On positivity, criticality and the spectral radius of the shuttle operator for elliptic operators

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Abstract

In this paper, we study the shuttle operator for a second order linear elliptic operator P on a noncompact manifold X. Zhao has introduced and studied the shuttle operator and its relation to the theory of positive solutions in the case of small perturbations of the Laplacian in $\mathbb{R}^n, n \geq 3$. Zhao was motivated by works of Chung and Varadhan which consider one-dimensional Schrödinger operators.

The main purpose of the paper is to extend the above studies. We prove that, in the general case, the spectral radius of the shuttle operator is strictly less, equal, or strictly greater than 1 if and only if the operator P is respectively subcritical, critical, or supercritical in X. We demonstrate the usefulness of the characterization for proving theorems. Our approach is purely analytic.

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RUNNING HEAD: positivity and shuttle operator

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1 Introduction

Let X be a smooth, noncompact, connected manifold and consider a second order linear elliptic operator P defined on X. Let Ω be a domain in X, and denote by $\mathcal{C}_P(\Omega)$ the convex cone of all classical, positive solutions of the equation

$$Pu = 0 \;\; in\, \Omega. \tag{1.1}$$

We recall ([11, 14], see also Section 2, for a more detailed definition) that P is a *subcritical operator* in Ω if it possesses a positive minimal Green function $G_P^{\Omega}(x, y)$ in Ω. The operator P is said to be *critical* in Ω if it is not subcritical, but $\mathcal{C}_P(\Omega) \neq \emptyset$. Finally, P is *supercritical* if $\mathcal{C}_P(\Omega) = \emptyset$.

Given an operator P and a domain Ω, there are several criteria to distinguish between the above three situations. These criteria depend on the behavior of the cone of positive solutions under perturbations of either the operator P, or the domain Ω (see for example [5, 6, 9, 10, 11, 14, 17] and the references therein). Thus, they are of perturbation nature. But, so far, there is no general intrinsic criterion depending only on the given operator P in Ω, which distinguishes between the three possibilities. Recall also, that we do not impose any boundary behavior at "infinity" for solutions in $\mathcal{C}_P(\Omega)$. Therefore, the natural topology in the theory of positive solutions is the compact open topology and, in general (especially in the nonselfadjoint case), we usually do not have any information in Banach spaces' terminology.

The aim of this paper is to present an intrinsic criterion, which distinguishes between subcriticality, criticality and supercriticality of the operator P in X. Moreover, this criterion depends only on the norm of a certain linear operator defined on a Banach space. More precisely, we introduce and study a linear operator S associated with the differential operator P in X. The operator S is called the *shuttle operator* and is defined on $C(\partial D)$, the Banach space of continuous functions on the boundary of a certain smooth, relatively compact subdomain D. It turns out (Theorem 4.1) that the norm and the spectral radius of the shuttle operator S is less, equal, or greater than 1 if and only if the operator P is respectively, subcritical, critical, or supercritical in X.

The shuttle operator was first introduced for Schrödinger operators in $\mathbb{R}^n$ by Zhao [16, 17]. Zhao was motivated by the works of Chung [2], and Chung and Varadhan [3], which consider the case of one-dimensional Schrödinger operators on $\mathbb{R}$. Using probabilistic methods, Zhao [17], and Gesztesy and Zhao [6], have studied the shuttle operator for Schrödinger operators in $\mathbb{R}^n$ with short range potentials (see also [5]). They proved the above criterion for this particular case using the Feynman-Kac formula and the explicit form and properties of the Green function of the Laplacian in a ball.

Our proof relies on the simple observation that the shuttle operator S is (in the nontrivial case) a positive compact operator. Therefore, the Krein-Rutman theorem implies, that there exists a simple principal eigenvalue $\nu_0 > 0$, which is equal to the norm (and also to the spectral radius) of S, and that the corresponding principal eigenfunction is strictly positive. It turns out, that the generalized maximum principle holds in X if and only if $\nu_0 \leq 1$ and that $\nu_0 < 1$ if and only if P admits a positive minimal Green function in X.

The outline of this paper is as follows. In Section 2, we give some basic definitions, collect some results and fix notations. In Section 3, we define the shuttle operator S, study some of its properties and use these results to prove certain sufficient conditions for subcriticality and criticality of diffusion operators (Corollary 3.10). Our main result (Theorem 4.1) is proved in Section 4. Finally, in Section 5, we use our criterion and prove the result known as "localization of binding" [13, 14, and the references therein]. Our approach here has the advantage that it can be used to prove, for the first time, the localization of binding for certain *nonselfadjoint* operators in the critical case . Using this approach, we prove the localization of binding for certain subcritical elliptic operators with periodic coefficients.

2 Preliminaries

In this section, we collect some general results concerning positivity properties of elliptic operators (see also, [9, 10, 11, 14, and the references therein]).

We consider a linear elliptic operator P of second order, which is defined on an n-dimensional manifold X. Here X is a connected, noncompact C^3-manifold which is countable at infinity. We assume that P acts on functions $u \in C^2(X)$. Thus, we suppose that in any coordinate system $(U; x_1, \ldots, x_n)$, the operator P is of the form

$$P(x, \partial_x) = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(x)\partial_i + c(x), \tag{2.1}$$

where $\partial_i = \partial/\partial x_i$.

We assume that for every $x \in X$, the quadratic form

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \tag{2.2}$$

is positive definite. The matrix $[a^{ij}]$, the inverse of $[a_{ij}]$, defines a Riemannian metric g on the manifold X:

$$g(\partial/\partial x_i, \partial/\partial x_j) = a^{ij}. \tag{2.3}$$

We denote by ρ and dx, the distance and volume element induced by the Riemannian metric (2.3). We denote by $B(x,r)$ the geodesic ball of radius r centered at x. Let $D \subset X$ be a domain. We use the notation $D^* = X \setminus \overline{D}$. If K is a compact set in D, then $D \setminus K$ is called a *neighborhood of infinity in* D. We denote by $\mathbf{1}$ the function on X taking the value one at every point.

We assume that the coefficients a_{ij}, b_i and c are *real* and locally Hölder continuous.

Let $\Omega \subset X$ be a domain, and let $\{\Omega_k\}_{k=1}^{\infty}$ be a sequence of smooth, relatively compact domains such that $\overline{\Omega_k} \subset \Omega_{k+1}$, and $\cup_{k=1}^{\infty}\Omega_k = \Omega$. Recall that $\mathcal{C}_P(\Omega)$ denotes the convex cone of all classical positive solutions of the equation $Pu = 0$ in Ω. If $\mathcal{C}_P(\Omega) \neq \emptyset$, then for every $k \geq 1$, the Dirichlet Green function $G_P^{\Omega_k}(x,y)$ exists and is positive. By the generalized maximum principle, $\{G_P^{\Omega_k}(x,y)\}_{k=1}^{\infty}$ is an increasing sequence which, by the Harnack inequality, converges uniformly in every compact subdomain of $\Omega \times \Omega \setminus \{(x,x) | x \in \Omega\}$, either to $G_P^{\Omega}(x,y)$, the positive *minimal Green function* of P in Ω, in which case P is said to be a *subcritical operator* in Ω, or to infinity, in which case P is said to be *critical* in Ω. The operator P is said to be *supercritical* in Ω if $\mathcal{C}_P(\Omega) = \emptyset$.

Definition 2.1 Let P be an elliptic operator defined in a domain $\Omega \subseteq X$, and let D^* be a neighborhood of infinity in Ω. A function $u \in C(\overline{D^*})$ is said to be a *positive solution of minimal growth in a neighborhood of infinity in* Ω if u satisfies the following two conditions:

(i) *The function* u is a positive solution of the equation $Pu = 0$ in D^*;

(ii) Let $(D_1)^* \subseteq D^*$ be a neighborhood of infinity in Ω with smooth boundary. If v is a continuous function on $\overline{(D_1)^*}$, and a positive solution of the equation $Pu = 0$ in $(D_1)^*$, and $u \leq v$ on ∂D_1, then $u \leq v$ on $(D_1)^*$.

Definition 2.2 Let P be a second order elliptic operator defined in a domain $\Omega \subseteq \mathbb{R}^n$. A function u is said to be a *ground state (in the sense of Agmon) of the operator P in the domain Ω with an eigenvalue zero* if u satisfies the following two conditions:

(i) u is a positive solution of the equation $Pu = 0$ in Ω;

(ii) u is a positive solution of minimal growth in a neighborhood of infinity in Ω.

Remark 2.3 1. Note that D^*, a neighborhood of infinity in Ω, is not necessarily a domain, but has a finite number of components. We denote the components of D^* by $\{D^*_{(i)}\}_{i=1}^{\ell}$. Thus, for every $1 \leq i \leq \ell$, a nonnegative solution of the equation $Pu = 0$ in D^* is either strictly positive in $D^*_{(i)}$, or identically zero there.

2. It turns out, that P is critical in Ω if and only if P admits a ground state (in the sense of Agmon) with an eigenvalue zero. Moreover, in the critical case, $\mathcal{C}_P(\Omega)$ is a one-dimensional cone. On the other hand, P is subcritical in Ω if and only if there exists a positive continuous supersolution ($Pu \geq 0$) of the equation $Pu = 0$ in Ω which is not a solution (see, for example, [1, page 92]).

The following lemma is standard and follows from the generalized maximum principle (see for example, [14])

Lemma 2.4 *Let P be an elliptic operator which is defined on a domain $\Omega \subseteq X$. Suppose that $\mathcal{C}_P(\Omega) \neq \emptyset$. Let $D \subset\subset \Omega$ be a smooth domain. Then*

(i) for every $f \in C(\partial D)$, there exists a unique function u_f which solves the Dirichlet problem

$$\begin{cases} Pu = 0 & \text{in } D, \\ u = f & \text{on } \partial D; \end{cases} \tag{2.4}$$

(ii) if f is a nonzero, nonnegative, continuous function on ∂D, then u_f is strictly positive in D;

(iii) if $f, g \in C(\partial D)$ such that $f \leq g$ on ∂D, then $u_f \leq u_g$ on D;

(iii) if $u \in \mathcal{C}_P(\Omega)$ and $|f| \leq Cu$ on ∂D, then $|u_f| \leq Cu$ on D;

(iv) for every $f \in C(\partial D)$, denote by T_D the map $f \mapsto u_f$. Then T_D is a bounded, linear operator from $C(\partial D)$ into $C(\overline{D})$.

We also need to solve the Dirichlet problem in a neighborhood of infinity.

Lemma 2.5 *Let P be an elliptic operator which is defined on X. Suppose there exists a neighborhood of infinity D_1^* in X such that $\mathcal{C}_P(D_1^*) \neq \emptyset$. Let $D_1 \subset\subset D \subset\subset X$ be a smooth domain. Then for every continuous, nonzero, nonnegative function f on ∂D, there exists a unique, nonzero, nonnegative function u_f^* which satisfies the following properties:*

(i) u_f^* is continuous in $\overline{D^*}$ and $u_f^* = f$ on ∂D;

(ii) u_f^* is a solution of the equation $Pu = 0$ in D^*;

(iii) if v is any other nonnegative function which satisfies (i)–(ii), then $u_f^* \leq v$ on D^*;

(iv) if $u \in \mathcal{C}_P(D_1^*)$ and $f \leq Cu$ on ∂D, then $u_f \leq Cu$ on D^*.

Moreover, u_f^* is strictly positive in the component $D_{(i)}^*$ of D^* provided that f is nonzero on $\partial D_{(i)}^* \cap \partial D$. In particular, if f is a strictly positive continuous function on ∂D, then u_f^* is a positive solution of minimal growth in a neighborhood of infinity in X.

Proof: Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of smooth, relatively compact domains such that $\overline{X_k} \subset X_{k+1}$ and $\cup_{k=1}^{\infty} X_k = X$. Denote by $D_{(i)}^*$, $i = 1, \ldots, \ell$, the components of D^*. We may assume that $\partial D \subset X_1$. Let $u \in \mathcal{C}_P(D_1^*)$, and let C be a positive constant such that $f \leq Cu$ on ∂D.

Set $k \geq 1$, and let $u_{f,k}$ be the solution of the following Dirichlet boundary value problem.

$$\begin{cases} Pu = 0 & \text{in } D^* \bigcap X_k, \\ u = f & \text{on } \partial D, \\ u = 0 & \text{on } \partial(D^* \bigcap X_k) \setminus \partial D. \end{cases} \tag{2.5}$$

Note that we actually solve the Dirichlet problem in each component $D_{(i)}^* \bigcap X_k$. Since $\mathcal{C}_P(D_1^*) \neq \emptyset$, it follows from Lemma 2.4 that the solution $u_{f,k}$ is well defined and is strictly positive in the component $D_{(i)}^* \bigcap X_k$, provided that f is nonzero on $\partial D_{(i)}^* \cap \partial D$. Moreover, by the generalized maximum principle, $\{u_{f,k}\}_{k=1}^{\infty}$ is a nondecreasing sequence which is bounded by Cu. Therefore, this sequence converges to a nonzero, nonnegative solution

u_f^*. Using standard arguments in the theory of elliptic equations, one easily checks that u_f^* satisfies all the other conditions of the lemma. $\square$For every $f \in C(\partial D), f \geq 0$, denote by T the mapping $f \mapsto u_f^*$. Then T can be extended uniquely as a linear operator T_{D^*} from $C(\partial D)$ into $C(\overline{D^*})$.

Remark 2.6 Using the Martin boundary theory, one can show that T_D and T_{D^*} are integral operators, and that their integral kernels are the corresponding Poisson kernels, which are positive and continuous in $D \times \partial D$ and $D^* \times \partial D^*$, respectively. Many properties of the shuttle operator may be deduced from these facts, but we prefer to give direct and more basic proofs of these properties.

Let Ω be an open set and let $K \subset\subset \Omega$ be a compact set. The restriction map $f \mapsto f|_K$ from $C(\overline{\Omega})$ into C(K) is denoted by $R_{(K,\Omega)}$. If $\overline{\Omega}$ is compact, then clearly, $R_{(K,\Omega)}$ is a bounded linear operator.

3 The shuttle operator

In this section, we define the shuttle operator and study some of its properties. We conclude the section with one application, Corollary 3.10, which gives certain sufficient conditions for subcriticality and criticality for diffusion operators.

Let P be an elliptic operator of the form (2.1) which is defined on X. We always assume that the following assumption holds:

(A) There exist four smooth, relatively compact subdomains X_k, $0 \leq k \leq 3$, such that $\overline{X_k} \subset X_{k+1}$, $k = 0, 1, 2$, and such that $\mathcal{C}_P(X_3) \neq \emptyset$ and $\mathcal{C}_P(X_0^*) \neq \emptyset$.

Remark 3.1 1. If assumption **(A)** is not satisfied, then we shall say that the spectral radius of the shuttle operator is infinity. In this case, it is clear that P is supercritical in X.

2. Assumption **(A)** does not imply that $\mathcal{C}_P(X) \neq \emptyset$. For example, take a smooth function V with compact support in $\mathbb{R}^n, n > 2$, such that $-\Delta + V(x)$ is critical in $\mathbb{R}^n$. Then for $R \in \mathbb{R}^n$ with $|R|$ sufficiently large, the operator $P_R = -\Delta + V(x) + V(x - R)$ is supercritical in $\mathbb{R}^n$, and therefore also in a sufficiently large ball. However, P_R satisfies assumption **(A)** in $\mathbb{R}^n$ (see Section 5 and [13]). On the other hand, in the selfadjoint case, if assumption **(A)** is satisfied, then the L^2-essential spectrum of P is nonnegative. Recall

that in the selfadjoint case, $\mathcal{C}_P(X) \neq \emptyset$ if and only if the L^2-spectrum of P is nonnegative.

Fix a sequence $\{X_k\}_{k=0}^{\infty}$, of smooth, relatively compact domains (with $X_k, 0 \leq k \leq 3$ as in assumption **(A)**) such that $\overline{X_k} \subset X_{k+1}$, and $\cup_{k=0}^{\infty} X_k = X$. Consider the operators, K_1 and K_2, defined by

$$K_1 = R_{(\partial X_2, X_1^*)} T_{X_1^*}, \qquad K_2 = R_{(\partial X_1, X_2)} T_{X_2}. \tag{3.1}$$

The shuttle operator $S : C(\partial X_1) \longrightarrow C(\partial X_1)$ is is defined as follows:

$$S = K_2 K_1 = R_{(\partial X_1, X_2)} T_{X_2} R_{(\partial X_2, X_1^*)} T_{X_1^*}. \tag{3.2}$$

Thus, for every $f \in C(\partial X_1)$, we solve the Dirichlet problem in the neighborhood of infinity, X_1^*, and use the restriction of the solution u_{f^*} to ∂X_2 as boundary data for the Dirichlet problem in X_2. Then, Sf is the restriction to ∂X_1 of the solution of the Dirichlet problem in X_2 with boundary data u_f^*. We denote the spectral radius of the operator S by $r(S)$.

Remark 3.2 We could change the order of the above procedure and solve first the Dirichlet problem on X_2 and then use the restriction of the solution to solve the Dirichlet problem in the neighborhood of infinity X_1^*. Thus, we may define a shuttle operator $S_1 : C(\partial X_2) \to C(\partial X_2)$ by $S_1 = K_1 K_2$. Since $r(K_1 K_2) = r(K_2 K_1)$, our main result (to be proved in Section 4) also holds true for S_1 and can be proved similarly. Note that S_1 has the advantage that already in the first step of the procedure, a nonzero nonnegative function f is mapped by K_2 to a strictly positive function. Therefore, by Lemma 2.5, the function $S_1 f$ is the restriction of a positive solution of minimal growth at infinity. We choose (3.2) as the definition of the shuttle operator in order to be consistent with the definition in [16].

Remark 3.3 The procedure in the definition of the shuttle operator may remind the reader the Schwartz alternating method (see, for example, [4]). This classical iterative method may be applied only in cases where the Dirichlet problem is solvable in a (bounded) domain X which is a union of two overlapping subdomains. On the other hand, the shuttle operator is well defined even in cases where the Dirichlet problem is not solvable in the domain X. See Remark 3.1.

Lemma 3.4 *(i) Let $w_0 \in \mathcal{C}_P(X_0^*)$. Denote $m_1 = \inf_{\partial X_1} w_0(x)$ and $M_2 = \sup_{\partial X_2} w_0(x)$. Let $K_1 = R_{(\partial X_2, X_1^*)} T_{X_1^*}$ and let $f \in C(\partial X_1)$. Then*

$$\|K_1 f\|_{C(\partial X_2)} \leq \frac{M_2}{m_1} \|f\|_{C(\partial X_1)} \, . \tag{3.3}$$

Moreover, K_1 is a compact operator from $C(\partial X_1)$ into $C(\partial X_2)$.

(ii) Let $w_3 \in \mathcal{C}_P(X_3)$. Denote $m_2 = \inf_{\partial X_2} w_3(x)$, $M_1 = \sup_{\partial X_1} w_3(x)$. Let $K_2 = R_{(\partial X_1, X_2)} T_{X_2}$, and let $f \in C(\partial X_2)$. Then

$$\|K_2 f\|_{C(\partial X_1)} \leq \frac{M_1}{m_2} \|f\|_{C(\partial X_2)}. \tag{3.4}$$

Moreover, K_2 is a compact operator from $C(\partial X_2)$ into $C(\partial X_1)$.

Proof: Estimate (3.3) follows directly from Lemma 2.5, while estimate (3.4) follows from Lemma 2.4. The compactness of K_1 and K_2 follows from (3.3) and (3.4) respectively using Harnack's inequality and standard Schauder interior estimates. $\square$

Corollary 3.5 *Suppose that $P\mathbf{1} = 0$ in X. Then $\|S\| \leq 1$.*

Proof: Since $S = K_2 K_1$, the proof follows directly from estimates (3.3) and (3.4), with $w_0 = w_3 = \mathbf{1}$. $\square$

Corollary 3.6 *Suppose that $\mathcal{C}_P(X) \neq \emptyset$. Then $r(S) \leq 1$.*

Proof: Let $h \in \mathcal{C}_P(X)$. Consider the elliptic operator P^h (the h-transform of P) defined by

$$P^h u = \frac{1}{h} P(hu). \tag{3.5}$$

Define the multiplication operator $Hf(x) = h(x)f(x)$ on $C(\partial X_1)$. It is easy to check that $P^h \mathbf{1} = 0$ and that the shuttle operator S^h for the operator P^h is given by

$$S^h f = H^{-1} S H f = \frac{S(hf)}{h}. \tag{3.6}$$

By Corollary 3.5, $\|S^h\| \leq 1$. Therefore, $r(S) = r(H^{-1}SH) = r(S^h) \leq \|S^h\| \leq 1$. $\square$

Definition 3.7 Let $\mathcal{B}$ be an ordered Banach space and let $\mathcal{K}$ be its positive cone.

(i) An operator L on $\mathcal{B}$ is said to be *positive* if $L\mathcal{K} \subseteq \mathcal{K}$.

(ii) Let $u_0 \in \mathcal{K} \setminus \{0\}$. An operator L is called *u_0-positive* if for any $u \in \mathcal{K} \setminus \{0\}$, there exist $\gamma, \delta > 0$ and $n \in \mathbb{N}$ such that

$$\gamma u_0 \leq L^n u \leq \delta u_0. \tag{3.7}$$

(iii) The cone $\mathcal{K}$ is *solid* if it contains an interior point. $\mathcal{K}$ is a *reproducing* cone if every $u \in \mathcal{B}$ can be represented as $u = v - w$, where $v, w \in \mathcal{K}$.

We consider in the Banach space $C(\partial X_1)$, the cone $\mathcal{K}$ of all nonnegative functions. This cone is a solid (and hence reproducing) cone.

Theorem 3.8 *(i) The shuttle operator S is a linear compact operator; it is u_0-positive with respect to the function $u_0 = \mathbf{1}$.*

(ii) The shuttle operator has a simple positive (principal) eigenvalue $\nu_0 = \|S\| = r(S)$ with a strictly positive (principal) eigenfunction v_0.

(iii) Up to a multiplicative constant, v_0 is the unique nonnegative eigenfunction of the operator S.

Proof: *(i)* The compactness of S follows from Lemma 3.4. Let $f \in C(\partial X_1)$ be a nonzero, nonnegative function. By Lemma 2.5, $K_1 f$ is positive at least on one component of ∂X_2. Therefore, Lemma 2.4 implies that Sf is a strictly positive continuous function on the compact set ∂X_1, and therefore the function Sf is bounded and bounded away from zero. Set $u_0 = \mathbf{1}$. Then f satisfies (3.7) with $L = S, n = 1, \gamma = \inf_{\partial X_1} Sf$, and $\delta = \sup_{\partial X_1} Sf$. Hence, S is u_0-positive with respect to the function $\mathbf{1}$.

(ii) It follows from Krasnosel'skii's theorem [7, Chapter 2] that S admits, up to a constant, a unique nonnegative continuous eigenfunction v_0. Moreover, the corresponding eigenvalue ν_0 is positive, simple and satisfies

$$\nu_0 = r(S) = \|S\|. \tag{3.8}$$

By part *(i)*, the operator S is u_0-positive with respect to $\mathbf{1}$. Therefore, v_0 is strictly positive.

(iii) Follows directly from Krasnosel'skii's theorem. □

Lemma 3.9 *Let $u \in \mathcal{C}_P(X)$. Then $Su \leq u$. Moreover, $Su = u$ if and only if u is a ground state. In particular, if P is critical, then $r(S) = 1$.*

Proof: By part *(iii)* of Lemma 2.5, we have $K_1 u \le u$. Therefore, part *(iii)* of Lemma 2.4 implies that $Su = K_2 K_1 u \le u$.

Suppose now that u is a ground state. Then u is a positive solution of minimal growth in a neighborhood of infinity. Thus, $u \le K_1 u$, and therefore, by the first part of the proof, $K_1 u = u$. The uniqueness of the Dirichlet problem in X_2 (Lemma 2.4) implies that $Su = u$. Moreover, by Theorem 3.8, we obtain $r(S) = 1$.

Suppose that $u \in \mathcal{C}_P(X)$, and u is not a ground state. In particular, u is not a positive solution of minimal growth at infinity. On the other hand, by Lemma 2.5, the function $T_{X_1^*} u$ is a positive solution of minimal growth at infinity and therefore $u - T_{X_1^*} u$ is a nonzero nonnegative solution in X_1^*. Using this and the uniqueness of the Dirichlet problem in X_2, it follows that $u - T_{X_2} K_1 u$ is a positive solution in X_2. In particular, $Su < u$ on ∂X_1. $\square$

As an application, we give a simple proof of the following corollary, which was proved in [14, theorems 6.1.1 and 6.1.2] using the Liapunov method.

Corollary 3.10 *Suppose that $P\mathbf{1} = 0$ in X.*

(i) Assume that there exist a bounded supersolution $u \in C(\overline{X_1^*})$ and $x_0 \in \partial X_2$ such that

$$u(x_0) < m_1 = \inf_{\partial X_1} u(x). \tag{3.9}$$

Then P is subcritical in X.

(ii) Assume that there exists a supersolution $u \in C(\overline{X_1^*})$ such that

$$\lim_{x \to \infty} u(x) = \infty. \tag{3.10}$$

Then P is critical in X.

Proof: *(i)* Without loss of generality, we may assume that $u > 0$ in X_1^*. Consider the positive solution $v(x) = m_1$. By Lemma 2.5, $K_1 v(x) \le v(x_0)$. Moreover, by the maximum principle and our assumption, $K_1 v(x_0) \le u(x_0) < m_1 = v(x_0)$. Therefore, by Lemma 2.4, $Sv < v$. Lemma 3.9 implies now that P is subcritical in X.

(ii) As in part *(i)*, we may assume that $u > 0$. Fix $x_1 \in X_1^* \bigcap X_2$ and denote $m_2 = \inf_{\partial X_2} u(x)$. Assume, contrary to the statement in *(ii)*, that P is subcritical. Then $\mathbf{1}(x) - T_{X_1^*}\mathbf{1}(x)$ is a positive solution in X_1^* and hence,

$$0 < \mathbf{1}(x) - T_{X_1^*}\mathbf{1}(x) \le 1 \le m_2^{-1} u(x) \tag{3.11}$$

on ∂X_2. On the other hand, $\mathbf{1}(x) - T_{X_1^*}\mathbf{1}(x) = 0$ on ∂X_1. Therefore, by the generalized maximum principle,

$$0 < \mathbf{1}(x) - T_{X_1^*}\mathbf{1}(x) \le m_2^{-1} u(x) \tag{3.12}$$

for every $x \in X_1^* \bigcap X_2$. In particular, we have

$$0 < 1 - T_{X_1^*}\mathbf{1}(x_1) \leq m_2^{-1}u(x_1). \tag{3.13}$$

But X_2 can be chosen arbitrarily large, and therefore, m_2 is also arbitrarily large, which contradicts (3.13). □

4 Positivity and the spectral radius

In this section we prove our main result.

Theorem 4.1 *Let P satisfy assumption* **(A)** *in X, and let S denote the shuttle operator for P in X. The operator P is subcritical, critical, or supercritical in X according to whether $r(S) < 1$, $r(S) = 1$, or $r(S) > 1$.*

Proof: It is enough to prove that $\mathcal{C}_P(X) \neq \emptyset$ if and only if $r(S) \leq 1$, and that $r(S) = 1$ if and only if P is critical.

If $\mathcal{C}_P(X) \neq \emptyset$, then by Corollary 3.6, $r(S) \leq 1$. On the other hand, by Lemma 3.9, if P is critical, then $r(S) = 1$.

Suppose now that $r(S) \leq 1$. Then there exists a principal eigenvalue $\nu_0 = r(S) \leq 1$, and a positive eigenfunction v_0 such that $Sv_0 = \nu_0 v_0 \leq v_0$. Let $v_1 = T_{X_1^*}v_0$ and $v_2 = T_{X_2}K_1 v_0$. Then v_1 and v_2 are positive functions. Recall that $v_2 = \nu_0 v_0 \leq v_0 = v_1$ on ∂X_1. On the other hand, $v_2 = v_1$ on ∂X_2. By the generalized maximum principle, $v_2 \leq v_1$ in $X_2 \setminus X_1$.

Define a function v on X by

$$v(x) = \begin{cases} v_1(x) & \text{if } x \in X_2^*, \\ v_2(x) & \text{if } x \in \overline{X_2}. \end{cases} \tag{4.1}$$

It is clear that v is a continuous positive function. We claim that v is a positive supersolution. Thus, we need to prove that for every $x \in X$, the function v satisfies the generalized maximum principle in a small ball centered at x. Since v is a positive solution in the open set $X_2^* \bigcup X_2$, we may assume that $x \in \partial X_2$. Therefore, let $B = B(x, r)$ be a ball such that $x \in \partial X_2$ and $B(x, r) \subset X_1^*$. Let w be the solution of the Dirichlet problem in B with boundary data v. Since $v_2 \leq v_1$ on $\partial B \bigcap X_2$, it follows from the generalized maximum principle that

$$w \leq v_1 \quad \text{in } B. \tag{4.2}$$

In particular, $w \leq v_1 = v_2$ on $B \bigcap \partial X_2$. Applying the generalized maximum principle in $B \bigcap X_2$ with w and v_2, we deduce that

$$w \leq v_2 \quad \text{in } B \bigcap X_2. \tag{4.3}$$

From (4.2), (4.3) and the definition of v, we deduce that $w \leq v$ in B. Thus, v is a positive, continuous, supersolution in X and hence, $\mathcal{C}_P(X) \neq \emptyset$ (see Remark 2.3).

Suppose now that $\nu_0 = 1$. Then $v_2 = v_1$ on $\partial(X_1^* \bigcap X_2)$ and therefore, $v_2 = v_1$ in $X_1^* \bigcap X_2$. Hence, $v = v_1$ in X_1^* and $v = v_2$ in X_2. Thus, v is a positive solution in X. But since v_1 is a positive solution of minimal growth in a neighborhood of infinity, it follows that v is a positive solution in X with minimal growth at infinity. Thus, v is a ground state. □

5 Localization of binding revisited

In this section, we use Theorem 4.1 to prove the result known as "localization of binding" (for the motivation and the state of art, see [13] and the references therein). The results here for the Laplacian are not new; nor do they give the asymptotic behavior of the ground state energy in the critical case . On the other hand, our approach here has the advantage that it can be used to prove, for the first time, the localization of binding in the critical case for certain *nonselfadjoint,* periodic operators (see, Theorem 5.3 and Remark 5.4).

Since our main aim in this section is illustrative, we shall assume for simplicity that the potentials have compact supports. Let $f \in C(\mathbb{R}^n)$ and $R \in \mathbb{R}^n$, and define the function $f^R \in C(\mathbb{R}^n)$ by

$$f^R(x) = f(x - R). \tag{5.1}$$

We need the following useful lemma.

Lemma 5.1 *Let $v_0 \in C(\partial X_1)$ be the principal eigenfunction of the shuttle operator S associated with the elliptic operator P in X. Suppose that $\|v_0\| = 1$. Let $D \subset\subset X_2$ be a fixed connected neighborhood of ∂X_1. Then $C \leq v_0(x)$, for every $x \in \partial X_1$, where $C > 0$ is the Harnack constant of the operator P in the domain $D \subset X_2$. In particular, C is a local constant which depends only on D, X_2, and the coefficients of P in X_2.*

Proof: Let ν_0 be the principal eigenvalue of the shuttle operator S. The function $\nu_0 v_0(x)$ is the restriction on ∂X_1 of a positive solution of the equation $Pu = 0$ in X_2. Since $\nu_0 v_0(x_0) = \nu_0$ for some $x_0 \in \partial X_1$, the Harnack inequality implies that $C = Cv_0(x_0) \le v_0(x)$. $\square$

Throughout the following two theorems , $C_i, i \ge 1$, will denote positive constants which do not depend on R. The first theorem concerns the localization of binding in the subcritical case.

Theorem 5.2 *Let $V_i(x),\ i = 1, 2$, be Hölder continuous functions with compact supports such that the operators $P_i = -\Delta + V_i(x),\ i = 1, 2$, are subcritical in $\mathbb{R}^n$, $n \ge 3$. There exists $R_0 > 0$ such that the operator*

$$P_R = -\Delta + V_1(x) + V_2^R(x) \tag{5.2}$$

is subcritical in $\mathbb{R}^n$, for all vectors $R \in \mathbb{R}^n \setminus B(0, R_0)$.

Proof: Without loss of generality, we assume that the supports of V_i are contained in $B(0, 1)$. Let $u_i \in \mathcal{C}_{P_i}(\mathbb{R}^n), u_i(0) = 1, i = 1, 2$. In [13] it was shown that $u_i(x) \sim U_i$ at infinity, where U_i are positive constants.

Let $R \in \mathbb{R}^n,\ |R| > 4$, and set $X_1 = B(0, 2), X_2 = B(0, |R|/2)$. Let $v_{0,R} \in C(\partial X_1)$ and $\nu_{0,R}$ denote the principal eigenfunction and the corresponding principal eigenvalue respectively of the shuttle operator S_R for the operator P_R in $\mathbb{R}^n$. We may assume that $\|v_{0,R}\| = 1$. Also, we denote by $K_{1,R}, K_{2,R}$ the corresponding compact operators such that $S_R = K_{1,R}K_{2,R}$. We need to prove that $\nu_{0,R} < 1$. Note that $G^{\mathbb{R}^n}_{-\Delta+V_2}(x - R, -R)$ is the positive Green function of the operator $-\Delta + V_2^R(x)$ in $\mathbb{R}^n$ with a pole at the origin. On ∂X_1, we have

$$v_{0,R}(x) \le 1 \le C_1 G^{\mathbb{R}^n}_{-\Delta+V_2}(x - R, -R). \tag{5.3}$$

Therefore, by Lemma 2.5, we deduce that for every $x \in \partial X_2$,

$$K_{1,R} v_{0,R}(x) \le C_1 G^{-\Delta+V_2}_{\mathbb{R}^n}(x - R, -R) \le C_2 |R|^{2-n}. \tag{5.4}$$

Using (5.4), we compare $K_{1,R} v_{0,R}(x)$ and $|R|^{2-n} u_1(x)$ on ∂X_2. Since $u_1(x) \sim U_1$ at infinity, we have

$$K_{1,R} v_{0,R}(x) \le C_3 |R|^{2-n} u_1(x), \qquad x \in \partial X_2. \tag{5.5}$$

Therefore, by Lemma 2.4 and Lemma 5.1, we have on ∂X_1,

$$\nu_{0,R} v_{0,R}(x) = S_R v_{0,R}(x) \le C_3 |R|^{2-n} u_1(x) \le C_4 |R|^{2-n} v_{0,R}(x). \tag{5.6}$$

Hence, for $|R|$ large enough, $\nu_{0,R} < 1$ and P_R is subcritical. $\square$

Next, we discuss localization of binding in the critical case. We consider an elliptic operator P on $\mathbb{R}^n, n \geq 3$, with $C^{1,\alpha}(\mathbb{R}^n)$ coefficients. We assume either that P is an elliptic operator in divergence form,

$$P(x, \partial_x) = -\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j), \tag{5.7}$$

or that P is a uniformly elliptic operator of the form,

$$P(x, \partial_x) = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j. \tag{5.8}$$

We assume that the coefficients of P are *periodic of period one*; that is, $a_{ij}(x + e_k) = a_{ij}(x)$, for $i, j, k \in \{1, 2, \ldots, n\}$, where e_k is the unit vector in the x_k direction. It is well known [8], that if P is of the form (5.7) and $n \geq 3$, then P is subcritical and the corresponding Green function satisfies

$$C^{-1}|x - y|^{2-n} \leq G^P_{\mathbb{R}^n}(x, y) \leq C|x - y|^{2-n}, \tag{5.9}$$

for every $x, y \in \mathbb{R}^n$, $x \neq y$, where C is a positive constant.

On the other hand, if P is of the form (5.8), then P is of Fuchsian type (see [12]). From [12] it follows that if $u \in \mathcal{C}_P(\mathbb{R}^n)$, then u is a constant function. In particular, the generalized principal eigenvalue of the periodic operator P is zero (see, [14, and the reference therein]). Since $n \geq 3$, it follows from a recent result of R. Pinsky, that P is subcritical (see [15]). Recall that P is of Fuchsian type. Let $0 < \alpha < \alpha_1 < 1 < \beta_1 < \beta$ be given. It follows from [12, Lemma 6.3] that there exists a positive constant C such that for every $u \in \mathcal{C}_P(B(0, \beta r) \setminus B(0, \alpha r))$, $r > 0$, and $x, y \in B(0, \beta_1 r) \setminus B(0, \alpha_1 r)$,

$$u(x) < Cu(y). \tag{5.10}$$

Let $M(r) = \max\{G^P_{\mathbb{R}^n}(x, 0) \mid |x| = r\}$ and $m(r) = \min\{G^P_{\mathbb{R}^n}(x, 0) \mid |x| = r\}$. It follows from (5.10) that for every $x \in B(0, \beta_1 r) \setminus B(0, \alpha_1 r)$, and $r > 0$,

$$C^{-1}M(r) \leq G^P_{\mathbb{R}^n}(x, 0) \leq Cm(r). \tag{5.11}$$

By the maximum principle,

$$\lim_{x \to \infty} G^P_{\mathbb{R}^n}(x, 0) = 0. \tag{5.12}$$

Note that if P is of the form (5.7), then (5.9) implies that (5.11) and (5.12) hold true also in this case.

Theorem 5.3 *Let P be an elliptic operator on $\mathbb{R}^n, n \geq 3$, of the form (5.7) or (5.8) with coefficients in $C^{1,\alpha}(\mathbb{R}^n)$. Assume that the coefficients of P are periodic with period one. Let $V_i \in C^\alpha(\mathbb{R}^n),\ i = 1, 2$ be functions with compact supports. Assume that the operators $P_i = P + V_i(x),\ i = 1, 2$, are critical in $\mathbb{R}^n,\ n \geq 3$. Then there exists an $R_0 > 0$ such that the operator*

$$P_R = P + V_1(x) + V_2^R(x) \tag{5.13}$$

is supercritical in $\mathbb{R}^n$, for all vectors $R \in \mathbb{Z}^n \setminus B(0, R_0)$.

Proof: Without loss of generality, we assume that the supports of V_i are contained in $B(0,1)$. Let $u_i \in \mathcal{C}_{P_i}(\mathbb{R}^n), u_i(0) = 1, i = 1, 2$, be the ground states of P_i. Then by (5.11) and [11, Lemma 3.6], we have

$$C_2^{-1} M(|x|) \leq C_3^{-1} G_{\mathbb{R}^n}^P(x,0) \leq u_i(x) \leq C_3 G_{\mathbb{R}^n}^P(x,0) \leq C_2 m(|x|), \tag{5.14}$$

for every $x \in B(0,3r) \setminus B(0, r/2)$ and $r > 2$, where $i = 1, 2$.

Let $R \in \mathbb{Z}^n,\ |R| > 4$, and set $X_1 = B(0,2), X_2 = B(0, |R|/2)$. Let $v_{0,R} \in C(\partial X_1)$ and $\nu_{0,R}$ denote the principal eigenfunction and the corresponding principal eigenvalue respectively of the shuttle operator S_R for the operator P_R in $\mathbb{R}^n$. We need to prove that $\nu_{0,R} > 1$. We may assume that $\|v_{0,R}\| = 1$. By Lemma 5.1, we have on ∂X_1,

$$C_1 \leq v_{0,R}(x). \tag{5.15}$$

Combining (5.15) and (5.14), it follows that

$$C_4 m(|R|)^{-1} u_2^R(x) \leq C_1 \leq v_{0,R}(x), \tag{5.16}$$

for every $x \in \partial X_1$. Note that $u_2^R(x)$ is a positive solution of minimal growth in a neighborhood of infinity for the equation $P_R u = 0$ in X. Moreover, it is a positive solution of this equation in X_1^*. Therefore, by the definition of a positive solution of minimal growth in a neighborhood of infinity and (5.14), we deduce that

$$C_5 \leq C_4 m(|R|)^{-1} u_2^R(x) \leq K_{1,R} v_{0,R}(x), \tag{5.17}$$

for every $x \in \partial X_2$. Using (5.17) and (5.14), we compare $K_{1,R} v_{0,R}(x)$ and $m(|R|)^{-1} u_1(x)$ on ∂X_2. We have,

$$C_6 m(|R|)^{-1} u_1(x) \leq C_5 \leq K_{1,R} v_{0,R}(x). \tag{5.18}$$

Therefore, by Lemma 2.4, Lemma 5.1, and the Harnack inequality, we have on ∂X_1,

$$C_7 m(|R|)^{-1} v_{0,R}(x) \leq C_6 m(|R|)^{-1} u_1(x) \leq S_R v_{0,R}(x) = \nu_{0,R} v_{0,R}(x). \quad (5.19)$$

By (5.12), $\lim_{r\to\infty} m(r) = 0$. Thus, for $|R|$ large enough, $\nu_{0,R} > 1$ and P_R is supercritical. $\square$

Remark 5.4 The proofs in the literature for the critical case all rely on the Rayleigh-Ritz variational formula, thus, they can only be applied to self-adjoint operators. The proof here relies on the fact that the periodic operator P is a subcritical, $\mathbb{Z}^n$-invariant operator in $\mathbb{R}^n$ whose Green function decays at infinity. Thus, the method here can be applied in some more general situations (see [13] for the general setting).

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