# ON NONEXISTENCE OF ANY $\lambda_0$-INVARIANT POSITIVE HARMONIC FUNCTION, A COUNTER EXAMPLE TO STROOCK'S CONJECTURE

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**Abstract**

Consider a sub-Markovian semigroup such that $\lambda_0$, the border number between recurrence and transience, equals zero. In 1982, D. W. Stroock conjectured that under general hypotheses on the semigroup the corresponding process always admits an invariant measure.

In this paper we present an example of a second order elliptic operator $P$ with a generalized principal eigenvalue $\lambda_0$ which equals zero such that the parabolic equation does not admit any positive invariant $P$-harmonic function and also any invariant measure. This gives a counter example to Stroock's conjecture for diffusion processes. We also present an example of a complete Riemannian manifold $M$ which does not admit any positive invariant harmonic function while $\lambda_0$, the bottom of the spectrum of $M$, is zero. This gives a partial answer to a question of Stroock and Sullivan.

## 1 Introduction

Let $X$ be a domain in $\mathbb{R}^n$ (or more generally, a smooth noncompact connected manifold) and consider a second order linear parabolic operator

$$Lu = u_t + Pu \tag{1.1}$$

defined on $X \times (0, \infty)$. Here $P$ is a time independent smooth elliptic operator of the form

$$P(x, \partial_x) = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(x)\partial_i + c(x), \tag{1.2}$$

where $\partial_i = \partial/\partial x_i$.

We denote the cone of all positive (classical) solutions of the elliptic equation $Pu = 0$ in $X$ by $\mathcal{C}_P(X)$. Let $\lambda_0(P, X)$ be the generalized principal eigenvalue given by

$$\lambda_0(P, X) = \sup\{\lambda \in \mathbb{R} \mid \mathcal{C}_{P-\lambda}(X) \neq \emptyset\}. \tag{1.3}$$

We always assume that $\lambda_0(P, X) > -\infty$. Thus, without loss of generality we may assume that $\lambda_0(P, X) = 0$. We denote by $P^*$ the formal adjoint of $P$.

Let $K_P^X(x, y, t)$ be the minimal positive heat kernel of the operator $L$ on $X \times (0, \infty)$ (see Section 2).

**Definition 1.1** *A function $u \in \mathcal{C}_P(X)$ is called an* **invariant solution** *of (P,X) if $u$ satisfies the integral equation*

$$\int_X K_P^X(x, y, t)u(y)dy = u(x) \text{ for for all } (x, t) \in X \times (0, \infty). \tag{1.4}$$

*A function $v$ is said to be an* **invariant density** *of (P,X) if $v \in \mathcal{C}_{P^*}(X)$ and $v$ satisfies the integral equation*

$$\int_X K_P^X(x, y, t)v(x)dx = v(y) \text{ for for all } (y, t) \in X \times (0, \infty). \tag{1.5}$$

**Remarks 1.2** $(i)$ An invariant solution is sometimes called also *complete*. If the constant function is invariant then one says that $L$ *conserves probability* and the diffusion process is said to be *stochastically complete*.

$(ii)$ A function $u$ is an invariant solution of $(P, X)$ if and only if it is an invariant density of $(P^*, X)$.

$(iii)$ It is well known that if $P$ is critical (i.e. the corresponding process is recurrent) or if the uniqueness of the positive Cauchy problem holds true for the operator $L$ in $X$ then every function $u \in \mathcal{C}_P(X)$ is invariant (see [13, 14, 15, 16, 17, 18], for other sufficient conditions for the existence of an invariant solution see [4, 6, 7, 9, 18]).

In the recent book of R. Pinsky [16] the following conjecture is stated (see also [13, 14, 17, 18]).

**Conjecture 1.3** *If* $\lambda_0(P, X) = 0$ *then* $L$ *admits an invariant solution and an invariant density on* $X$.

This conjecture goes back to D. Stroock [17] who stated it in the more general situation of sub-Markovian semigroups. Later on, Stroock and Sullivan [18], posed the problem on a Riemannian manifold and the corresponding heat equation.

The aim of this note is to construct counter examples to Conjecture 1.3. In Section 3 we present an elliptic operator $P$ in $\mathbb{R}^n, n > 2$ with $\lambda_0(P, \mathbb{R}^n) = 0$ such that the corresponding parabolic operator does not admit an invariant solution and an invariant density on $\mathbb{R}^n$. Moreover, $P$ can be chosen to be symmetric.

In section 4 we construct a complete Riemannian manifold such that its heat operator does not admit any $\lambda_0$-invariant solution. We use the same idea as in the first example, potential theory and some spectral properties of the Laplace-Beltrami operator on the standard 2-dimensional jungle gym $JG^2$ in $\mathbb{R}^3$.

Finally, in Section 5 we discuss briefly two conjectures which turned out to be closely related to the example presented in Section 4. The first conjecture deals with the Liouville property for a Riemannian product $X \times K$, where $X$ is a complete Riemannian manifold which satisfies the Liouville property (with $\lambda_0(-\Delta, X) = 0$) and $K$ is a compact Riemannian manifold. The second conjecture concerns exact long time asymptotics of the heat kernel.

## 2 Preliminaries

Let $X$ be a noncompact connected countable at infinity smooth (Riemannian) manifold of dimension $n$ and let $Lu = u_t + Pu$ be a second order parabolic linear operator on $X \times (0, \infty)$. Here $P$ is a time independent smooth elliptic operator with real coefficients which in any coordinate system $(U; x_1, \ldots, x_n)$ has the form (1.2). We denote by $B_r(x)$ the open geodesic ball of radius $r$ centered at $x$.

Let $\Omega \subset X$ be a domain and let $\{\Omega_k\}_{k=1}^{\infty}$ be a sequence of smooth relatively compact domains such that $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\cup_{k=1}^{\infty} \Omega_k = \Omega$. Assume that

$\mathcal{C}_P(\Omega) \neq \emptyset$ then for every $k \geq 1$ the Dirichlet Green function $G_P^{\Omega_k}(x,y)$ exists and is positive. By the generalized maximum principle $\{G_P^{\Omega_k}(x,y)\}_{k=1}^{\infty}$ is an increasing sequence which, by the elliptic Harnack inequality, converges uniformly in every compact subdomain either to $G_P^{\Omega}(x,y)$, the positive *minimal Green function* of $P$ in $\Omega$ and $P$ is said to be *subcritical operator* in $\Omega$, or to infinity and in this case $P$ is *critical* in $\Omega$. The operator $P$ is said to be *supercritical* if $\mathcal{C}_P(\Omega) = \emptyset$. Similarly, one defines $K_P^{\Omega}(x,y,t)$ the *positive (minimal) heat kernel* of the parabolic operator $L$ in $\Omega$ (for more details see [10, 16]).

Let $\lambda_0(P,\Omega)$ be the generalized principal eigenvalue defined by (1.3). One easily sees [16] that $\lambda_0(P,\Omega) = \lambda_0(P^*,\Omega)$ and that $P$ is subcritical (critical) in $\Omega$ if and only if $P^*$ is subcritical (critical) in $\Omega$. Moreover, if $\{\Omega_k\}_{k=1}^{\infty}$ is the above increasing sequence of domains such that $\bigcup_{k=1}^{\infty} \Omega_k = \Omega$ then

$$\lambda_0(P,\Omega) = \lim_{k\to\infty} \lambda_0(P,\Omega_k). \tag{2.1}$$

Throughout this note we suppose that $\lambda_0(P,X) = 0$. Moreover, we assume that the elliptic operator $P$ is subcritical in $X$. Recall also that in the subcritical case

$$G_P^X(x,y) = \int_0^{\infty} K_P^X(x,y,t)dt. \tag{2.2}$$

Let $u \in \mathcal{C}_P(X)$. It follows from the generalized maximum principle that either

$$\int_X K_P^X(x,y,t)u(y)dy = u(x) \text{ for some (and hence for all) } x \in X \tag{2.3}$$

or

$$\int_X K_P^X(x,y,t)u(y)dy < u(x) \text{ for some (and hence for all) } x \in X. \tag{2.4}$$

It is well known [4] that Inequality (2.4) holds true if and only if there exists $\lambda < 0$ such that

$$-\lambda \int_X G^X_{P-\lambda}(x,y)u(y)dy < u(x) \text{ for some (and hence for all) } x \in X. \quad (2.5)$$

Furthermore, we have ([12, 15])

**Lemma 2.1** *Assume that the operator $P$ is subcritical in $X$ and let $u \in \mathcal{C}_P(X)$. Then $u$ is not an invariant solution of $(P, X)$ if*

$$\int_X G^X_P(x,y)u(y)dy < \infty \quad (2.6)$$

*for some $x \in X$.*

**Proof**: It is enough to show that if (2.6) holds true then there exists $\lambda < 0$ such that Inequality (2.5) is satisfied. Assume on the contrary that for every $\lambda < 0$

$$-\lambda \int_X G^X_{P-\lambda}(x,y)u(y)dy = u(x) \text{ for some (and hence for all) } x \in X. \quad (2.7)$$

Divide (2.7) by $\lambda$ and let $\lambda$ tend to zero. Using the Lebesgue monotone convergence theorem we obtain

$$\int_X G^X_P(x,y)u(y)dy = \infty \quad (2.8)$$

which contradicts our assumption. □

Let $P$ be an elliptic operator and let $\rho$ be a positive smooth function on $X$. We denote by $P_\rho = \rho(x)P$. The following is a key lemma for the construction of our examples.

**Lemma 2.2** *Consider the parabolic operator* $Lu = u_t + \rho(x)Pu$ *defined on* $X$. *Assume that* $P$ *is subcritical in* $X$ *and that* $\rho$ *is a positive smooth function. Then* $u \in \mathcal{C}_P(X)$ *is not an invariant solution of the parabolic operator* $Lu = u_t + P_\rho u$ *in* $X$ *if*

$$\int_X \frac{G_P^X(x,y)u(y)}{\rho(y)}dy < \infty \qquad (2.9)$$

*for some* $x \in X$. *Furthermore,* $\lambda_0(P_\rho, X) = 0$ *if and only if* $\mathcal{C}_{P-\lambda/\rho(x)}(X) = \emptyset$ *for all* $\lambda > 0$.

**Proof**: Consider the elliptic operator $P_\rho = \rho(x)P$ then $P_\rho$ is subcritical in $X$ and the positive minimal Green function of $P_\rho$ in $X$ is given by

$$G_{P_\rho}^X(x,y) = \frac{G_P^X(x,y)}{\rho(y)}. \qquad (2.10)$$

Moreover, $u \in \mathcal{C}_P(X)$ if and only if $u \in \mathcal{C}_{P_\rho}(X)$. Hence, the non-invariance of $u$ follows directly from Lemma 2.1.

The second assertion of the lemma follows from definition of $\lambda_0$ and from the simple observation that for every $\lambda \in \mathbb{R}$, $u \in \mathcal{C}_{P_\rho - \lambda}(X)$ if and only if $u \in \mathcal{C}_{P-\lambda/\rho(x)}(X)$. □

# 3 A diffusion operator in $\mathbb{R}^n$ with no positive $\lambda_0$-invariant solution

In this section we construct a counter example to Conjecture 1.3. We construct a parabolic operator in $\mathbb{R}^n, n \geq 3$ of the form $Lu = u_t + \rho(x)\Delta u$ with $\lambda_0(-\rho(x)\Delta, \mathbb{R}^n) = 0$ and with no invariant solution (density). Note that by the classical Liouville theorem

$$\mathcal{C}_{-\rho(x)\Delta}(\mathbb{R}^n) = \mathcal{C}_{-\Delta}(\mathbb{R}^n) = \{u \mid u(x) = c,\ c > 0\}. \tag{3.1}$$

Therefore, by Lemma 2.2 it is enough to find a positive function $\rho$ such that

$$\int_{\mathbb{R}^n} \frac{|y|^{2-n}}{\rho(y)} dy < \infty \tag{3.2}$$

and such that $\mathcal{C}_{-\Delta-\lambda/\rho(x)}(\mathbb{R}^n) = \emptyset$ for all $\lambda > 0$.

Set $x_i = (2^i + i, 0, \ldots, 0) \in \mathbb{R}^n, i = 1, 2, \ldots$ . Let $0 < \rho_0(x)$ be a smooth function on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \frac{|y|^{2-n}}{\rho_0(y)} dy < \infty. \tag{3.3}$$

For $i \geq 1$ consider a smooth function on $\mathbb{R}^n$ such that $0 \leq V_i(x) \leq 1$ and such that

$$V_i(x) = \begin{cases} 1 & \text{if } x \in B_{i/2}(x_i) \\ 0 & \text{if } x \notin B_i(x_i)\ . \end{cases} \tag{3.4}$$

Set

$$\frac{1}{\rho(x)} = V(x) = \frac{1}{\rho_0(x)} + \sum_{i=1}^{\infty} V_i(x). \tag{3.5}$$

It is easy to check that the function $\rho$ defined in (3.5) satisfies (3.2). On the other hand,

$$\lim_{i\to\infty} \lambda_0(-\Delta, B_{i/2}(x_i)) = \lim_{i\to\infty} \lambda_0(-\Delta, B_{i/2}(x)) = \lambda_0(-\Delta, \mathbb{R}^n) = 0. \tag{3.6}$$

Hence, for a fixed $\lambda > 0$ there exists $i_0 \geq 1$ such that $\mathcal{C}_{-\Delta-\lambda}(B_{i/2}(x_i)) = \emptyset$ for every $i \geq i_0$. Since $V(x) = 1/\rho(x) \geq 1$ on $B_{i/2}(x_i)$ it follows that $\mathcal{C}_{-\Delta-\lambda V(x)}(B_{i/2}(x_i)) = \emptyset$ for every $i \geq i_0$. Also, from $\mathcal{C}_{-\Delta-\lambda V(x)}(\mathbb{R}^n) \subset$

$\mathcal{C}_{-\Delta-\lambda V(x)}(B_{i/2}(x_i))$, we have $\mathcal{C}_{-\Delta-\lambda/\rho(x)}(\mathbb{R}^n) = \emptyset$. But $\lambda$ was an arbitrary positive number, therefore, Lemma 2.2 implies that $\lambda_0(-\rho(x)\Delta, \mathbb{R}^n) = 0$.

Now we show that $(-\rho(x)\Delta, \mathbb{R}^n)$ does not admit also an invariant density. The formal adjoint of the operator $P = -\rho(x)\Delta$ is given by $P^*v = -\Delta(\rho(x)v)$. By the classical Liouville theorem $\mathcal{C}_{P^*}(\mathbb{R}^n)$ is a one-dimensional cone spanned by the positive solution $v(x) = 1/\rho(x)$. Consider the operator $P^*$ in $\mathbb{R}^n$ and its positive solution $v$. Then condition (2.6) of Lemma 2.1 for the non-invariance of the positive solution $v$ reads again as condition (3.2) which was just shown to hold. Hence, $v$ is not a positive invariant solution of $P^*$ in $\mathbb{R}^n$.

**Remark 3.1** It is easy to see that the operator $Pu = -\rho(x)\Delta(\rho(x)u)$ is a symmetric subcritical elliptic operator on $\mathbb{R}^n, n \geq 3$. Moreover, $\mathcal{C}_P(\mathbb{R}^n)$ is a one-dimensional cone spanned by the positive solution $v(x) = 1/\rho(x)$. Using the same function $\rho$ and the same consideration as in the above example we infer that $\mathcal{C}_{-\Delta-\lambda/\rho^2(x)}(\mathbb{R}^n) = \emptyset$ for all $\lambda > 0$ and

$$\int_{\mathbb{R}^n} \frac{|y|^{2-n}}{\rho^2(y)} dy < \infty. \tag{3.7}$$

It follows that $\lambda_0(P, \mathbb{R}^n) = 0$ and the *symmetric* operator $P$ does not admit any positive invariant solution.

# 4 A complete Riemannian manifold with no $\lambda_0$-invariant harmonic function

In this section we construct a complete Riemannian manifold which is conformal to the standard 2-dimensional jungle gym $JG^2$ such that its heat operator does not admit any $\lambda_0$-invariant solution.

The standard 2-dimensional jungle gym $JG^2$ in $\mathbb{R}^3$ is constructed by (i) considering the integer lattice $\mathbb{Z}^3$ in $\mathbb{R}^3$, (ii) connecting points, for which precisely one of their coordinates differs by 1 and the other two coordinates are equal, by a line segment parallel to the coordinate axis, (iii) considering the surface consisting of all points with distance from the 1–dimensional network to be precisely equal to $\epsilon$, for some given small $\epsilon > 0$, and finally (iv) smoothing out the corners in a bounded periodic fashion.

The natural action of the group $\mathbb{Z}^3$ on $JG^2$ is properly discontinuous and free. Therefore, the quotient space $N = JG^2/\mathbb{Z}^3$ has a natural structure of a smooth compact 2-dimensional manifold. We introduce a Riemannian structure on $N$ and obtain a regular unramified Riemannian covering $(JG^2, g)$, here $g$ is the lifted metric. The lifted Laplace-Beltrami operator $-\Delta_g$ is a $\mathbb{Z}^3$-equivariant operator on $JG^2$.

Since $\mathbb{Z}^3$ is abelian it follows [10, 11] that $JG^2$ satisfies the positive Liouville theorem, that is, $JG^2$ does not admit any nonconstant positive harmonic function. On the other hand, $\lambda_0(-\Delta_g, JG^2) = 0$ ([2, 10]). Moreover, T. Lyons and D. Sullivan [11] proved that $-\Delta_g$ is subcritical on $JG^2$.

Denote by $d(\cdot,\cdot)$ the distance function on $(JG^2, g)$. In their recent paper [1] Benjamini, Chavel and Feldman proved that

$$G_{-\Delta_g}^{JG^2}(x_0, y) \leq C d(x_0, y)^{-1} \tag{4.1}$$

if $d(x_0, y) > 1$ (see also [3]).

Let $\rho(x)$ be a positive smooth function on $JG^2$. Consider a conformal change of the metric $g \mapsto \hat{g} = (\rho(x))^{-1} g$. Since $JG^2$ is a 2-dimension manifold the Laplace-Beltrami operator with respect to the new Riemannian structure is given by $-\Delta_{\hat{g}} = -\rho(x)\Delta_g$.

So, we are actually in the same situation as in the Euclidean space but now our weighted operator $-\rho(x)\Delta_g$ is the Laplace-Beltrami operator on $(JG^2, \hat{g})$. Since $JG^2 \subset \mathbb{R}^3$ we may even use the same weight function $\rho(x)$ (see (3.5)) with a smooth positive function $\rho_0$ such that

$$\rho_0(x) = (|x| \log |x|)^2, \qquad |x| \geq 2, \tag{4.2}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^3$. One can check that with this weight function $\rho$, the Riemannian manifold $(JG^2, \hat{g})$ is complete. As in Section 3, we have that $\lambda_0(-\rho(x)\Delta_g, JG^2) = 0$ while 1 is the unique (up to a multiplicative constant) positive $\Delta_{\hat{g}}$-harmonic function. On the other hand,

$$\int_{JG^2} \frac{G^{JG^2}_{-\Delta_g}(x_0, y)}{\rho(y)} dy < \infty, \tag{4.3}$$

where $dy$ is the volume element with respect to the metric $g$ and $x_0$ is some fixed point in $JG^2$. By Lemma 2.2 the constant function is not an invariant solution of the Laplace-Beltrami operator $-\Delta_{\hat{g}}$ on $JG^2$.

# 5 Some remarks

It seems to us that it is worth to point out here that the example constructed in Section 4 is closely related to two other conjectures. The first conjecture deals with the Liouville property and in the second the exact long time asymptotics of the heat kernel is considered.

Recall that a Riemannian manifold $X$ has the *Liouville property* if there is no nonconstant bounded harmonic function on $X$. The following conjecture was posed by A. A. Grigor'yan in a conference on diffusion processes held in Northwestern University on summer 1994:

**Conjecture 5.1** *If the Liouville property holds on a complete manifold $X$ (with $\lambda_0(-\Delta, X) = 0$) then it holds also on any Riemannian product $X \times K$, where $K$ is a compact manifold.*

Professor A. A . Grigor'yan has kindly pointed out to the author that the example of Section 4 is also a counter example to Conjecture 5.1. The proof below is based on a result of A. A. Grigor'yan [8].

Let $(K, \kappa)$ be a compact Riemannian manifold. Consider the positive eigenvalues $\lambda_i$ and the corresponding eigenfunctions $v_{\lambda_i}(k),\ i \geq 1$ of the Laplace-Beltrami operator $-\Delta_\kappa$ on $K$.

Let $(X, \hat{g})$ be the jungle gym $JG^2$ with the conformal Riemannian metric $\hat{g} = (\rho(x))^{-1}g$, where $g$ is a $\mathbf{Z}^3$-invariant Riemannian metric on $JG^2$ and $\rho$ is the weight function of Section 4. Let $G^{JG^2}_{-\Delta_{\hat{g}}}(x, y)$ be the corresponding Green function. We have shown that

$$\int_X G^{JG^2}_{-\Delta_{\hat{g}}}(x_0, y)dy < \infty \ , \tag{5.1}$$

where $dy$ is the volume element with respect to the metric $\hat{g}$. Thus by Theorem 2.1 of [8], for every $\lambda > 0$ there exists a positive bounded solution $u_\lambda(x)$ of the equation $(-\Delta_{\hat{g}} + \lambda)u = 0$ in $X$. It can be easily checked that for every $i \geq 1$

the function $u_{\lambda_i}(x)v_{\lambda_i}(k)$ is a nonconstant bounded harmonic function on the Riemannian product $(X, \hat{g}) \times (K, \kappa)$. Hence, Conjecture 5.1 is false.

We conclude this section with a short remark on the following recent conjecture which was posed by E. B. Davies (see [5]).

**Conjecture 5.2** *Let $X$ be a Riemannian manifold and let $K(x, y, t)$ be the heat kernel associated with the Laplace-Beltrami operator on $X$. Let $p \in X$ be any chosen point. Then the limit*

$$a(x, y) = \lim_{t \to \infty} \frac{K(x, y, t)}{K(p, p, t)} \tag{5.2}$$

*exists and positive for all $x, y \in X$.*

It can be easily shown that if the limit in (5.2) exists then the function $a(x, y)$ is a positive $\lambda_0$-harmonic function in each variable separately. Moreover, it was shown in [5, Theorem 25] that under a certain uniform-integrability condition if the limit in (5.2) exists then $a(x, y)$ is a positive $\lambda_0$-invariant solution in each variable separately.

Hence, for the manifold $(JG^2, \hat{g})$ of Section 4 either Conjecture 5.2 is false or $a(x, y)$ is the constant function (which was shown to be a noninvariant harmonic function) and the heat kernel fails to satisfy the uniform-itegrability condition.

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