyset$*. Then there exists a positive Perron-solution*

$$u \in C^2(\bar{\Omega} \setminus (K \cup \Gamma)) \cap C((\overline{\Omega \setminus K}) \setminus \Gamma)$$

of the problem

$$Pu = 0 \text{ in } \Omega \setminus K, \ \ Bu = 0 \text{ on } \partial\Omega \setminus \Gamma, \ \ u = g \text{ on } \partial K. \tag{4.4}$$

Moreover, u *is the minimal positive solution of this problem, namely, if* w *is a positive (classical) solution of (4.4), then* $u \leq w$*. In particular,* u *depends neither on the supersolution* v*, nor on the system of the neighborhoods.*

Proof: Note that although on ∂K we impose the Dirichlet boundary condition, we do not consider ∂K as part of the singular set Γ, since it is a disjoint smooth component of the boundary.

First, we prove the solvability of problem (4.4) under the stronger assumption that there exists $v \in \mathcal{SH}_P(\Omega)$ such that $Bv > 0$ on $\partial\Omega \setminus \Gamma$.

Set $\Omega' = \Omega \setminus K$. We say that (4.4) is **locally solvable** if for each $y \in \Omega' \cup (\partial\Omega \setminus \Gamma)$, there is a relatively open subset $N = N(y)$ of $\Omega' \cup (\partial\Omega \setminus \Gamma)$ with $y \in N$ and $\bar{N} \cap (\Gamma \cup \partial K) = \emptyset$, such that for every $h \in C(\bar{N})$ there is a unique solution $(h)_y \in C^2(N) \cap C(\bar{N})$ of

$$Pu = 0 \text{ in } N \cap \Omega, \ Bu = 0 \text{ on } N \cap \partial\Omega, \text{ and } \ u = h \text{ on } \partial' N, \tag{4.5}$$

where $\partial' N = \partial N \cap \Omega'$. A **Perron-subsolution** of (4.4) is $w \in C(\overline{\Omega'} \setminus \Gamma)$ satisfying

$$\limsup_{x\to\Gamma} \frac{w}{v} \leq 0, \ \limsup_{x\to\infty} \frac{w}{v} \leq 0, \ w \leq g \text{ on } \partial K,$$

and for each $y \in \Omega' \cup (\partial\Omega \setminus \Gamma)$ if $w \leq h$ on $\partial' N(y)$, then $w \leq (h)_y$ in $N(y)$. A **Perron-supersolution** is defined similarly. The set of all Perron-subsolutions (supersolutions) of (4.4) is denoted by $S^-(\Omega')$ $(S^+(\Omega'))$.

As in Theorem 4.1, one can prove that

$$u(x)=\sup\{w^-(x)|\ w^-\in S^-(\Omega')\}$$

is a $C^2(\bar\Omega\setminus(K\cup\Gamma))$ solution of (4.4). Note that for every $y\in\partial K$ and $w^\pm\in S^\pm(\Omega')$ we have

$$\left(\frac{w^-}{v}(y)-\frac{w^+}{v}(y)\right)\le\frac{g}{v}(y)-\frac{g}{v}(y)=0.$$

Furthermore, since ∂K is smooth, by a standard local barrier argument, u satisfies the boundary condition on ∂K.

Suppose now that $Bv\ge 0$ on $\partial\Omega\setminus\Gamma$, we use the first part of the proof and solve the sequence of problems:

$$Pu_k=0 \text{ in } \Omega\setminus K,\ \ (B+\frac{1}{k})u_k=0 \text{ on } \partial\Omega\setminus\Gamma,\ \ u_k=g \text{ on } \partial K. \qquad (4.6)$$

Since $Cv\ge g$ on ∂K for some $C>0$, the sequence $\{u_k\}$ satisfies $0\le u_k\le Cv$. Hence, $\{u_k\}$ has a subsequence converging to a solution u of (4.4). Note that the condition $u=g>0$ on ∂K implies that u is positive on $\Omega\setminus K$.

It remains to prove the minimality of u. If w is a nonnegative solution of (4.4), then $w\in S_k^+$ of problem (4.6). Hence, $u_k\le w$ and therefore, $u\le w$. □

Proposition 4.5 *1. Let N be an open set in Ω, $y\in\bar N\cap\Gamma$, $v\in\mathcal{SH}_P$, and u be the Perron-solution of (4.2) or (4.4). Then*

$$\lim_{\substack{x\to y\\ x\in\bar N\setminus\Gamma}} v(x)=0 \quad\Rightarrow\quad \lim_{\substack{x\to y\\ x\in\bar N\setminus\Gamma}} u(x)=0.$$

2. Suppose further that $v\in C(\bar N)$, and $v(y)=0$ for every $y\in\bar N\cap\Gamma$. Then $\tilde u\in C(\bar N)$, where

$$\tilde u(x)=\begin{cases} u(x) & x\in\Omega\setminus\Gamma\\ 0 & x\in\bar N\cap\Gamma.\end{cases}$$

Proof: 1. Take $w\in S^-$. For k sufficiently large, $kv\in S^+$, hence, $w(x)\le kv(x)$ in $\bar N\setminus\Gamma$. Recall that $0\in S^-$. Therefore,

$$0\le u(x)=\sup\{w(x)|\ w\in S^-\}\le kv(x) \quad \text{in } \bar N\setminus\Gamma.$$

Since $\lim_{\substack{x\to y\\ x\in\bar N\setminus\Gamma}} v(x)=0$, it follows that $\lim_{\substack{x\to y\\ x\in\bar N\setminus\Gamma}} u(x)=0$.

2. Let $y \in \bar{N} \cap \Gamma$, and take $x_n \to y$. If $\{x_n\} \subseteq \bar{N} \setminus \Gamma$, then by part 1, $\lim_{x_n \to y} \tilde{u}(x) = \lim_{x_n \to y} u(x) = 0$. On the other hand, if $x_n \in \bar{N} \cap \Gamma$, then $\tilde{u}(x_n) = 0$, and, $\lim\limits_{x_n \to y} \tilde{u}(x) = 0$. Thus, $\tilde{u}$ is continuous in $\bar{N}$. □

5 The generalized principal eigenvalue

The aim of this section is to generalize the notion of the (classical) principal eigenvalue to our problem, and to study its properties. To this end, we prove that the sets $S = \{t|\ \mathcal{H}_{P_t}(\Omega) \neq \emptyset\}$ and $\{t|\ \mathcal{SH}_{P_t}(\Omega) \neq \emptyset\}$ are equal by showing that the existence of a positive supersolution implies the existence of a positive solution of the homogeneous degenerate mixed boundary value problem.

Definition 5.1 Let $P_t = P - tW$, $t \in \mathbb{R}$. The **generalized principal eigenvalue** λ_0 of the boundary value problem

$$P_t u = f \text{ in } \Omega, \text{ and } Bu = g \text{ on } \partial\Omega \setminus \Gamma$$

is defined by

$$\lambda_0 = \lambda_0(\Omega, P, W, B) := \sup\{t \in \mathbb{R}|\ \mathcal{H}_{P_t}(\Omega) \neq \emptyset\}, \tag{5.1}$$

where $\mathcal{H}_P$ is defined by (1.8).

Theorem 5.2 *Let P and B be operators of the forms (1.1) and (1.2) satisfying (1.3)-(1.5). If Ω is bounded suppose further that $\Gamma \neq \emptyset$. Let $\mathcal{SH}_P(\Omega)$, $\mathcal{SH}^0_P(\Omega)$, $\mathcal{H}_P(\Omega)$, and $\mathcal{H}^0_P(\Omega)$ be the sets as defined in Definition 1.2. Then*

$$\{t \in \mathbb{R}|\ \mathcal{SH}_{P_t}(\Omega) \neq \emptyset\} = \{t \in \mathbb{R}|\ \mathcal{H}_{P_t}(\Omega) \neq \emptyset\} =$$
$$\{t \in \mathbb{R}|\ \mathcal{SH}^0_{P_t}(\Omega) \neq \emptyset\} = \{t \in \mathbb{R}|\ \mathcal{H}^0_{P_t}(\Omega) \neq \emptyset\}.$$

Proof: It is obvious that

$$\{t|\ \mathcal{H}^0_{P_t}(\Omega) \neq \emptyset\} \subseteq \{t|\ \mathcal{H}_{P_t}(\Omega) \neq \emptyset\} \subseteq \{t|\ \mathcal{SH}_{P_t}(\Omega) \neq \emptyset\},$$

and

$$\{t|\ \mathcal{H}^0_{P_t}(\Omega) \neq \emptyset\} \subseteq \{t|\ \mathcal{SH}^0_{P_t}(\Omega) \neq \emptyset\} \subseteq \{t|\ \mathcal{SH}_{P_t}(\Omega) \neq \emptyset\}.$$

Therefore, it suffices to prove that

$$\{t|\ \mathcal{SH}_{P_t}(\Omega) \neq \emptyset\} \subseteq \{t|\ \mathcal{H}^0_{P_t}(\Omega) \neq \emptyset\}.$$

Suppose that $\mathcal{SH}_{P_t}(\Omega) \neq \emptyset$. Take $\{p_k\}$ a sequence of points in Ω which converges either to $p \in \Gamma$ (if $\Gamma \neq \emptyset$), or to infinity (if Ω is unbounded). Set $\varepsilon_k = \min\{\frac{1}{k}, \frac{\mathrm{dist}(p_k,\partial\Omega)}{2}\}$, and $B_k = B_{\varepsilon_k}(p_k)$.

By Theorem 4.4, there exists $w_k \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma)$ that satisfies

$$w_k > 0,\ P_t w_k = 0 \text{ in } \Omega \setminus B_k,\ \ Bw_k = 0 \text{ on } \partial\Omega \setminus \Gamma,\ \ w_k = 1 \text{ on } \partial B_k. \quad (5.2)$$

Fix $x_1 \in \Omega \setminus \cup_{k=1}^{\infty} B_k$ and define $v_k(x) = \frac{w_k(x)}{w_k(x_1)}$. Using the local Harnack inequality and the interior Schauder estimate, we infer that

$$|v_k|_{2,\alpha;K'} \leq C|v_k|_{0;K} = CM_K,$$

for every $K' \subset\subset\ K \subset\subset \Omega$.

Let K be a compact set in $\bar{\Omega}\setminus\Gamma$ such that $K \cap \Omega \neq \emptyset$. Recall that $Bv_k = 0$ on $K \cap \partial\Omega$. By the local and up to the boundary Harnack inequalities [3], and the up to the boundary weighted Schauder estimates [14, Lemma 1], there exists a positive constant N_K such that for every k large enough

$$|v_k|^{(0)}_{2,\alpha;K} \leq C|v_k|_{0;K} \leq CN_K.$$

Using the Arzelà-Ascoli theorem for $\{v_k\}$ and its derivatives up to order 2, and by applying the diagonal method, we may extract a subsequence which converges in every relatively compact subdomain of $\bar{\Omega} \setminus \Gamma$ to a function v_0 which satisfies

$$v_0 > 0 \text{ in } \Omega,\ P_t v_0 = 0 \text{ in } \Omega, \text{ and } Bv_0 = 0 \text{ on } \partial\Omega \setminus \Gamma.$$

Therefore, $v_0 \in \mathcal{H}^0_{P_t}(\Omega)$. Thus, $\{t|\ \mathcal{SH}_{P_t}(\Omega) \neq \emptyset\} \subseteq \{t|\ \mathcal{H}^0_{P_t}(\Omega) \neq \emptyset\}$. □

Remark 5.3 Under some assumptions, the solution $v_0 \in \mathcal{H}^0_{P_t}$ that was constructed in Theorem 5.2 satisfies the homogeneous Dirichlet boundary condition on Lipschitz portions of Γ. More precisely, let N be an open set in Ω, such that $\bar{N} \cap \partial\Omega \subseteq \Gamma \setminus \{p\}$, where p is the boundary point in the proof of Theorem 5.2. Suppose that $\Gamma \cap \bar{N}$ is a Lipschitz portion of $\partial\Omega$. Take

$u_0 \in \mathcal{SH}_{P_t}$ and suppose that $u_0 \in C(\bar{N})$, and $u_0(y) = 0$ for every $y \in \Gamma \cap \bar{N}$. Then $\tilde{v}_0 \in C(\bar{N})$, where

$$\tilde{v}_0(x) = \begin{cases} v_0(x) & x \in \Omega \setminus \Gamma \\ 0 & x \in \bar{N} \cap \Gamma. \end{cases}$$

Indeed, let $\{v_k\}$ be the normalized sequence that converges to v_0 in the proof of Theorem 5.2. Fix $y \in \Gamma \cap N$ and let $B_{8r}(y)$ be a ball contained in N. By Proposition 4.5, the extended functions $\tilde{v}_k(x)$ are in $C(B_{8r}(y) \cap \bar{\Omega})$. Recall that $P_t \tilde{v}_k = 0$ in $B_{8r}(y) \cap \Omega$ and $\tilde{v}_k = 0$ on $B_{8r}(y) \cap \partial\Omega \subseteq \Gamma$. According to the boundary Harnack principle [5], there exists $C > 0$, such that for every $k \geq 1$, and every $x \in B_r(y) \cap \Omega$

$$v_k(x) < C \frac{v_1(x) v_k(x_0)}{v_1(x_0)},$$

where $x_0 \in \partial B_r(y) \cap \Omega$. The sequence $\{v_k\}$ is locally uniformly bounded in Ω, in particular, $M^{-1} \leq v_k(x_0) < M$. So, $v_k(x) < C v_1(x)$, and $0 < v_0(x) \leq C v_1(x)$. In view of Proposition 4.5, $\lim_{\substack{x \to y \\ x \in \bar{N}}} \tilde{v}_1(x) = 0$, therefore, $\lim_{\substack{x \to y \\ x \in \bar{N}}} \tilde{v}_0(x) = 0$. Thus, $\tilde{v}_0$ is continuous in $\bar{N}$.

Proposition 5.4 *Let Ω be a smooth bounded domain and $\Gamma = \emptyset$ (the regular oblique derivative problem). Then*

$$\{t |\ \mathcal{H}_{P_t}(\Omega) \neq \emptyset\} = \{t |\ \mathcal{SH}^0_{P_t}(\Omega) \neq \emptyset\} = \{t |\ \mathcal{SH}_{P_t}(\Omega) \neq \emptyset\}.$$

Proof: Clearly,

$$\{t |\ \mathcal{SH}^0_{P_t}(\Omega) \neq \emptyset\} \subseteq \{t |\ \mathcal{SH}_{P_t}(\Omega) \neq \emptyset\},$$

$$\{t |\ \mathcal{H}_{P_t}(\Omega) \neq \emptyset\} \subseteq \{t |\ \mathcal{SH}_{P_t}(\Omega) \neq \emptyset\}.$$

We need to prove the opposite inclusions. Suppose that there exists $u \in \mathcal{SH}_{P_t} \setminus \mathcal{SH}^0_{P_t}$. We assert that $\mathcal{SH}^0_{P_t} \neq \emptyset$. By the generalized maximum principle (see Remark 2.2) and Hopf's Lemma, $u > 0$ on $\bar{\Omega}$, therefore, for $k = 1, 2, \ldots$, we have

$$(P_t + \frac{1}{k})u \geq \delta_k > 0 \text{ in } \Omega, \quad (B + \frac{1}{k})u \geq \varepsilon_k > 0 \text{ on } \partial\Omega.$$

Take $f \in C_0^\alpha(\Omega)$ such that $f \gneqq 0$. According to Theorem 4.1, there exist positive solutions w_k of the problems

$$(P_t + \frac{1}{k})w_k = f \text{ in } \Omega,\ (B + \frac{1}{k})w_k = 0 \text{ on } \partial\Omega.$$

We distinguish between two cases:
(a) suppose that $\{w_k\}$ is not locally uniformly bounded, then there exists $x_0 \in \Omega$ such that $w_k(x_0) \to \infty$. Define $\tilde{w}_k = \frac{w_k}{w_k(x_0)}$. By a standard elliptic argument, there exists a subsequence $\{\tilde{w}_k\}$ that converges to a nonnegative function $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ which satisfies

$$P_t w = 0 \text{ in } \Omega,\ Bw = 0 \text{ on } \partial\Omega.$$

Since $w(x_0) = 1$, it follows that $w > 0$ in Ω and $w \in \mathcal{H}^0_{P_t}(\Omega)$.
(b) Suppose that $\{w_k\}$ is locally uniformly bounded. It follows from the Schauder estimates and the Arzelá-Ascoli theorem that $\{w_k\}$ has a subsequence that converges to a nonnegative function w which satisfies

$$P_t w = f \gneqq 0 \text{ in } \Omega,\ Bw = 0 \text{ on } \partial\Omega.$$

By the maximum principle, $w > 0$ in $\bar{\Omega}$. Thus, $\mathcal{SH}^0_{P_t} \neq \emptyset$.

Similarly, if $u \in \mathcal{SH}_{P_t} \backslash \mathcal{H}_{P_t}$, take $g \in C_0^{1,\alpha}(\partial\Omega)$, $g \gneqq 0$, and $k \geq 1$. By solving

$$(P_t + \frac{1}{k})v_k = 0 \text{ in } \Omega, \quad (B + \frac{1}{k})v_k = g \text{ on } \partial\Omega,$$

and repeating the above argument, we deduce that $\mathcal{H}_{P_t} \neq \emptyset$. Thus, these three sets are equal. □

Remark 5.5 For the regular oblique derivative problem (and for the Dirichlet problem) in a smooth bounded domain, and for $W > 0$ in $\bar{\Omega}$, we only have

$$\{t|\ \mathcal{H}^0_{P_t}(\Omega) \neq \emptyset\} \subseteq \{t|\ \mathcal{H}_{P_t}(\Omega) \neq \emptyset\}.$$

Indeed, by [2, Theorem 4.3], if $W > 0$ in $\bar{\Omega}$, then $\{t|\ \mathcal{H}^0_{P_t} \neq \emptyset\} = \{\lambda_0^c\}$, where λ_0^c is the classical principal eigenvalue. Moreover, by [2, Theorem 4.4] and Proposition 5.4, $S = \{t|\ \mathcal{SH}_{P_t} \neq \emptyset\} = (-\infty, \lambda_0^c]$.

Remark 5.6 If there exists a constant $m \in \mathbb{R}$ such that $c - mW \geq 0$, then $\lambda_0(\Omega, P, W, B) \geq m$. Clearly, the assumption $c - mW \geq 0$ implies that for any positive constant k

$$(P - mW)k = (c - mW)k \geq 0 \text{ in } \Omega, \quad \text{and } Bk \geq 0 \text{ on } \partial\Omega \setminus \Gamma.$$

By Theorem 5.2 and Proposition 5.4, $\mathcal{H}_{P_m}(\Omega) \neq \emptyset$. Hence, $\lambda_0 \geq m$.

6 The set $S = \{t \in \mathbb{R} |\ \mathcal{H}_{P_t}(\Omega) \neq \emptyset\}$

In this section we show that the set

$$S = S(\Omega, P, W, B) = \{t \in \mathbb{R} |\ \mathcal{H}_{P_t}(\Omega) \neq \emptyset\} \tag{6.1}$$

is a closed convex set. First, we need two auxiliary lemmas:

Lemma 6.1 *If $\mathcal{SH}_P \neq \emptyset$, then for each $x_0 \in \Omega$ there exists a positive function $w \in C^2(\Omega \setminus \{x_0\}) \cap C^1(\bar{\Omega} \setminus (\{x_0\} \cup \Gamma))$, such that*

$$Pw = 0 \text{ in } \Omega \setminus \{x_0\}, \quad Bw = 0 \text{ on } \partial\Omega \setminus \Gamma.$$

In other words, $w \in \mathcal{H}^0_P(\Omega \setminus \{x_0\})$.

Proof: Suppose that $\mathcal{SH}_P \neq \emptyset$. Take $x_0 \in \Omega$. For $0 < \varepsilon < \varepsilon_0$, let B_ε be a ball centered at x_0 such that $B_\varepsilon \subset\subset \Omega$. By Theorem 4.4, for each $0 < \varepsilon < \varepsilon_0$, there exists a positive minimal solution u_ε of the boundary value problem

$$Pu = 0 \text{ in } \Omega \setminus B_\varepsilon,\ Bu = 0 \text{ on } \partial\Omega \setminus \Gamma,\ u = 1 \text{ on } \partial B_\varepsilon.$$

Take $x_1 \in \Omega \setminus B_{\varepsilon_0}$ and define $w_\varepsilon(x) = u_\varepsilon(x)/u_\varepsilon(x_1)$. Using the local and up to the boundary Harnack inequalities, we infer that w_ε are uniformly bounded in compacts of $\bar{\Omega} \setminus (\Gamma \cup \{x_0\})$, and by a standard elliptic argument (as in Theorem 5.2), there exists a subsequence, denoted by $\{w_{\varepsilon_k}\}$ that converges as $\varepsilon_k \to 0$ to w which is a positive solution of

$$Pw = 0 \text{ in } \Omega \setminus \{x_0\}, \quad Bw = 0 \text{ on } \partial\Omega \setminus \Gamma. \qquad \square$$

Lemma 6.2 *Suppose that there exists a positive function $w \in \mathcal{H}^0_P(\Omega \setminus \{x_0\})$ for some $x_0 \in \Omega$. Then $\mathcal{H}_P(\Omega) \neq \emptyset$ and*

$$w = \alpha G^D_\Omega(\cdot, x_0) + \tilde{w}, \tag{6.2}$$

where α is a nonnegative constant, $G^D_\Omega(\cdot, x_0)$ is the minimal (Dirichlet) positive Green function of the operator P in Ω with a pole at x_0, and $\tilde{w} \in \mathcal{H}_P(\Omega) \cup \{0\}$.

Moreover, if $\alpha = 0$, then $\tilde{w} \in \mathcal{H}^0_P(\Omega)$, and if $\alpha > 0$ and $\partial\Omega \setminus \Gamma \neq \emptyset$, then $\tilde{w} \in \mathcal{H}_P(\Omega) \setminus \mathcal{H}^0_P(\Omega)$. The case in which $\tilde{w} = 0$ may occur only for the (generalized) Dirichlet boundary value problem ($\partial\Omega = \Gamma$).

Proof: If $\limsup_{x \to x_0} w(x) < \infty$, then using [10], w has a removable singularity at x_0, and we obtain (6.2), with $\alpha = 0$ and with $\tilde{w}$ which is the continuous extension of w. In this case $\tilde{w} \in \mathcal{H}_P^0(\Omega)$.

Otherwise, take an increasing sequence $\{\Omega_k\}_{k=1}^{\infty}$ of smooth bounded domains that exhausts Ω such that $\Omega_k \subset\subset \Omega_{k+1} \subset\subset \Omega$, and $\Omega = \cup_{k=1}^{\infty} \Omega_k$. It follows from [24] that

$$w|_{\Omega_k \setminus \{x_0\}} = \alpha_k G_{\Omega_k}^D(\cdot, x_0) + w_k,$$

where $\alpha_k > 0$, $G_{\Omega_k}^D(x, x_0)$ is the positive minimal Green function of the Dirichlet boundary value problem in Ω_k, and $w_k > 0$ satisfies $Pw_k = 0$ in Ω_k, and $w_k = w$ on $\partial\Omega_k$. Hence, for a subsequence $k_n \to \infty$, we have

$$w = \alpha G_{\Omega}^D(\cdot, x_0) + \tilde{w} \quad \text{in } \Omega,$$

where α is a positive constant and $\tilde{w} \geq 0$ satisfies $P\tilde{w} = 0$ in Ω.

Obviously, if $\partial\Omega = \Gamma$ (the generalized Dirichlet problem), then either $\tilde{w} = 0$, or $\tilde{w} \in \mathcal{H}_P(\Omega) = \mathcal{H}_P^0(\Omega)$. Note that in both these cases $\mathcal{H}_P \neq \emptyset$, since for the generalized Dirichlet problem, it is well known that the existence of the positive minimal Green function G_{Ω}^D implies that $\mathcal{H}_P \neq \emptyset$.

Suppose now that $\partial\Omega \setminus \Gamma \neq \emptyset$. Then $G_{\Omega}^D(\cdot, x_0) = 0$ on $\partial\Omega \setminus \Gamma$, and, by Hopf's Lemma, $BG_{\Omega}^D(\cdot, x_0) < 0$ on $\partial\Omega \setminus \Gamma$. On the other hand,

$$0 = Bw = \alpha BG_{\Omega}^D(\cdot, x_0) + B\tilde{w} \quad \text{on } \partial\Omega \setminus \Gamma.$$

Since $\alpha > 0$, it follows that $B\tilde{w} > 0$. Consequently,

$$\tilde{w} > 0, \ P\tilde{w} = 0, \text{ in } \Omega, \text{ and } B\tilde{w} > 0 \quad \text{ on } \partial\Omega \setminus \Gamma.$$

Thus, $\tilde{w} \in \mathcal{H}_P \setminus \mathcal{H}_P^0$. □

Lemma 6.3 *The set S is a closed interval.*

Proof: For the case where $\Gamma = \partial\Omega$ (the generalized Dirichlet problem), this result is known (see [23]). Our proof is quite similar and covers also this case.

Closedness: If $\{t_k\}$ is a sequence in S that converges to t_0, then by Lemma 6.1, there exists a sequence of normalized positive solutions w_k of the problem

$$P_{t_k} w_k = 0 \quad \text{ in } \Omega \setminus \{x_0\}, \qquad Bw_k = 0 \quad \text{ on } \partial\Omega \setminus \Gamma.$$

By standard elliptic arguments, there exists a subsequence of $\{w_k\}$ that converges to a positive solution w of the problem

$$P_{t_0}w = 0 \quad \text{in } \Omega \setminus \{x_0\}, \qquad Bw = 0 \quad \text{on } \partial\Omega \setminus \Gamma.$$

According to Lemma 6.2, $\mathcal{H}_{P_{t_0}}(\Omega) \neq \emptyset$ and S is closed.

Convexity: Let $t_0, t_1 \in S$. For any $0 < \alpha < 1$ denote by t_α the convex combination $t_\alpha = \alpha t_1 + (1-\alpha)t_0$. Let $u_0 \in \mathcal{H}_{P_{t_0}}$ and $u_1 \in \mathcal{H}_{P_{t_1}}$, and set $u_\alpha = (u_0)^{1-\alpha}(u_1)^\alpha$. We claim that $u_\alpha \in \mathcal{SH}_{P_{t_\alpha}}$. Indeed,

$$\begin{aligned} P_{t_\alpha}u_\alpha = (1-\alpha)u_\alpha(u_0)^{-1}P_{t_0}u_0 + \alpha u_\alpha(u_1)^{-1}P_{t_1}u_1 + \\ \alpha(1-\alpha)u_\alpha\left(\frac{u_1}{u_0}\right)^2 \sum_{i,j=1}^{n} a_{ij}\partial_i\left(\frac{u_0}{u_1}\right)\partial_j\left(\frac{u_0}{u_1}\right) \geq 0, \end{aligned} \tag{6.3}$$

$$Bu_\alpha = (1-\alpha)u_\alpha(u_0)^{-1}Bu_0 + \alpha u_\alpha(u_1)^{-1}Bu_1 \geq 0. \tag{6.4}$$

Therefore, $u_\alpha \in \mathcal{SH}_{P_{t_\alpha}}(\Omega)$. By Theorem 5.2 and Proposition 5.4, we deduce that $\mathcal{H}_{P_{t_\alpha}}(\Omega) \neq \emptyset$, and S is convex. Note that if $P_{t_1} \neq P_{t_0}$, or more generally, if u_0 and u_1 are linearly independent, then $u_\alpha \in \mathcal{SH}_{P_{t_\alpha}}(\Omega) \setminus \mathcal{H}_{P_{t_\alpha}}(\Omega)$. $\square$

Remark 6.4 Lemma 6.3 can be slightly extended. Let $\mathcal{H}_{P_t,B_t}$ be the set

$$\begin{aligned} \mathcal{H}_{P_t,B_t} = \{u \in C^2(\Omega) \cup C^1(\bar{\Omega} \setminus \Gamma) |\ u > 0 \text{ in } \Omega, \\ P_t u = 0 \text{ in } \Omega \text{ and } B_t u \geq 0 \text{ on } \partial\Omega \setminus \Gamma\}, \end{aligned}$$

where $B_t u = t\gamma u + \beta\frac{\partial u}{\partial \nu}$, and define $\tilde{S} = \{t \geq 0 |\ \mathcal{H}_{P_t,B_t} \neq \emptyset\}$. Then, as in the proof of Lemma 6.3, it can be shown that $\tilde{S}$ is a closed interval.

Lemma 6.5 *1. If $\mathcal{SH}_P \neq \emptyset$, and $W \gneqq 0$, then $S = (-\infty, \lambda_0]$, where $0 \leq \lambda_0 < \infty$ is the generalized principal eigenvalue.*

2. If $\mathcal{SH}_P \neq \emptyset$, and W changes its sign in Ω, then S is the bounded interval $[\lambda_-, \lambda_0]$, where $\lambda_- := \inf\{t \in S\}$, and $-\infty < \lambda_- \leq 0 \leq \lambda_0 < \infty$.

Proof: 1. Let

$$S^D := \{t \in \mathbb{R} \mid \exists\, u_t > 0 \text{ s.t. } P_t u_t = 0 \ \text{ in } \Omega\}$$

be the corresponding set for the generalized Dirichlet problem ($\Gamma = \partial\Omega$). Obviously, $S \subseteq S^D$. It is known that S^D is bounded from above (see [21, theorem 4.4]), therefore, S is bounded from above. Since $0 \in S$, it follows that $0 \leq \lambda_0 = \sup S < \infty$. By Lemma 6.3, S is a closed interval, hence, there exists $u_0 \in \mathcal{H}_{P_{\lambda_0}}$. For every $t < \lambda_0$, we have $u_0 \in \mathcal{SH}_{P_t}$. According to Theorem 5.2 and Proposition 5.4, $\mathcal{H}_{P_t} \neq \emptyset$. Hence, $S = (-\infty, \lambda_0]$.

2. By Lemma 6.3 and our assumptions, S is a closed nonempty interval. Recall that $S \subseteq S^D$. In addition, S^D is a bounded set [21, theorem 4.7]. Therefore, S is the closed bounded interval $S = [\lambda_-, \lambda_0]$. $\square$

7 Basic properties of λ_0

With the aid of the preceding section's results, we obtain various properties of the generalized principal eigenvalue. In particular, we wish to show that in regular problems, the generalized and the classical principal eigenvalue are equal. We also show the continuity, convexity and monotonicity of λ_0.

Lemma 7.1 *Let Ω be a C^2-bounded domain. Assume that P is uniformly elliptic in $\bar{\Omega}$, the coefficients of P are in $C^\alpha(\bar{\Omega})$, and $W \in C^\alpha(\bar{\Omega})$ is a positive function. Suppose that there exists a classical principal eigenvalue λ_0^c with classical principal eigenfunction $u_0^c \in C^2(\Omega) \cap C^1(\partial\Omega \setminus \Gamma) \cap C(\bar{\Omega})$ satisfying*

$$Pu_0^c = \lambda_0^c W u_0^c \quad in \;\; \Omega, \quad Bu_0^c = 0 \quad on \;\; \partial\Omega \setminus \Gamma, \quad u_0^c = 0 \quad on \; \Gamma. \tag{7.1}$$

Then $\lambda_0^c = \lambda_0$.

Proof: As $u_0^c \in \mathcal{H}_{P_{\lambda_0^c}}(\Omega)$, it follows that $\lambda_0^c \leq \lambda_0$. Suppose that $\lambda_0^c < \lambda_0$. Take $\lambda_0^c < t \leq \lambda_0$ and $u_t \in \mathcal{H}_{P_t}(\Omega)$. Then $u_t > 0$ satisfies

$$P_{\lambda_0^c} u_t = (t - \lambda_0^c) W u_t > 0 \;\text{ in } \Omega, \quad Bu_t \geq 0 \;\text{ on } \partial\Omega \setminus \Gamma.$$

By Lemma 2.3, there exists $\tilde{u} \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma)$ that satisfies

$$\inf_\Omega \tilde{u} > 0, \quad P_{\lambda_0^c} \tilde{u} > 0 \text{ in } \;\Omega, \quad B\tilde{u} > 0 \text{ on } \;\partial\Omega \setminus \Gamma.$$

We use the **the ε-maximal method**. Denote by $\varepsilon_0 = \sup\{\varepsilon > 0 | \; \tilde{u} - \varepsilon u_0^c \geq 0\}$, and let $\hat{u} = \tilde{u} - \varepsilon_0 u_0^c$. Clearly, $0 < \varepsilon_0 < \infty$. It follows that

$$\hat{u} \geq 0, \quad P_{\lambda_0^c} \hat{u} > 0 \text{ in } \;\Omega, \quad B\hat{u} > 0 \text{ on } \;\partial\Omega \setminus \Gamma,$$

which implies that $\hat{u} > 0$ on $\bar{\Omega} \setminus \Gamma$. Moreover, $\liminf_{x \to \Gamma} \hat{u} > 0$, hence, $\inf_\Omega \hat{u} > 0$. By replacing $\tilde{u}$ with $\hat{u}$, we have $\varepsilon_1 > 0$ such that $\hat{u} - \varepsilon_1 u_0^c \geq 0$, which contradicts the maximality of ε_0. □

Next, we establish the uniqueness of the Zaremba boundary value problem.

Lemma 7.2 *Let Ω be a C^2-bounded domain. Assume that P is uniformly elliptic in $\bar{\Omega}$, and that the coefficients of P are in $C^\alpha(\bar{\Omega})$. Suppose that $u \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma)$ satisfies*

$$u > 0 \ \ \textit{in } \Omega, \ Pu > 0 \ \textit{in } \Omega, \ Bu \geq 0 \ \textit{on } \partial\Omega \setminus \Gamma. \tag{7.2}$$

If $v \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma) \cap C(\bar{\Omega})$ is a solution of

$$Pv = 0 \ \textit{in } \Omega, \ Bv = 0 \ \textit{on } \partial\Omega \setminus \Gamma, \ \textit{and } v|_\Gamma = 0, \tag{7.3}$$

then $v = 0$.

Proof: By Lemma 2.3, there exists $\tilde{u} \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma)$ which satisfies

$$\inf_\Omega \tilde{u} > 0, \ P\tilde{u} > 0 \text{ in } \Omega, \ B\tilde{u} > 0 \text{ on } \partial\Omega \setminus \Gamma.$$

Take v that satisfies (7.3). Suppose on the contrary that there is a point $x_0 \in \Omega$ such that $v(x_0) > 0$. We use again the ε-maximal method. Define

$$\varepsilon_0 = \sup\{\varepsilon > 0 | \ \tilde{u} - \varepsilon v \geq 0 \ \text{ in } \ \Omega\}.$$

Clearly, $0 < \varepsilon_0 < \infty$. Set $w = \tilde{u} - \varepsilon_0 v$, we have

$$w \geq 0 \ \text{ in } \Omega, \ Pw = P\tilde{u} > 0 \text{ in } \Omega, \ Bw = B\tilde{u} > 0 \ \text{ on } \partial\Omega \setminus \Gamma,$$

and $\liminf_{x \to \Gamma} w = \liminf_{x \to \Gamma} \tilde{u} > 0$ on Γ. By the generalized maximum principle and Hopf's lemma, $w > 0$ on $\bar{\Omega} \setminus \Gamma$. Hence, $\inf_\Omega w > 0$. Therefore, by replacing $\tilde{u}$ with w, there is a positive ε_1 such that $w - \varepsilon_1 v \geq 0$ in Ω, which contradicts the definition of ε_0. Consequently, $v = 0$ as required. □

Remark 7.3 The assumption that Ω is bounded is essential for the uniqueness (see Theorem 5.2). The requirement that $v|_\Gamma = 0$ is also essential even if Γ is a singleton, as it was shown recently by G. M. Lieberman [16].

Under some assumptions, the principal eigenfunction for the degenerate oblique boundary problem is unique. First, we need the following:

Lemma 7.4 *Suppose that Ω is a C^2-bounded domain and let $w, v \in C^1(\bar{\Omega})$ satisfy $w, v > 0$ in $\bar{\Omega} \setminus \Gamma$, $\frac{\partial w}{\partial \nu}, \frac{\partial v}{\partial \nu} < 0$ on Γ, and $w, v = 0$ on Γ. Then $\frac{w}{v} \in C(\bar{\Omega})$ and $\frac{w}{v} > 0$ in $\bar{\Omega}$, where on Γ, $\frac{w}{v}$ is defined as $\frac{\partial w}{\partial \nu} / \frac{\partial v}{\partial \nu}$.*

Proof: By dividing the Taylor polynomials of the functions w and v, the proof is immediate. □

Lemma 7.5 *Assume that Ω is a $C^{2,\alpha}$-bounded domain. Let P be a uniformly elliptic operator in $\bar{\Omega}$ with $C^{\alpha}(\bar{\Omega})$ coefficients, and*

$$Bu = \gamma(x)u + \beta(x)\frac{\partial u}{\partial \nu}$$

a degenerate oblique boundary operator defined on $\partial\Omega$ with $C^{1,\alpha}(\partial\Omega)$ coefficients such that $\beta, \gamma \geq 0$, and $\beta + \gamma > 0$. Let $w, v \in C^{2,\alpha}(\Omega) \cap C^1(\bar{\Omega})$ be positive solutions of the problem

$$Pu = 0 \quad \text{in } \Omega, \quad Bu = 0 \quad \text{on } \partial\Omega.$$

Then $w = \varepsilon_0 v$, where ε_0 is some positive constant.

Proof: Set $\Gamma = \{x \in \partial\Omega |\ \beta(x) = 0\}$. Clearly, w, v fulfill the requirements of Lemma 7.4, therefore, $\frac{w}{v} \in C(\bar{\Omega})$ and $\frac{w}{v} > 0$ on $\bar{\Omega}$. Consequently, there exist k_0, k_1 such that $0 < k_0 < \frac{w}{v} < k_1 < \infty$ in $\bar{\Omega}$. Using again the ε-maximal method, we conclude that $w = \varepsilon_0 v$. □

Next, we study monotonicity, continuity and concavity properties of λ_0 as a function of the coefficients of P and B, and of the domain Ω.

Lemma 7.6 *(1) For $i = 1, 2$, let Ω_i be a domain in $\mathbb{R}^n$, Γ_i a closed subset of $\partial\Omega_i$, and $\partial\Omega_i \setminus \Gamma_i$ a $C^{2,\alpha}$-portion of $\partial\Omega_i$. Assume that*

$$\Omega_1 \subseteq \Omega_2, \ \partial\Omega_1 \setminus \Gamma_1 \subseteq \partial\Omega_2 \setminus \Gamma_2 \text{ and } B_2|_{\partial\Omega_1 \setminus \Gamma_1} = B_1.$$

Then $\lambda_0^2 \leq \lambda_0^1$, where $\lambda_0^i := \lambda_0(\Omega_i, P, W, B_i)$, $i = 1, 2$.

(2) Suppose that Ω is unbounded. Let $\{\Omega_k\}_{k=1}^{\infty}$ be an increasing sequence of bounded domains such that $\cup_{k=1}^{\infty} \Omega_k = \Omega$. Denote $\partial'\Omega_k = \overline{\partial\Omega_k \cap \Omega}$, and $\Gamma_k = (\Gamma \cap \partial\Omega_k) \cup \partial'\Omega_k$. Assume that

$$\bigcup_{k=1}^{\infty} \bar{\Omega}_k = \bar{\Omega}, \quad \bar{\Omega}_k \subset\subset \bar{\Omega}_{k+1}, \quad (\partial\Omega_k \setminus \partial'\Omega_k) \subset (\partial\Omega_{k+1} \setminus \partial'\Omega_{k+1}).$$

Let B_k be given boundary operators on $\partial\Omega_k \setminus \Gamma_k$, and suppose that for $k \geq 1$, $B_{k+1}|_{\partial\Omega_k \setminus \Gamma_k} = B_k$. Then $\lambda_0^k \to \lambda_0$, where $\lambda_0^k = \lambda_0(\Omega_k, P, W, B_k)$ and $\lambda_0 = \lambda_0(\Omega, P, W, B)$.

Proof: (1) Clearly, $\mathcal{H}_{P_t,B_2}(\Omega_2) \subseteq \mathcal{H}_{P_t,B_1}(\Omega_1)$, thus, $\lambda_0^2 \leq \lambda_0^1$.

(2) According to part (1), $\lambda_0 \leq \lambda_0^k$, and $\{\lambda_0^k\}$ is a decreasing sequence. Therefore, $\lambda_0 \leq \tilde{\lambda} := \lim_{k\to\infty} \lambda_0^k$.

First, suppose that $\tilde{\lambda} < \infty$. In the light of Theorem 5.2 (note that $\Gamma_k \neq \emptyset$), for each λ_0^k there exists a normalized solution $u_0^k \in \mathcal{H}^0_{P_{\lambda_0^k}}(\Omega_k)$. The sequence $\{u_0^k\}$ contains a subsequence converging to a function $u_0 \in \mathcal{H}^0_{P_{\tilde{\lambda}}}(\Omega)$. It follows that, $\tilde{\lambda} \leq \lambda_0$. Hence, $\lambda_0 = \lim_{k\to\infty} \lambda_0^k$.

If $\tilde{\lambda} = \infty$, then $\lambda_k^0 = \infty$ for every $k \geq 1$. It follows from Theorem 5.2 that for every $\lambda > 0$, there exists a normalized function $u_{k,\lambda} \in \mathcal{H}^0_{P_\lambda}(\Omega_k)$. The sequence $\{u_{k,\lambda}\}$ has a subsequence converging to $u_\lambda \in \mathcal{H}^0_{P_\lambda}(\Omega)$. Since λ is an arbitrarily large number, it follows that $\lambda_0 = \infty$. □

Remark 7.7 (1) We note that for the Neumann boundary value problem, λ_0 is not a continuous function of the domain in the usual sense (see [7]).

(2) If Ω is bounded and $\Gamma = \emptyset$, then in general, the *standard* monotonicity of λ_0 with respect to the domains does not hold true, as we show in the following example (see also [7]).

Example 7.8 Take $\Omega_1 = B_1(0)$, $\Omega_2 = B_2(0)$, $P = -\Delta + V$, where $V|_{\Omega_1} = 0$, $V|_{\Omega_2 \setminus \bar{\Omega}_1} > 0$, and $B = \frac{\partial}{\partial n}$. Then $\lambda_0(\Omega_1) = 0$, but $\lambda_0(\Omega_2) > 0$.

Lemma 7.9 *(a) Suppose that $c_1 \leq c_2$, $\gamma_1 \leq \gamma_2$, $\beta_1 \geq \beta_2$ and denote*

$$P_i = P + c_i, \quad B_i = \gamma_i + \beta_i \frac{\partial}{\partial \nu} \quad \textit{and} \quad \lambda_0^i = \lambda_0(\Omega, P_i, W, B_i),$$

for $i = 1, 2$. Then $\lambda_0^2 \geq \lambda_0^1$.

(b) λ_0 is a concave function of the coefficient c.

(c) Let $\{c_k\}_{k=0}^\infty$ and c_0 be $C^\alpha_{loc}(\bar{\Omega} \setminus \Gamma)$ functions such that $|c_k|_{0;\Omega'} \leq M(\Omega')$ for every $\Omega' \subset\subset \bar{\Omega} \setminus \Gamma$, and $c_k(x) \to c_0(x)$ for all $x \in \bar{\Omega} \setminus \Gamma$. Set $P_k = P + c_k$ and $\lambda_0^k = \lambda_0(\Omega, P_k, W, B)$, $k \geq 0$. Then $\limsup_{k\to\infty} \lambda_0^k \leq \lambda_0^0$.

(d) If $W > 0$ and $|\frac{c_k(x)-c_0(x)}{W(x)}|_{0;\Omega} \to 0$, then $\lambda_0^k \to \lambda_0^0$. Moreover, λ_0 is a Lipschitz continuous function of c with Lipschitz constant 1 with respect to the norm $|||f||| := |\frac{f}{W}|_{0;\Omega}$.

Proof: (a) Clearly, $\mathcal{H}_{P_1-\lambda_0^1 W,B_1}(\Omega) \subset \mathcal{SH}_{P_2-\lambda_0^1 W,B_2}(\Omega)$. By Theorem 5.2 and Proposition 5.4, $\mathcal{H}_{P_2-\lambda_0^1 W,B_2} \neq \emptyset$, and consequently $\lambda_0^2 \geq \lambda_0^1$.

(b) For $0 \leq \alpha \leq 1$, denote

$$P_\alpha = P + \alpha c_1 + (1-\alpha)c_0, \quad \text{and } \lambda_0^\alpha = \lambda_0(\Omega, P_\alpha, W, B).$$

Let $u_i \in \mathcal{H}_{P_i-\lambda_0^i W}$, where $i=0,1$. Set $u_\alpha = (u_0)^{1-\alpha}(u_1)^\alpha$. As shown in Lemma 6.3, $[P_\alpha - (\alpha\lambda_0^1 + (1-\alpha)\lambda_0^0)W]u_\alpha \geq 0$, and $Bu_\alpha \geq 0$. Hence $\lambda_0^\alpha \geq \alpha\lambda_0^1 + (1-\alpha)\lambda_0^0$.

(c) Let $\tilde{\lambda} := \limsup_{k\to\infty} \lambda_0^k$. By taking a subsequence we may assume that $\lambda_0^k \to \tilde{\lambda}$. We wish to prove that $\tilde{\lambda} \leq \lambda_0^0$.

Suppose first that $\tilde{\lambda} < \infty$. According to Lemma 6.1, there exists a normalized sequence of positive solution $u_k \in \mathcal{H}^0_{P_k-\lambda_0^k W}(\Omega \setminus \{x_0\})$. As c_k are locally uniformly bounded, by standard elliptic arguments, the sequence $\{u_k\}$ has a subsequence converging to $\tilde{u} \in \mathcal{H}^0_{P_0-\tilde{\lambda}W}(\Omega \setminus \{x_0\})$. Lemma 6.2 implies that $\mathcal{H}_{P_0-\tilde{\lambda}W}(\Omega) \neq \emptyset$. Hence, $\tilde{\lambda} \leq \lambda_0^0$.

If $\tilde{\lambda} = \infty$, then for every $\lambda > 0$ there exists k_λ such that $\mathcal{H}_{P_k-\lambda W} \neq \emptyset$ for every $k > k_\lambda$. It follows as above that $\mathcal{H}_{P_0-\lambda W}(\Omega) \neq \emptyset$. Inasmuch as λ is an arbitrarily large number, it follows that $\lambda_0^0 = \infty$.

(d) If $|\frac{c_k-c_0}{W}|_{0;\Omega} \to 0$ and $W > 0$, then for every $0 < \varepsilon < 1$ there exists k_ε such that for every $k_\varepsilon < k \in \mathbb{N}$,

$$c_0 - \varepsilon W \leq c_k \leq c_0 + \varepsilon W.$$

Now, let $u_k \in \mathcal{H}_{P_{\lambda_0^k}}$, $k \geq 0$. Then for $k > k_\varepsilon$,

$$(P_k - (\lambda_0^0 - \varepsilon)W)u_0 \geq (P_0 - \lambda_0^0 W)u_0 = 0 \ \text{ in } \Omega,$$

and

$$(P_0 - (\lambda_0^k - \varepsilon)W)u_k \geq (P_k - \lambda_0^k W)u_k = 0 \ \text{ in } \Omega.$$

It follows that

$$\lambda_0^0 - \varepsilon \leq \lambda_0^k \leq \lambda_0^0 + \varepsilon,$$

for every $k \geq k_\varepsilon$. Therefore, $\lim_{k\to\infty} \lambda_0^k = \lambda_0^0$, and λ_0 is a Lipschitz continuous function of c with Lipschitz constant 1. $\square$

Similarly, we show that λ_0 is a monotone, continuous function of W.

Lemma 7.10 *(a) Suppose that $W_1 \leq W_2$ and denote $\lambda_0^i = \lambda_0(\Omega, P, W_i, B)$ for $i = 1, 2$. If $\mathcal{SH}_P(\Omega) \neq \emptyset$, then $\lambda_0^2 \leq \lambda_0^1$. On the other hand, if $W_1 \geq 0$ and $\mathcal{SH}_P(\Omega) = \emptyset$, then $\lambda_0^2 \geq \lambda_0^1$.*

(b) Let $\{W_k\}_{k=1}^\infty$ and W be $C^\alpha_{loc}(\bar\Omega \setminus \Gamma)$ functions such that $W_k(x) \to W(x)$ for every $x \in \Omega$, and $|W_k|_{0;\Omega'} \leq M(\Omega')$, for every $\Omega' \subset\subset \bar\Omega \setminus \Gamma$. Denote by $\lambda_0^k = \lambda_0(\Omega, P, W_k, B)$, and $\lambda_0 = \lambda_0(\Omega, P, W, B)$. Then $\limsup_{k\to\infty} \lambda_0^k \leq \lambda_0$.

(c) If $W > 0$ and $|\frac{W_k(x)}{W(x)} - 1|_{0;\Omega} \to 0$, then $\lambda_0^k \to \lambda_0$.

Proof: (a) Suppose that $\mathcal{SH}_P(\Omega) \neq \emptyset$. Clearly, $\lambda_0^i \geq 0$ for $i = 1, 2$. It follows that $\mathcal{SH}_{P-\lambda_0^2 W_2}(\Omega) \subset \mathcal{SH}_{P-\lambda_0^2 W_1}(\Omega)$. Hence $\lambda_0^2 \leq \lambda_0^1$.

On the other hand, suppose that $W_1 \geq 0$ and $\mathcal{SH}_P(\Omega) = \emptyset$. It follows that $\lambda_0^i \leq 0$ for $i = 1, 2$, and $\mathcal{SH}_{P-\lambda_0^1 W_1}(\Omega) \subset \mathcal{SH}_{P-\lambda_0^1 W_2}(\Omega)$. Thus, $\lambda_0^1 \leq \lambda_0^2$.

(b) Assume first that $\tilde\lambda := \limsup_{k\to\infty} \lambda_0^k < \infty$. By taking a subsequence, we may assume that $\lambda_0^k \to \tilde\lambda$. Since $\mathcal{H}_{P-\lambda_0^k W_k} \neq \emptyset$, it follows as in Lemma 7.9 that $\mathcal{H}_{P-\tilde\lambda W} \neq \emptyset$. Thus, $\tilde\lambda \leq \lambda_0$.

Suppose now that $\tilde\lambda = \infty$. We may assume that $\lambda_0^k \to \infty$. Then for every $\lambda > 0$ there exists k_λ such that $\mathcal{H}_{P-\lambda W_k} \neq \emptyset$ for every $k > k_\lambda$. It follows that $\mathcal{H}_{P-\lambda W}(\Omega) \neq \emptyset$, which implies that $\lambda_0 = \infty$.

(c) If $W > 0$ and $|\frac{W_k}{W} - 1|_{0;\Omega} \to 0$, then for every $0 < \varepsilon < 1$ there exists k_ε such that for every $k_\varepsilon < k \in \mathbb{N}$,

$$0 \leq (1-\varepsilon)W \leq W_k \leq (1+\varepsilon)W.$$

Now let $u_0 \in \mathcal{H}_{P_{\lambda_0}}$. Suppose that $\lambda_0 \geq 0$; then

$$(P - \frac{\lambda_0}{1+\varepsilon} W_k)u_0 \geq (P - \frac{\lambda_0}{1+\varepsilon}(1+\varepsilon)W)u_0 = 0 \quad \text{in } \Omega.$$

Hence, $\frac{\lambda_0}{1+\varepsilon} \leq \lambda_0^k$ for every $k > k_\varepsilon$. Similarly, if $\lambda_0 < 0$, then $\frac{\lambda_0}{1-\varepsilon} \leq \lambda_0^k$ for every $k > k_\varepsilon$. Consequently, $\lambda_0 \leq \liminf_{k\to\infty} \lambda_0^k$ and $\lim_{k\to\infty} \lambda_0^k = \lambda_0$. $\square$

Corollary 7.11 *Fix a singular set $\Gamma \subset \partial\Omega$. Assume that $\vec\nu$ is the conormal direction. Let λ_0^N, λ_0 and λ_0^D be the generalized principal eigenvalues of the*

Neumann problem ($\gamma = 0$), the generalized mixed boundary value problem, and the generalized Dirichlet problem ($\Gamma^D = \partial\Omega$), respectively. Then

$$\lambda_0^N \le \lambda_0 \le \lambda_0^D.$$

We conclude this section with the following Protter-Weinberger type variational principle for λ_0 (see [19]).

Theorem 7.12 *Suppose that P and B are operators of the forms (1.1) and (1.2) satisfying (1.3)-(1.5). Let $W \in C^\alpha(\bar{\Omega} \setminus \Gamma)$, $W > 0$ in Ω, and let*

$$\mathcal{K}=\mathcal{K}(\Omega)=\{u\in C^2(\Omega)\cap C^1(\bar{\Omega}\setminus\Gamma) |\ u>0 \text{ in } \bar{\Omega}\setminus\Gamma,\ Bu\ge 0\}. \tag{7.4}$$

Then

$$\lambda_0 = \lambda_0(\Omega, P, W, B) = \sup_{u\in\mathcal{K}} \inf_{x\in\Omega} \left\{ \frac{Pu}{Wu} \right\}. \tag{7.5}$$

Proof: Let $u\in\mathcal{H}_{P_{\lambda_0}}$. Then $u\in\mathcal{K}$, and for every $x\in\Omega$, $\lambda_0=\frac{Pu}{Wu}(x)$. Hence

$$\lambda_0 \le \sup_{u\in\mathcal{K}} \inf_{x\in\Omega} \left\{ \frac{Pu}{Wu} \right\}. \tag{7.6}$$

Let $u \in \mathcal{K}$ and denote $\mu = \inf_{x\in\Omega}\{\frac{Pu}{Wu}\}$. If $\mu = -\infty$, then obviously $\mu \le \lambda_0$. Otherwise, the fact that $\mu \le \frac{Pu}{Wu}$ in Ω implies that $u \in \mathcal{SH}_{P_\mu}(\Omega)$. Therefore $\mu \le \lambda_0$. So $\inf_{x\in\Omega}\left\{\frac{Pu}{Wu}\right\} \le \lambda_0$ for every $u \in \mathcal{K}$. Consequently

$$\sup_{u\in\mathcal{K}} \inf_{x\in\Omega} \left\{ \frac{Pu}{Wu} \right\} \le \lambda_0. \tag{7.7}$$

Combining (7.6) and (7.7), we obtain (7.5). □

Remark 7.13 Suppose in addition to the assumptions of Theorem 7.12 that W and the coefficients of P are in $C(\bar{\Omega})$, and the following slightly stronger version of the Protter-Weinberger variational principle holds true:

$$\lambda_0 = \sup_{u\in\mathcal{K}\cap C^2(\bar{\Omega})} \inf_{x\in\Omega} \left\{ \frac{Pu}{Wu} \right\}. \tag{7.8}$$

Then as in [19], one obtains the following Donsker-Varadhan variational principle

$$\lambda_0 = \inf_{\mu\in M(\bar{\Omega})} \sup_{u\in\mathcal{K}\cap C^2(\bar{\Omega})} \left\{ \int_\Omega \frac{Pu}{Wu}\mu(dx) \right\},$$

where $M(\bar{\Omega})$ is the space of probability measures on $\bar{\Omega}$. Note that if $\partial\Omega \in C^{2,\alpha}$, Γ is either empty or a smooth closed manifold of dimension $n-1$, and W and the coefficients of P are in $C^\alpha(\bar{\Omega})$, then (7.8) holds true.

8 Criticality theory

This section, defines critical, subcritical, and supercritical operators, and also examines various criteria for these cases. First, we define a positive solution of minimal growth at infinity in Ω with respect to (P, B). This notion was introduced by S. Agmon [1] for the generalized Dirichlet problem.

Definition 8.1 (a) Let K be a compact set in Ω. The set $\Omega \setminus K$ is called **a neighborhood of infinity in** Ω.

(b) Let $\Omega \setminus K$ be a neighborhood of infinity in Ω. By the sets $\mathcal{SH}_{P,B}(\Omega \setminus K)$, $\mathcal{H}_{P,B}(\Omega \setminus K)$, $\mathcal{H}^0_{P,B}(\Omega \setminus K)$, we mean the corresponding sets of positive (super)solutions in $\Omega \setminus K$, where we consider ∂K as part of the singular set of $\partial(\Omega \setminus K)$. For example, $u \in \mathcal{H}^0_{P,B}(\Omega \setminus K)$ if

$$u > 0 \text{ in } \Omega \setminus K, \ Pu = 0 \text{ in } \Omega \setminus K, \ \text{ and } Bu = 0 \text{ on } \partial\Omega \setminus \Gamma.$$

(c) A function $u \in \mathcal{H}^0_{P,B}(\Omega \setminus K)$ is a **positive solution of the operator** (P, B) **of minimal growth at infinity in** Ω (in short, positive solution of minimal growth), if for every smooth $K \subset\subset K' \subset\subset \Omega$ and for every $w \in \mathcal{SH}_{P,B}(\Omega \setminus K') \cap C(\overline{(\Omega \setminus K')} \setminus \Gamma)$, satisfying $u \leq w$ on $\partial K'$, we have $u \leq w$ in $\Omega \setminus K'$.

Remark 8.2 Clearly, $u \in \mathcal{H}^0_{P,B}(\Omega \setminus K)$ is a positive solution of minimal growth at infinity in Ω if and only if for every smooth $K \subset\subset K' \subset\subset \Omega$ and $w \in \mathcal{SH}_{P,B}(\Omega \setminus K') \cap C(\overline{(\Omega \setminus K')} \setminus \Gamma)$ there exists $C > 0$ such that $u \leq Cw$ in $\Omega \setminus K'$.

Next, we define the ground state and the Green function.

Definition 8.3 (a) A positive solution $u \in \mathcal{H}^0_{P,B}(\Omega)$ of minimal growth at infinity in Ω is called a **ground state** of the operator (P, B) in Ω.

(b) Let $y \in \Omega$, and let $G^D_{B_\varepsilon(y)}(x, z)$ be the minimal positive (Dirichlet) Green function of the operator P in $B_\varepsilon(y)$, for some $B_\varepsilon(y) \subset \Omega$. A positive solution $u_y \in \mathcal{H}^0_{P,B}(\Omega \setminus \{y\})$ which has a minimal growth at infinity in Ω and satisfies

$$\lim_{x \to y} \frac{u_y(x)}{G^D_{B_\varepsilon(y)}(x, y)} = 1,$$

is called a **Green function** of the operator (P, B) in Ω with a pole at $\{y\}$. We denote it by $G^{P,B}_\Omega(x, y)$ or simply, $G^B_\Omega(x, y)$.

We are ready now to generalize the definition of subcriticality which was introduced in [18, 21, 29] for the generalized Dirichlet problem.

Definition 8.4 We say that (P, B) is a **critical** operator in Ω if (P, B) has a ground state. The operator (P, B) is **subcritical** in Ω if $\mathcal{SH}_{P,B}(\Omega) \neq \emptyset$, but (P, B) does not admit a ground state. The operator (P, B) is **supercritical** in Ω if $\mathcal{SH}_{P,B}(\Omega) = \emptyset$.

Theorem 8.5 *Suppose that $\mathcal{SH}_P(\Omega) \neq \emptyset$, and let K be a smooth nonempty compact set in Ω. Then there exists a positive solution in $D = \Omega \setminus K$ of the operator (P, B) of minimal growth at infinity in Ω.*

Proof: According to Theorem 4.4, there exists a positive Perron-solution u_0 of the problem:

$$Pu_0 = 0 \text{ in } D,\ Bu_0 = 0 \text{ on } \partial\Omega \setminus \Gamma,\ \text{and } u_0 = 1 \text{ on } \partial K. \tag{8.1}$$

We claim that u_0 is the desired solution. Let $K \subset\subset K' \subset\subset \Omega$, and denote $D' = \Omega \setminus K'$. Let $w \in \mathcal{SH}_{P,B}(\Omega \setminus K') \cap C(\overline{D'} \setminus \Gamma)$, satisfying $u_0 \leq w$ on $\partial K'$. We need to prove that $u_0 \leq w$ in D'.

Consider the degenerate mixed boundary value problem:

$$Pu = 0 \text{ in } D',\ Bu = 0 \text{ on } \partial\Omega \setminus \Gamma,\ \text{and } u = u_0 \text{ on } \partial K'. \tag{8.2}$$

Let $S^{\pm}(D)$ and $S^{\pm}(D')$ be the sets of Perron-supersolutions (subsolutions) of problems (8.1) and (8.2), respectively. By Theorem 4.4, we may assume that the system of neighborhoods of local solvability of problem (8.2) is a subset of the corresponding system of neighborhoods of problem (8.1). Moreover, we may also take the same reference positive supersolution $v \in \mathcal{SH}_P(\Omega)$ for the two problems.

We claim that $S^-(D) \subseteq S^-(D')$. Take $w^- \in S^-(D)$, clearly, $\limsup \frac{w^-}{v} \leq 0$ on Γ and at ∞. Since u_0 is the supremum of Perron-subsolutions in $S^-(D)$, it follows that $w^- \leq u_0$ on $\partial K'$. Let $y \in \overline{D'} \setminus (\Gamma \cup \partial K')$. As the neighborhood $N = N(y)$ of local solvability in D' does not intersect $\partial K'$, if $w^- \leq h$ on $\partial' N$, then $w^- \leq (h)_y^D = (h)_y^{D'}$ in N, where $(h)_y^D$ and $(h)_y^{D'}$ are the local solutions for the problems (8.1) and (8.2), respectively. Hence, $w^- \in S^-(D')$.

For each $w^{\pm} \in S^{\pm}(D')$, we have $w^- \leq w^+$, therefore it is enough to show that $w \in S^+(D')$. Evidently, $\liminf(w/v) \geq 0$ on Γ and at ∞. To see that w is a Perron-supersolution, we consider again $y \in \overline{D'} \setminus (\Gamma \cup \partial K')$ and $N = N(y)$. Let h be a continuous function such that $w \geq h$ on $\partial' N$. Inasmuch as $\overline{N(y)} \cap (\Gamma \cup \partial K') = \emptyset$ and $(h)_y = h$ on $\partial' N$, we have

$$Pw \geq (h)_y \text{ in } N, \quad Bw \geq (h)_y \text{ on } N \cap (\partial\Omega \setminus \Gamma), \quad w \geq (h)_y \text{ on } \partial' N.$$

Thus $w \geq (h)_y$ in $N(y)$, and $w \in S^+(D')$. Hence, $w \geq w^-$ in D', for all $w^- \in S^-(D')$. In particular, $w \geq w^-$ for all $w^- \in S^-(D)$. Consequently, $w \geq u_0$ in D'. □

Remark 8.6 It can be easily shown that the Perron-solution for the problem (8.2) is equal to u_0.

Theorem 8.7 *Suppose that $\mathcal{SH}_P(\Omega) \neq \emptyset$. Then for every $y \in \Omega$ there exists $w \in \mathcal{H}_P^0(\Omega \setminus \{y\})$ such that w is a positive solution of minimal growth at infinity in Ω.*

Proof: Let $w \in \mathcal{H}_P^0(\Omega \setminus \{y\})$ be the positive solution which was constructed in Lemma 6.1. We claim that w is the desired solution. Let $K \subset\subset \Omega$ be a smooth compact set such that $y \in \text{int} K$, and $u \in \mathcal{SH}_P(\Omega \setminus K) \cap C(\overline{\Omega \setminus K} \setminus \Gamma)$. Since the sequence of positive solutions $\{w_\varepsilon\}$ of the proof of Lemma 6.1 converges uniformly on ∂K to the continuous function w, it follows that w_ε are uniformly bounded there. As $u > 0$ on ∂K, there exists a positive constant k such that for every $\varepsilon > 0$, $w_\varepsilon \leq ku$ on ∂K. By Theorem 8.5, w_ε are positive solutions of minimal growth, and $w_\varepsilon(x) \leq ku(x)$ in $\Omega \setminus K$. Letting $\varepsilon \to 0$, we obtain $w(x) \leq ku(x)$ in $\Omega \setminus K$. □

Our aim now is to prove in several steps theorems 1.3 and 1.4. Since these theorems are known to hold for the generalized Dirichlet problem, we will assume in the sequel that $\Gamma \neq \partial\Omega$. First we prove the following lemma.

Lemma 8.8 *If $\dim \mathcal{SH}_P = 1$, then $\mathcal{SH}_P = \mathcal{H}_P^0$. In particular, $\dim \mathcal{H}_P^0 = 1$.*

Proof: Suppose that $\mathcal{SH}_P = \{Cv \mid C > 0\}$. By theorem 5.2 and Proposition 5.4, $\mathcal{SH}_P^0 \neq \emptyset$, and $\mathcal{H}_P \neq \emptyset$. It follows that

$$\mathcal{SH}_P^0 = \mathcal{H}_P = \{Cv \mid C > 0\}.$$

Therefore, $Bv = 0$ on $\partial\Omega \setminus \Gamma$ and $Pv = 0$ in Ω . Consequently, $v \in \mathcal{H}_P^0$. □

Theorem 8.9 *The operator P is critical in Ω if and only if* $\dim \mathcal{SH}_P = 1$.

Proof: Assume that P is critical, and let u_0 be a ground state. Suppose that $w \in \mathcal{SH}_P$, and let K be a smooth nonempty compact set in Ω. Take $C > 0$ such that $u_0 \leq Cw$ on ∂K. As u_0 is of minimal growth, it follows that $u_0 \leq Cw$ in $\Omega \setminus K$. By the generalized maximum principle, we infer that $u_0 \leq Cw$ in K, and hence, in Ω. Using the ε-maximal method it follows that $w = \varepsilon_1 u_0$ for some $\varepsilon_1 > 0$ which implies that dim $\mathcal{SH}_P = 1$.

Suppose that dim $\mathcal{SH}_P = 1$. Lemma 8.8 implies that

$$\mathcal{SH}_P = \mathcal{H}_P^0 = \{Cv | \ C > 0\}. \tag{8.3}$$

We claim that v is a ground state. According to Lemma 8.7, there exists a positive function $w \in \mathcal{H}_P^0(\Omega \setminus \{x_0\})$ which is of minimal growth at infinity in Ω. Furthermore, by Lemma 6.2, we have $w = \alpha G_\Omega^D(\cdot, x_0) + \tilde{w}$, where α is a nonnegative constant, $G_\Omega^D(x, y)$ is the Green function with respect to the generalized Dirichlet boundary value problem, and either $\tilde{w} \in \mathcal{H}_P^0(\Omega)$ (for $\alpha = 0$), or $\tilde{w} \in \mathcal{H}_P(\Omega) \setminus \mathcal{H}_P^0(\Omega)$ (for $\alpha > 0$). In view of (8.3), $\alpha = 0$. It follows that $\tilde{w} = w$ is a positive solution in Ω of minimal growth at infinity in Ω. Thus, $w = Cv$ is a ground state. □

In the following lemmas, we prove that the operator (P, B) is subcritical in Ω if and only if it admits a Green function.

Lemma 8.10 *If (P, B) is subcritical in Ω, then for every $y \in \Omega$ there exists a unique Green function $G_\Omega^B(x, y)$ of the operator (P,B) with a pole at y.*

Proof: We may assume that $\Gamma \neq \partial\Omega$. Suppose that (P, B) is a subcritical operator and fix $y \in \Omega$. By Lemma 8.7, there exists a positive solution in $\Omega \setminus \{y\}$ of minimal growth at infinity of Ω, denoted by $u_y(x)$. According to Lemma 6.2, $u_y(x) = \alpha G_\Omega^D(x, y) + u(x)$, where $\alpha \geq 0$ and $u \in \mathcal{H}_P(\Omega)$. We proved in Lemma 8.9 that if $\alpha = 0$, then (P, B) is a critical operator. Consequently, for a subcritical operator, $\alpha > 0$. Therefore, the function $G_\Omega^B(x, y) = \frac{u_y(x)}{\alpha}$ is a Green function of the operator (P,B) with a pole at y.

We need to prove the uniqueness. Suppose that there exists another Green function $F(x, y)$, with a pole at y. Since $\lim_{x \to y} \frac{F(x,y)}{G_\Omega^B(x,y)} = 1$, it follows that for every $0 < \varepsilon < 1$, there exists $r_\varepsilon > 0$ such that $G^B \geq \varepsilon F$ in $\overline{B_{r_\varepsilon}(y)}$. As a result of F being a positive solution of minimal growth at infinity of Ω, it follows that $G^B \geq \varepsilon F$ in $\Omega \setminus B_{r_\varepsilon}(y)$, hence, $G^B \geq \varepsilon F$ in Ω. Thus, $G^B \geq F$ in Ω. Similarly $G^B \leq F$ in Ω. Consequently, $F(x,y) = G^B_\Omega(x,y)$. $\square$

Lemma 8.11 *(P,B) is subcritical in Ω if and only if $\mathcal{SH}_P(\Omega)\setminus\mathcal{H}^0_P(\Omega)\neq\emptyset$.*

Proof: $\Leftarrow$ Suppose that $\mathcal{SH}_P(\Omega)\setminus\mathcal{H}^0_P(\Omega)\neq\emptyset$. If $\dim \mathcal{SH}_P(\Omega) = 1$, then by Corollary 8.8, $\mathcal{SH}_P(\Omega) = \mathcal{H}^0_P(\Omega)$ which contradicts our assumption. Hence, $\dim \mathcal{SH}_P(\Omega) > 1$, which implies that $\mathcal{H}_P(\Omega) \neq \emptyset$. By Theorem 8.9, (P,B) is subcritical in Ω.

$\Rightarrow$ Suppose that (P,B) is a subcritical operator in Ω and recall that we may assume that $\Gamma\neq\partial\Omega$. In view of Lemma 8.10, there exists a Green function $G^B_\Omega(\cdot,y)$ with a pole at $\{y\}$. By Lemma 6.2, $G^B_\Omega(\cdot,y) = \alpha G^D_\Omega(\cdot,y) + w(\cdot)$, where $\alpha>0$ and $w \in \mathcal{H}_P(\Omega)\setminus\mathcal{H}^0_P(\Omega)$. Thus, $\mathcal{SH}_P(\Omega)\setminus\mathcal{H}^0_P(\Omega)\neq\emptyset$. $\square$

Lemma 8.12 *The operator (P,B) is a subcritical operator in Ω if and only if for every $y\in\Omega$ the operator (P,B) admits a (unique) Green function with a pole at y.*

Proof: $\Rightarrow$ follows from Lemma 8.10.

$\Leftarrow$ Suppose that $F(x,y)$ is a Green function of (P,B). We may assume that $\Gamma \neq \partial\Omega$. In view of Lemma 6.2, $F(x,y) = \alpha G^D_\Omega(x,y) + \tilde{w}$, where $\alpha \geq 0$ and $w \in \mathcal{H}_P(\Omega)$. Clearly, $\alpha > 0$, and by Lemma 6.2, $\tilde{w} \in \mathcal{H}_P(\Omega) \setminus \mathcal{H}^0_P(\Omega)$. In the light of Lemma 8.11, (P,B) is subcritical. $\square$

In the following lemma and theorem, we present additional characterizations of criticality and subcriticality.

Lemma 8.13 *(P,B) is subcritical in Ω if and only if $\mathcal{SH}_P(\Omega)\setminus\mathcal{H}_P(\Omega)\neq\emptyset$.*

Proof: $\Leftarrow$ If $\mathcal{SH}_P \setminus \mathcal{H}_P \neq \emptyset$, then $\mathcal{SH}_P \setminus \mathcal{H}^0_P \neq \emptyset$, and by Lemma 8.11, (P,B) is subcritical in Ω.

$\Rightarrow$ Suppose that (P,B) is subcritical. According to Theorem 8.9, there exist u_0 and u_1 in $\mathcal{SH}_P$, such that $u_0 \neq Cu_1$. We have

$$P(\sqrt{u_0 u_1}) = \tfrac{1}{2}\left(\tfrac{u_1}{u_0}\right)^{1/2} Pu_0 + \tfrac{1}{2}\left(\tfrac{u_0}{u_1}\right)^{1/2} Pu_1 +$$

$$\frac{1}{4}u_0^{-3/2}u_1^{5/2}\sum_{i=1}^{n} a_{ij}\partial_i\left(\frac{u_0}{u_1}\right)\partial_j\left(\frac{u_0}{u_1}\right) \geq \frac{1}{4}u_0^{-3/2}u_1^{5/2}\Lambda_0\left|\nabla\left(\frac{u_0}{u_1}\right)\right|^2 \gneqq 0,$$

where $\Lambda_0 = \Lambda_0(x)$ is the ellipticity constant at x. In addition,

$$B\sqrt{u_0 u_1} = \frac{1}{2}\left(\frac{u_1}{u_0}\right)^{1/2} Bu_0 + \frac{1}{2}\left(\frac{u_0}{u_1}\right)^{1/2} Bu_1 \geq 0.$$

Consequently, $\sqrt{u_0 u_1} \in \mathcal{SH}_P \setminus \mathcal{H}_P$. $\square$

Lemma 8.14 *The operator (P, B) is subcritical in Ω, if and only if there exists $W \gneqq 0$ such that $P - W$ is subcritical.*

Proof: If (P, B) is subcritical, then Lemma 8.13 implies that there exists $u \in \mathcal{SH}_P \setminus \mathcal{H}_P$. Define $W = \frac{Pu}{2u}$, hence, $(P - W)u \gneqq 0$ in Ω and $Bu \geq 0$ on $\partial\Omega \setminus \Gamma$. According to Lemma 8.13, $P - W$ is subcritical.

On the other hand, if there exists $W \gneqq 0$ such that $P - W$ is subcritical, then there exists $u \in \mathcal{SH}_{P-W}$, and it follows that $u \in \mathcal{SH}_P \setminus \mathcal{H}_P$. Thus, by Lemma 8.13, P is subcritical. $\square$

Lemma 8.15 *If (P, B) is not a supercritical operator in Ω, then for every $W \gneqq 0$, the operator $P + W$ is subcritical.*

Proof: Let $u \in \mathcal{SH}_P$. Clearly, $u \in \mathcal{SH}_{P+W} \setminus \mathcal{H}_{P+W}$, and by Lemma 8.13, $P + W$ is subcritical. $\square$

Theorem 8.16 *The operator (P, B) is a critical in Ω if and only if for every $W \gneqq 0$, $P + W$ is subcritical and $P - W$ is supercritical.*

Proof: $\Rightarrow$ Suppose that $W \gneqq 0$. If P is a critical operator in Ω, then by Lemma 8.15, $P + W$ is subcritical.

Moreover, if for some $W \gneqq 0$, the operator $P - W$ is not supercritical, then, according to Lemma 8.15, the operator $P = (P-W)+W$ is subcritical. Consequently, if P is critical, then $P - W$ is supercritical for all $W \gneqq 0$.

$\Leftarrow$ Take $W_0 \gneqq 0$ in Ω. By our assumption, for every $\varepsilon > 0$, $P + \varepsilon W_0$ is subcritical. Hence, $P + \varepsilon W_0$ admits a positive normalized solution $u_\varepsilon \in \mathcal{H}_{P_\varepsilon}$. By Lemma 6.3, $\mathcal{H}_P \neq \emptyset$. Therefore, P is not supercritical.

In view of Lemma 8.14, if P is subcritical, there exists $W \gneqq 0$ such that $P - W$ is subcritical. Thus, under our assumption, P is critical. $\square$

Lemma 8.17 *The operator (P, B) is subcritical in Ω if and only if for any compact set $K \neq \emptyset$ in Ω there exists $u \in \mathcal{SH}_P(\Omega)$ such that $Pu > 0$ on K.*

Proof: If there exists $u \in \mathcal{SH}_P$ such that $Pu > 0$ in K, then, by Lemma 8.13, (P, B) is a subcritical operator.

Suppose that (P, B) is subcritical in Ω. Take $f \in C_0^\alpha(\Omega)$, $f \geq 0$, such that $f > 0$ in K. We claim that there exists $u_0 \in \mathcal{SH}_P$ satisfying $Pu_0 = f$ in Ω.

Recall our assumption that $\partial\Omega \setminus \Gamma \neq \emptyset$. Take $u \in \mathcal{H}_P$. Then for each $k \geq 1$ $(P + \frac{1}{k})u > 0$ in Ω, and $(B + \frac{1}{k})u > 0$ on $\partial\Omega \setminus \Gamma$. By Theorem 4.1, there exists a positive Perron-solution $u_k \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma)$ satisfying

$$(P + \frac{1}{k})u_k = f \text{ in } \Omega \text{ and } (B + \frac{1}{k})u_k = 0 \text{ on } \partial\Omega \setminus \Gamma. \tag{8.4}$$

If $\{u_k\}_{k=1}^\infty$ is locally uniformly bounded in $\bar{\Omega} \setminus \Gamma$, then there exists a subsequence that converges to a positive solution u_0 of the problem

$$Pu = f \text{ in } \Omega \text{ and } Bu = 0 \text{ on } \partial\Omega \setminus \Gamma,$$

and the proof is completed.

Otherwise, we have a point $x_0 \in \bar{\Omega} \setminus \Gamma$ such that $u_k(x_0) \to \infty$. Set $w_k(x) = \frac{u_k(x)}{u_k(x_0)}$. Hence, the sequence $\{w_k\}$ has a subsequence that converges to a positive function $w_0 \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \Gamma)$ satisfying

$$Pw_0 = 0 \text{ in } \Omega \text{ and } Bw_0 = 0 \text{ on } \partial\Omega \setminus \Gamma. \tag{8.5}$$

We shall show that in this case w_0 is a ground state, which contradicts our assumption that the operator is subcritical.

First, it can be shown as in the proof of Theorem 8.5 that w_k is a positive solution of minimal growth with respect to the operator $(P + \frac{1}{k}, B + \frac{1}{k})$.

Now let K' be a smooth compact set such that $\operatorname{supp}(f) \subset\subset K'$ and denote $D = \Omega \setminus K'$. Let $w \in C(\bar{D} \setminus \Gamma)$ such that

$$w > 0, \quad Pw \geq 0 \text{ in } D \quad \text{ and } Bw \geq 0 \text{ on } \partial\Omega \setminus \Gamma.$$

There exists $C > 0$ such that $w_k \leq Cw$ on $\partial K'$. Obviously,

$$(P + \frac{1}{k})Cw \geq 0 \text{ in } D, \ (B + \frac{1}{k})Cw \geq 0 \text{ on } \partial\Omega \setminus \Gamma.$$

Hence, $Cw \geq w_k$ in D and therefore $Cw \geq w_0$ in D. Accordingly, w_0 is a positive solution of problem (8.5) of minimal growth in Ω, and hence it is a ground state. $\square$

In the following theorem, we prove that for W with a compact support, the subcriticality is an "open" property.

Theorem 8.18 *The operator* (P, B) *is subcritical in* Ω *if and only if for every* $W \in C_0^\alpha(\Omega)$, $W \neq 0$ *there exists* $\varepsilon_0 > 0$ *such that* $(P - \varepsilon W, B)$ *is subcritical in* Ω *for every* $|\varepsilon| < \varepsilon_0$.

Proof: $\Rightarrow$ Suppose that P is subcritical and let $W \in C_0^\alpha(\Omega)$, $W \neq 0$. If $W \leq 0$, then $(P - \varepsilon W, B)$ is subcritical in Ω, for every nonnegative ε. Otherwise, define $W_+ = \max\{W, 0\}$ and $K^+ = \operatorname{supp}(W_+)$. By Lemma 8.17, there exists $u \in \mathcal{SH}_P$ satisfying $Pu > 0$ in K^+. Take $\varepsilon_+ = \frac{\min_{K^+}(Pu)}{2\max_{K^+}(W_+u)}$. Hence, for every $0 < \varepsilon \leq \varepsilon_+$,

$$(P - \varepsilon W)u \geq (P - \varepsilon W_+)u \geq (P - \varepsilon_+ W_+)u \gneqq 0 \text{ in } \Omega, \ Bu \geq 0 \text{ on } \partial\Omega.$$

Consequently, $P - \varepsilon W$ is subcritical for every $0 < \varepsilon < \varepsilon_+$.

By replacing W by $-W$ we obtain $\varepsilon_- < 0$ such that $P - \varepsilon W$ is subcritical for every $\varepsilon_- < \varepsilon < 0$.

$\Leftarrow$ Take $W \in C_0^\alpha(\Omega)$, $W \gneqq 0$. By our assumption, there exists $\varepsilon > 0$ such that $P - \varepsilon W$ is subcritical. Let $u_\varepsilon \in \mathcal{SH}_{P-\varepsilon W}(\Omega)$; then $u_\varepsilon \in \mathcal{SH}_P \setminus \mathcal{H}_P$, and P is subcritical. □

Proof of theorems 1.3 and 1.4: These two theorems follow directly from theorems 8.9, 8.16 and 8.18 and lemmas 8.11 and 8.12. □

We conclude the paper with some applications.

Corollary 8.19 *Suppose that* $W \in C^\alpha(\bar{\Omega} \setminus \Gamma)$, $W \neq 0$. *For every* $t \in \operatorname{int} S$ *the operator* (P_t, B) *is subcritical in* Ω. *Moreover, if* $W \in C_0^\alpha(\Omega)$, $W \neq 0$, *then for* $t \in \partial S$ *the operator* (P_t, B) *is critical in* Ω.

Proof: The first assertion follows from the proof of Lemma 6.3 and Lemma 8.13. The second assertion follows from Theorem 8.18. □

Corollary 8.20 *(a) Suppose that* $c_1 \geq c_2$ *in* Ω, $\gamma_1 \geq \gamma_2$ *and* $\beta_1 \leq \beta_2$ *on* $\partial\Omega \setminus \Gamma$, *and assume that not all the above inequalities are equalities. For* $i = 1, 2$, *denote* $P_i = P + c_i$, $B_i = \gamma_i + \beta_i \frac{\partial}{\partial \nu}$. *If* (P_2, B_2) *is not supercritical in* Ω, *then* (P_1, B_1) *is subcritical in* Ω.

(b) Suppose that $\Omega_1 \subsetneqq \Omega_2$ *such that* $\partial\Omega_1 \setminus \Gamma_1 \subset \partial\Omega_2 \setminus \Gamma_2$, *and assume that the boundary operators satisfy* $B_2|_{\partial\Omega_1 \setminus \Gamma_1} = B_1$. *If* (P, B_2) *is not supercritical in* Ω_2, *then* (P, B_1) *is subcritical in* Ω_1.

Proof: (a) If $u \in \mathcal{H}_{P_2,B_2}(\Omega)$, then $u \in \mathcal{SH}_{P_1,B_1}(\Omega) \setminus \mathcal{H}^0_{P_1,B_1}(\Omega)$. By Lemma 8.13, the operator (P_1, B_1) is subcritical in Ω.

(b) Suppose that (P, B_2) is not supercritical in Ω_2. Take $W \gneqq 0$ with a compact support in $\Omega_2 \setminus \Omega_1$. According to Lemma 8.15, $(P + W, B_2)$ is subcritical in Ω_2. Let $K \subset\subset \Omega_1$. By Lemma 8.17, there exists a function $u \in \mathcal{SH}_{P+W,B_2}(\Omega_2)$ satisfying $(P + W)u > 0$ in K. Since $W = 0$ in Ω_1, it follows that $u \in \mathcal{SH}_{P,B_1}(\Omega_1)$ and $Pu > 0$ in $K \subset \Omega_1$. In view of Lemma 8.17, (P, B_1) is subcritical operator in Ω_1. □

We now show that in general, if $\Omega_1 \subset \Omega_2$, then statement (b) of Lemma 8.20 does not hold, and that $\dim \mathcal{H}^0_P = 1$ does not imply criticality.

Example 8.21 Let $\Omega_R = \mathbb{R}^n \setminus B_R(0)$, where $R > 0$ and $n \geq 2$. Consider the Neumann problem:

$$-\Delta u = 0 \quad \text{in} \;\; \Omega_R, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on} \;\; \partial B_R. \tag{8.6}$$

Then $\dim \mathcal{H}^0_P(\Omega_R) = 1$ for every $R > 0$. Moreover, for $n = 2$ the operator $(-\Delta, \frac{\partial}{\partial n})$ is critical in Ω_R for all $R > 0$ (although it is subcritical for the Dirichlet problem or if $\gamma \gneqq 0$), while for $n \geq 3$ the operator $(-\Delta, \frac{\partial}{\partial n})$ is subcritical operator in Ω for all $R > 0$.

Corollary 8.22 *Suppose that $\Omega_1 \subseteq \Omega_2$ such that $\partial\Omega_1 \setminus \Gamma_1 \subseteq \partial\Omega_2 \setminus \Gamma_2$. Suppose also that $c_1 \geq c_2$ in Ω_1, $\gamma_1 \geq \gamma_2$ and $\beta_1 \leq \beta_2$ on $\partial\Omega_1 \setminus \Gamma_1$. For $i = 1, 2$, denote $P_i = P + c_i$, $B_i = \gamma_i + \beta_i \frac{\partial}{\partial \nu}$, and suppose that (P_2, B_2) is subcritical in Ω_2. Denote by $G^{P_i,B_i}_{\Omega_i}$ the Green function of (P_i, B_i) in Ω_i. Then*

$$G^{P_1,B_1}_{\Omega_1}(\cdot, y) \leq G^{P_2,B_2}_{\Omega_2}(\cdot, y) \qquad \text{in } \Omega_1.$$

Proof: (a) By Corollary 8.20, (P_1, B_1) is subcritical in Ω_1. Obviously,

$$\lim_{x \to y} \frac{G^{P_1,B_1}_{\Omega_1}(\cdot, y)}{G^{P_2,B_2}_{\Omega_2}(\cdot, y)} = 1.$$

Hence, for every $0 < \varepsilon < 1$ there exists $r_\varepsilon > 0$ such that $G^{P_1,B_1}_{\Omega_1}(\cdot, y) \leq (1 + \varepsilon) G^{P_2,B_2}_{\Omega_2}(\cdot, y)$ in $B_{r_\varepsilon}(y)$. Since

$$P_1 G^{P_2,B_2}_{\Omega_2}(\cdot, y) \geq 0 \text{ in } \Omega_1 \setminus B_{r_\varepsilon}, \;\; B_1 G^{P_2,B_2}_{\Omega_2}(\cdot, y) \geq 0 \text{ on } \partial\Omega_1 \setminus \Gamma_1,$$

and $G_{\Omega_1}^{P_1,B_1}(\cdot,y)$ has minimal growth at infinity of Ω_1, it follows that $G_{\Omega_1}^{P_1,B_1}(\cdot,y) \leq (1+\varepsilon)G_{\Omega_2}^{P_2,B_2}(\cdot,y)$ in $\Omega_1 \setminus B_{r_\varepsilon}(y)$, and hence in Ω_1. The conclusion follows easily by letting $\varepsilon \to 0$. □

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References

[1] S. Agmon, On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds, in "Methods of Functional Analysis and Theory of Elliptic Equations", ed. D. Greco, (Naples, 1982), 19–52, Liguori, Naples, 1983.

[2] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18** (1976), 620–709.

[3] H. Berestycki, L. A. Caffarelli and L. Nirenberg, Uniform estimates for regularization of free boundary problems, in: "Analysis and Partial Differential Equations", ed. C. Sadosky, Dekker, New-York, 1990, 567–619.

[4] H. Berestycki, L. Nirenberg and S. R. S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains, *Comm. Pure. App. Math.* **47** (1994), 47–92.

[5] L. A. Caffarelli, E. B. Fabes, S. Mortola and S. Salsa, Boundary behaviour of nonnegative solutions of elliptic operators in divergence form, *Indiana Univ. Math. J.* **30** (1981), 621–640.

[6] M. Chicco, A maximum principle for mixed boundary value problem for elliptic equations in non-divergence form, *Boll. Un. Math. Ital. B (7)* **11** (1997), 531–538.

[7] R. Courant and D. Hilbert, "Methods of Mathematical Physics", Wiley-Intenscience, New-York, 1962.

[8] D. Daners, Robin boundary value problems on arbitrary domains, *Trans. Amer. Math. Soc.* **352** (2000), 4207–4236.

[9] D. Gilbarg and L. Hörmander, Intermediate Schauder estimates, *Arch. Rational Mech. Anal.* **74** (1980), 297–318.

[10] D. Gilberg and J. Serrin, On isolated singularities of solutions of second order elliptic differential equations, *J. Analyse Math.* **4** (1955/6), 309–340.

[11] P. Guan and E. Sawyer, Regularity estimates for the oblique derivative problem, *Ann. of Math. (2)* **137** (1993), 1–70.

[12] A. I. Ibragimov and E. M. Landis, Zaremba's problem for elliptic equation in the neighborhood of a singular point or at infinity I-II, *Appl. Anal.* **67** (1997), 269–282, ibid **69** (1998), 333–347.

[13] T. M. Kerimov, A mixed boundary value problem for a second order linear elliptic equation, *Diff. Eq.* **13** (1977), 322–324.

[14] G. M. Lieberman, The Perron process applied to oblique derivative problems, *Adv. in Math.* **55** (1985), 161–172.

[15] G. M. Lieberman, Mixed boundary value problem for elliptic and parabolic differential equations of second order, *J. Math. An. App.* **113** (1986), 422–440.

[16] G. M. Lieberman, Nonuniqueness for some linear oblique derivative problems for elliptic equations, *Comm. Math. Univ. Carolin.* **40** (1999), 477–481.

[17] J. López-Gómez, The maximum principle and the existence of principal eigenvalue for some linear weighted boundary value problem, *J. Differential Equations* **127** (1996), 263–294.

[18] M. Murata, Structure of positive solutions to $-\Delta u + Vu = 0$ in $\mathbb{R}^n$, *Duke Math. J.* **53** (1986), 869–943.

[19] R. D. Nussbaum and Y. Pinchover, On variational principles for the generalized principal eigenvalue of second order elliptic operators and some applications, *J. Anal. Math.* **59** (1992), 161–177.

[20] B. P. Paneah, "The Oblique Derivative Problem. The Poincaré-Problem", Mathematical Topics **17**, Wiley-VCH Verlag, Berlin, 2000.

[21] Y. Pinchover, On positive solutions of second-order elliptic equations, stability results, and classification, *Duke Math. J.* **57** (1988), 955–980.

[22] Y. Pinchover, Criticality and ground states for second-order elliptic equations, *J. Differential Equations* **80** (1989), 237–250.

[23] Y. Pinchover, On criticality and ground states for second-order elliptic equations II, *J. Differential Equations* **87** (1990), 353–364.

[24] Y. Pinchover, On positive Liouville theorems and asymptotic behavior of solutions of Fuchsian type elliptic operators, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **11** (1994), 313–341.

[25] R. G. Pinsky, "Positive Harmonic Function and Diffusion", Cambridge University Press, Cambridge, 1995.

[26] M. H. Protter and H. F. Weinberger, "Maximum Principles in Differential Equations", Springer-Verlag, N.Y, 1984.

[27] T. Saadon (Suez), On the generalized principal eigenvalue of degenerate boundary value problems for second-order elliptic equations, Ph. D. thesis, Technion - Israel Institute of Technology, 2000.

[28] J. Serrin, A remark on the preceding paper of Herbert Amann, *Arch Rational Mech. Anal.* **44** (1971/72), 182–186.

[29] B. Simon, Large time behavior of the L^p norm of the Schrödinger semigroups, *J. Funct. Anal.* **40** (1981), 66–83.

[30] K. Taira, Existence and uniqueness theorems for semilinear elliptic boundary value problems, *Adv. Differential Equations* **2** (1997), 509–534.