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Binding of Schrödinger particles through conspiracy of potential wells in $\mathbb{R}^4$

1 Introduction

In this paper we discuss the ground state energy $E(R)$ of the Schrödinger operator

$$P_R = -\Delta + V_1(x) + V_2(x-R) \ \ x, R \in \mathbb{R}^n, \tag{1.1}$$

where the potentials V_i are small perturbations of the Laplacian in $\mathbb{R}^n, n \geq 3$ and the operators $P_i = -\Delta + V_i(x)$ are nonnegative. We review briefly the case $n = 3$ which was studied in [3, 7, 10, 15, 16, 17] and the case $n \geq 5$ (which was studied in [10]). Our main aim here is to treat for the first time the case $n = 4$.

Recall [1] that a Schrödinger operator P defined in a domain $\Omega \subseteq \mathbb{R}^n$ is nonnegative if and only if the convex cone $\mathcal{C}_P(\Omega)$ of all positive solutions of the equation $Pu = 0$ in Ω is not empty.

Definition 1.1 *Let P be a second order elliptic linear operator defined in a domain $\Omega \subseteq \mathbb{R}^n$. A function u is said to be a ground state (in the sense of Agmon) of the operator P in the domain Ω with an eigenvalue zero if the following two conditions hold:*

(i) u is a positive solution of the equation $Pu = 0$ in Ω.

(ii) If v is any positive solution of the equation $Pu = 0$ in some subdomain $\Omega_1 = \Omega \setminus K_1, K_1 \subset\subset \Omega$ then there exist a positive constant C and a

subdomain $\Omega_2 = \Omega \setminus K_2, K_2 \subset\subset \Omega$ *such that* $u \le Cv$ *in* Ω_2.

In other words, a ground state is a positive global solution of the equation $Pu = 0$ which has minimal growth at a neighborhood of infinity in Ω.

It is well known that if the operator P admits a positive solution in Ω then either P admits *a minimal (Dirichlet) positive Green function* $G_P^\Omega(x, y)$ and in this case we say that P is *subcritical* in Ω, or P admits a ground state in the sense of Agmon with an eigenvalue zero and in this case P is said to be *critical* in Ω. The operator P is *supercritical* in Ω if $\mathcal{C}_P(\Omega) = \emptyset$. (For more details see [8, 9, 10].)

The motivation for studying the ground state energy $E(R)$ comes from a remarkable phenomenon known as the Efimov effect for a three-body Schrödinger operator. Such an operator H takes (in the center-of-mass frame) the following form

$$H = H_0 + \sum_{1 \le j < k \le 3} V_{jk}(r_j - r_k),$$

where the operator H_0 (the free Hamiltonian) and the operator H act on the space $L^2(\mathbb{R}^6)$, V_{jk} are short-range potentials in $\mathbb{R}^3$ and r_j denotes the position vector of the j-th particle. Suppose that all the three two-body subsystems admit ground states in the sense of Agmon with eigenvalues zero (so, V_{jk} are *critical potentials*) then by the Efimov effect the three-particle system has an *infinite* number of negative L^2-eigenvalues accumulating to zero. Note that this effect holds true even if each pair potential V_{jk} is a function with compact support.

A variational method for proving the Efimov effect was given in [7] by Yu. N. Ovchinnikov and I. M. Sigal and was generalized by H. Tamura in [16, 17] (see also [3, 15] and related results in [2, 18] and in the references therein). The proof relies on the following

(*i*) It is well known that the Schrödinger operator $-\Delta+C(1+|x|)^{-2}$ on $\mathbb{R}^n$ has an infinite number of negative eigenvalues provided that $C < -1/4$.

(*ii*) Suppose that the functions $V_i, i = 1, 2$ are short range critical potentials in $\mathbb{R}^3$. Then the ground state energy $E(R)$ of the operator P_R in $\mathbb{R}^3$ satisfies

$$\lim_{R\to\infty} |R|^2 E(R) = -\beta < -1/4. \tag{1.2}$$

(*iii*) The number of the negative eigenvalues of the three-body Hamiltonian H is not less than the number of the negative eigenvalues of a certain Schrödinger operator on $\mathbb{R}^3$ with a potential, the leading order term of which is $E(R)$ (hence, the effective interaction is long-range).

So, the study of the ground state energy $E(R)$ of such a type of perturbation is a natural problem in the spectral theory of Schrödinger operators. (See also [5, and the references therein], where some spectral and scattering properties of P_R were studied.) It is not known whether the Efimov effect holds true in higher dimensions or for many-body Schrödinger operators.

In [10] we show that (*ii*) is a very special three-dimensional fact (and therefore, the Efimov effect is conceivably also a special three-dimensional effect). More precisely, we have ([10], Theorem 4.3 and compare it with the remarks in pages 84 and 87 in [3])

Theorem 1.2 *Let $n \geq 5$. Suppose that the potentials $V_i,\ i = 1, 2$ are two Hölder continuous functions which satisfy the following decay assumption*

$$|V_i(x)| \leq \frac{C}{\langle x \rangle^{\beta}},\ x \in \mathbb{R}^n, \tag{1.3}$$

where $\langle x \rangle = 1+|x|$, $\beta > n-2$ and C some positive constant. Assume that the operators $P_i = -\Delta + V_i(x),\ i = 1, 2,$ are critical in $\mathbb{R}^n$. Then:

(i) There exists $R_0 > 0$ such that the operator P_R is supercritical for all vectors $R \in \mathbb{R}^n \backslash B(0, R_0)$.

(ii) There exists a positive constant C such that the ground state energy $E(R)$ of the operator P_R satisfies

$$-C|R|^{2-n} \leq E(R) \leq -C^{-1}|R|^{2-n} \quad \textit{for all } |R| \geq R_0. \tag{1.4}$$

Let v_i be the ground states of the operators $P_i,\ i = 1, 2$. The lower bound in (1.4) is proved using the function $v_R(x) = v_1(x) + v_2(x - R)$ as a test function in the Protter-Weinberger variational principle (see [6]):

$$E(R) = \sup_{u>0} \inf_{x\in\mathbb{R}^n} \left\{ \frac{P_R u(x)}{u(x)} \right\}. \tag{1.5}$$

We remark here that the lower bound in (1.4) is valid also for $n = 3, 4$. The upper bound in (1.4) is proved using the (same) function v_R as a test function in the Rayleigh-Ritz quotient. Note that the ground states v_i behave at infinity like $|x|^{(2-n)}$. Hence, $v_i \in L^2(\mathbb{R}^n)$ if and only if $n > 4$. Therefore, the cases $n = 3, 4$ need a refined argument. Our method for these special cases is based on the very delicate asymptotic behavior near $\lambda = 0$ of the ground states of the critical operators $P_{i,\lambda} = -\Delta - \lambda + \alpha_i(\lambda)v_i(x)$, where $\lambda < 0,\ |\lambda|$ is sufficiently small and $\alpha_i(\lambda) \searrow 1$ as $\lambda \nearrow 0$. We shall present here the proof for the case $n = 4$ which was never studied before. Nevertheless, using our approach one also obtains an alternative (and simpler) proof for the case $n = 3$.

Assume now that the operators P_i are nonnegative in $\mathbb{R}^n, n \geq 3$ but do not admit (Agmon) ground states (so, P_i are *subcritical* in $\mathbb{R}^n$). The space of functions

$$K_n^\infty = \left\{ V \in C^\alpha(\mathbb{R}^n) \middle| \lim_{M\to\infty} \sup_{x\in\mathbb{R}^n} \int_{|y|>M} \frac{|V(y)|}{|x-y|^{n-2}}\, dy = 0 \right\} \tag{1.6}$$

is called *the Kato class at infinity* (see [10, and the references therein]).

In [10] the following theorem was proved (see also [3, 7, 11, 12, 15] where potentials with compact supports were considered).

Theorem 1.3 *Let $V_i(x) \in K_n^\infty,\ i = 1, 2,$ be two functions such that the operators $P_i = -\Delta + V_i(x),\ i = 1, 2,$ are subcritical in $\mathbb{R}^n$, $n \geq 3$. There exists $R_0 > 0$ such that the operator*

$$P_R = -\Delta + V_1(x) + V_2(x - R) = -\Delta + V_1(x) + V_2^R(x)$$

is subcritical for all vectors $R \in \mathbb{R}^n \setminus B(0, R_0)$.

The results described in theorems 1.2 and 1.3 are known as “localization of binding”. These results demonstrate again that subcriticality is a stable property while criticality is not (for further examples see [8, 9]).

In [10] the “localization of binding” was proved in the more general case of a similar type of perturbation of a subcritical operator P defined on a manifold X such that P is an equivariant operator under a given group action on X. More precisely, Theorem 1.2 (the critical case) was extended to the case of a symmetric subcritical equivariant operator P such that the corresponding critical operators $P_i = P + V_i(x), V_i \in C_0^\infty(X)$ admit L^2 ground states while a generalization of Theorem 1.3 (the subcritical case) was proved for general (not necessarily symmetric) equivariant subcritical elliptic operator P such that P_i are subcritical.

Remark 1.4 Recently, using the shuttle operator, alternative proofs of the "localization of binding" for both the critical and the subcritical case were given in [11]. The advantage of this approach is that it applies for the first time to the nonsymmetric critical case.

Remark 1.5 Throughout the paper, we usually do not indicate explicitly the quantities that a given constant C depends on. Moreover, in a given context, the same letter C may be used to denote different constants depending on the same set of arguments.

2 The critical case in $\mathbb{R}^4$

In this section we assume that the operators $P_i = -\Delta + V_i(x), i = 1, 2$, are *critical* in $\mathbb{R}^4$. For simplicity we assume also that the potentials V_i are Hölder continuous functions with compact supports. We consider the family of operators

$$P_R = -\Delta + V_1(x) + V_2(x - R) = -\Delta + V_1(x) + V_2^R(x).$$

Our main aim is to study the ground state energy $E(R)$ of the operator P_R. We shall prove in Theorem 2.3 that the operator P_R is supercritical in $\mathbb{R}^n$ for $|R|$ large enough and that $E(R)$ is bounded above by $-C|R|^{-2}(\log |R|)^{-1}$ as $|R| \to \infty$ (compare with the remarks in pages 84 and 87 in [3]).

Given the differential operator P_R one can define the maximal realization H_R of P_R in $L^2(\mathbb{R}^n)$ as follows (see [1]):

$$\begin{aligned} D(H_R) &= \left\{u | u \in L^2(\mathbb{R}^n) \cap H^1_{loc}(\mathbb{R}^n), V_R u \in L^1_{loc}(\mathbb{R}^n), P_R u \in L^2(\mathbb{R}^n)\right\} \\ H_R u &= P_R u \ \text{ for } u \in D(H_R), \end{aligned} \tag{2.1}$$

where P_R acts on u in the distribution sense. The operator H_R is the *only* self-adjoint realization of P in $L^2(\mathbb{R}^n)$. The ground state energy $E(R)$ is equal to the bottom of the spectrum of H_R and therefore,

$$E(R) = \inf_{u \in D(H_R)} \left\{ \frac{(P_R u, u)}{\|u\|^2_{L^2(\mathbb{R}^n)}} \right\}. \tag{2.2}$$

Lemma 2.1 *Assume that the operators* $P_i = -\Delta + V_i(x), i = 1, 2,$ *are critical in* $\mathbb{R}^4$. *For* $\lambda \leq 0$ *define the function*

$$\alpha_i(\lambda) = \sup\left\{s > 0 | \mathcal{C}_{-\Delta - \lambda + sV_i(x)}(\mathbb{R}^4) \neq \emptyset\right\}, \quad i = 1, 2. \tag{2.3}$$

Then:

(i) The functions $\alpha_i(\lambda)$ *are decreasing continuous strictly concave functions in* $(-\infty, 0]$ *and* $\alpha_i(0) = 1$.

(ii) There exists a constant $c < 0$ *such that*

$$\alpha_i(\lambda) \sim 1 + c|\lambda| \log|\lambda| \tag{2.4}$$

as $\lambda \nearrow 0$.

(iii) The operators $P_{i,\lambda} = -\Delta - \lambda + \alpha_i(\lambda) V_i(x), \ \ i = 1, 2$ *are critical in* $\mathbb{R}^4$ *for all* $\lambda \leq 0$.

(iv) Denote by $v_{i,\lambda}(x)$ the ground states of the operator $P_{i,\lambda}$ in $\mathbb{R}^4$ such that $v_{i,\lambda}(0) = 1$. There exists $C > 0$ such that for all $\lambda < 0$ and $|x| > 1$

$$C^{-1}\frac{|\lambda|^{1/2}}{|x|}K_1(|\lambda|^{1/2}|x|) \le v_{i,\lambda}(x) \le C\frac{|\lambda|^{1/2}}{|x|}K_1(|\lambda|^{1/2}|x|), \qquad (2.5)$$

where K_ν is the modified Bessel function of order ν.

Proof: Parts (i) and (iii) follow from Theorem 3.8 and Corollary 3.10 in [9], while (ii) follows from Section 6 of [4].

(iv) It is well known that the Green function of the operator $-\Delta - \lambda$ in $\mathbb{R}^n$ is given by

$$G^{\mathbb{R}^n}_{-\Delta-\lambda}(x,y) = (2\pi)^{-n/2}\left(\frac{|\lambda|^{1/2}}{|x-y|}\right)^{\nu} K_\nu(|\lambda|^{1/2}|x-y|), \qquad (2.6)$$

where $\nu = (n-2)/2$ and $\lambda < 0$. Therefore, (2.5) follows from the generalized maximum principle and the Harnack inequality (see also Lemma 3.6 in [8]). $\square$

The key estimates for studying the asymptotic behavior of $E(R)$ are given in the following lemma.

Lemma 2.2 *Let $V_i(x) \in C^\alpha(\mathbb{R}^4)$, $i = 1, 2$ be two potentials with compact supports. Suppose that the operators $P_i = -\Delta + V_i(x), i = 1, 2,$ are critical in $\mathbb{R}^4$. Consider the one-parameter family of operators $P_{i,\lambda} = -\Delta - \lambda + \alpha_i(\lambda)V_i(x)$, $\lambda < 0$, where $\alpha_i(\lambda)$ is defined by (2.3). Denote by $v_{i,\lambda}(x)$ the ground states of $P_{i,\lambda}$ with $v_{i,\lambda}(0) = 1$. For every $R \in \mathbb{R}^4 \setminus B(0, 2)$ let $\lambda < 0$,*

$$|\lambda| = |\lambda(R)| = (|R| \log |R|)^{-2}. \qquad (2.7)$$

Then there exist positive constants C and R_0 such that for all $R \in \mathbb{R}^4 \setminus B(0, R_0)$ the following inequalities hold:

(i)

$$\int_{\mathbb{R}^4} v_{2,\lambda}^R(x)V_1(x)v_{1,\lambda}(x)dx \le -C^{-1}\langle R\rangle^{-2}, \tag{2.8}$$

(ii)

$$\int_{\mathbb{R}^4} (v_{1,\lambda}(x))^2|V_2^R(x)|dx \le C\langle R\rangle^{-4}, \tag{2.9}$$

(iii)

$$C^{-1}\log|R| \le \int_{\mathbb{R}^4} (v_{1,\lambda}(x))^2 dx \le C\log|R|, \tag{2.10}$$

(iv)

$$C^{-1}\log\log|R| \le \int_{\mathbb{R}^4} v_{1,\lambda}(x)v_{2,\lambda}^R(x)dx \le C\log\log|R|. \tag{2.11}$$

Proof: (i) The function $v_{2,\lambda}$ satisfies the integral equation

$$-\alpha_2(\lambda)\int_{\mathbb{R}^4} G^{\mathbb{R}^4}_{-\Delta-\lambda}(x-R,y)V_2(y)v_{2,\lambda}(y)dy = v_{2,\lambda}^R(x). \tag{2.12}$$

The asymptotic behavior of the modified Bessel function of order $\nu > 0$ near $z = 0$ is given by $K_\nu(z) \sim 1/2\Gamma(\nu)(1/2z)^{-\nu}$. Hence, for $|\lambda| = (|R|\log|R|)^{-2}$ Equation (2.6) implies that

$$G^{\mathbb{R}^4}_{-\Delta-\lambda}(R,0) \sim C|R|^{-2}, \tag{2.13}$$

and

$$\lim_{|R|\to\infty} \frac{G^{\mathbb{R}^4}_{-\Delta-\lambda}(x-R,y)}{G^{\mathbb{R}^4}_{-\Delta-\lambda}(x-R,0)} = 1. \tag{2.14}$$

Dividing (2.12) by $G^{\mathbb{R}^4}_{-\Delta-\lambda}(x-R,0)$ and using Lemma 2.1 (iv) and (2.14) we obtain for $|\lambda| = (|R|\log|R|)^{-2}$

$$\lim_{|R|\to\infty} \frac{v_{2,\lambda}^R(x)}{G^{\mathbb{R}^4}_{-\Delta-\lambda}(R,x)} =$$

$$-\lim_{|R|\to\infty}\alpha_2(\lambda)\int_{\mathbb{R}^4}\frac{G^{\mathbb{R}^4}_{-\Delta-\lambda}(x-R,y)}{G^{\mathbb{R}^4}_{-\Delta-\lambda}(x-R,0)}V_2(y)v_{2,\lambda}(y)dy = -\int_{\mathbb{R}^4}V_2(y)v_{2,0}(y)dy = C>0. \tag{2.15}$$

Thus,

$$\int_{\mathbb{R}^4} v^R_{2,\lambda}(x)V_1(x)v_{1,\lambda}(x)dx = C\int_{\mathbb{R}^4} G^{\mathbb{R}^4}_{-\Delta-\lambda}(R,x)V_1(x)v_{1,\lambda}(x)dx +$$
$$\int_{\mathbb{R}^4}\left(\frac{v^R_{2,\lambda}(x)}{G^{\mathbb{R}^4}_{-\Delta-\lambda}(R,x)}-C\right)G^{\mathbb{R}^4}_{-\Delta-\lambda}(R,x)V_1(x)v_{1,\lambda}(x)dx =$$
$$-C(\alpha_1(\lambda))^{-1}v_{1,\lambda}(R) +$$
$$\int_{\mathbb{R}^4}\left(\frac{v^R_{2,\lambda}(x)}{G^{\mathbb{R}^4}_{-\Delta-\lambda}(R,x)}-C\right)G^{\mathbb{R}^4}_{-\Delta-\lambda}(R,x)V_1(x)v_{1,\lambda}(x)dx. \tag{2.16}$$

It follows from parts (ii) and (iv) of Lemma 2.1 and (2.13) that there exists $C_1>0$ such that

$$-C(\alpha_1(\lambda))^{-1}v_{1,\lambda}(R) < -C_1\langle R\rangle^{-2}, \quad R\in\mathbb{R}^4. \tag{2.17}$$

Also, by (2.15) we can choose $|R|$ large enough such that

$$\left|\frac{v^R_{2,\lambda}(x)}{G^{\mathbb{R}^4}_{-\Delta-\lambda}(R,x)}-C\right| \tag{2.18}$$

is so small for all x in the compact support of V_1 to guarantee that

$$\int_{\mathbb{R}^4}\left|\frac{v^R_{2,\lambda}(x)}{G^{\mathbb{R}^4}_{-\Delta-\lambda}(R,x)}-C\right|G^{\mathbb{R}^4}_{-\Delta-\lambda}(R,x)|V_1(x)|v_{1,\lambda}(x)dx \le \frac{C_1}{3\langle R\rangle^2}. \tag{2.19}$$

Now, Using estimates (2.17) and (2.19) in (2.16) we obtain

$$\int_{\mathbb{R}^n} v^R_2(x)V_1(x)v_1(x)dx \le -\frac{C_1}{3\langle R\rangle^2}.$$

(ii) Estimate (2.9) follows easily from (2.13) and Lemma 2.1 (iv).

(*iii*) It follows from the Harnack inequality and Lemma 2.1 (iv) that we need only to estimate the integral

$$I = \int_{|x|\geq 1} \frac{|\lambda|}{|x|^2} \left(K_1(|\lambda|^{1/2}|x|)\right)^2 dx = c|\lambda| \int_1^\infty r \left(K_1(|\lambda|^{1/2}r)\right)^2 dr \tag{2.20}$$

Using formula (1.12.3.2) in [13, page 48] we have

$$I = 0.5c|\lambda| \left\{ \left(K_1'(|\lambda|^{1/2})\right)^2 - (1 + 1/|\lambda|) \left(K_1(|\lambda|^{1/2})\right)^2 \right\}. \tag{2.21}$$

Recall that $K_1'(z) = -0.5(K_0(z) + K_2(z))$ and

$$\begin{aligned} K_\nu(z) &\sim \frac{\Gamma(\nu)}{2} \left(\frac{z}{2}\right)^{-\nu} \quad \text{if } \nu > 0, \text{ and} \\ K_0(z) &\sim -\log z \qquad \text{as } z \searrow 0. \end{aligned} \tag{2.22}$$

Therefore, $I \sim -C \log|\lambda|$ which is of order $\log|R|$.

(*iv*) Let

$$f(r) = G^{\mathrm{IR}^4}_{-\Delta-\lambda}(x,0) = (2\pi)^{-2} \frac{|\lambda|^{1/2}}{|x|} K_1(|\lambda|^{1/2}|x|). \tag{2.23}$$

We need to estimate the function $h(R) = (f * f)(R)$ which by the Fourier transform is equivalent of estimating the radial Fourier integral

$$\int_{\mathrm{IR}^4} \frac{e^{-i\xi\cdot R}}{(|\xi|^2 + |\lambda|)^2} d\xi = \frac{C}{|R|} \int_0^\infty \frac{r^2 J_1(Rr)}{(r^2 + |\lambda|)^2} dr, \tag{2.24}$$

where J_1 is the Bessel function of order 1 (see [14, page 142]). Using the integration formula (2.12.4.28) of [13, page 179] we find that

$$h(R) = C K_0(|\lambda|^{1/2}|R|) \sim C \log\log|R|. \tag{2.25}$$

$\square$

Now, we prove the theorem.

Theorem 2.3 *Let $n = 4$. Assume that the potentials V_i, $i = 1, 2$ are two Hölder continuous functions with compact supports. Assume that the operators $P_i = -\Delta + V_i(x)$, $i = 1, 2$, are critical in $\mathbb{R}^4$. Then:*

(i) There exists $R_0 > 0$ such that the operator P_R is supercritical for all vectors $R \in \mathbb{R}^4 \backslash B(0, R_0)$.

(ii) There exists a positive constant C such that the ground state energy $E(R)$ of the operator P_R satisfies

$$-C|R|^{-2} \leq E(R) \leq -C^{-1}|R|^{-2}(\log|R|)^{-1} \quad \text{for all } |R| \geq R_0. \quad (2.26)$$

Proof: The operator P_R is supercritical if and only if $E(R) < 0$. Therefore, in order to prove the theorem it is enough to prove Inequality (2.26).

Let v_i be the ground states of the operators P_i, $i = 1, 2$. The lower bound of (2.26) was proved in [10] for all $n \geq 3$ using the function $v_1(x) + v_2^R(x)$ as a test function in the Protter-Weinberger variational principle (1.5). Hence, it remains to prove only the upper bound of (2.26).

Upper bound: Let $v_{i,\lambda}$ be the ground states of the operators $P_{i,\lambda} = -\Delta - \lambda + \alpha_i(\lambda)v_i(x)$, where $\lambda < 0$, $v_{i,\lambda}(0) = 1$ and the functions α_i are defined by (2.3). Fix $R \in \mathbb{R}^4$ and let $\lambda < 0$,

$$|\lambda| = (|R| \log|R|)^{-2}. \quad (2.27)$$

Consider the function

$$u_{R,\lambda}(x) = v_{1,\lambda}(x) + v_{2,\lambda}^R(x) \quad (2.28)$$

as a test function in the Rayleigh-Ritz quotient for the operator H_R (see (2.2)).

$$P_R u_{R,\lambda}(x) = (-\Delta + V_1(x) + V_2^R(x))u_{R,\lambda}(x)$$

$$
\begin{aligned}
&= (1-\alpha_1(\lambda))V_1(x)v_{1,\lambda}(x) + V_2^R(x)v_{1,\lambda}(x) + \lambda u_{R,\lambda}(x) + \\
&\quad V_1(x)v_{2,\lambda}^R(x) + (1-\alpha_2(\lambda))V_2^R(x)v_{2,\lambda}^R(x). \qquad (2.29)
\end{aligned}
$$

Hence, using (2.2) we have

$$
E(R) \leq \frac{(P_R u_{R,\lambda}, u_{R,\lambda})}{\|u_{R,\lambda}\|^2_{L^2(\mathbb{R}^4)}} = \lambda + \frac{\sum_{i=1}^8 I_i}{\|u_{R,\lambda}\|^2_{L^2(\mathbb{R}^4)}}, \qquad (2.30)
$$

where

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^4} v_{1,\lambda}(x)V_2^R(x)v_{2,\lambda}^R(x)dx, \\
I_2 &= \int_{\mathbb{R}^4} v_{1,\lambda}(x)V_1(x)v_{2,\lambda}^R(x)dx, \\
I_3 &= (1-\alpha_2(\lambda))\int_{\mathbb{R}^4} v_{1,\lambda}(x)V_2^R(x)v_{2,\lambda}^R(x)dx, \\
I_4 &= (1-\alpha_1(\lambda))\int_{\mathbb{R}^4} v_{1,\lambda}(x)V_1(x)v_{2,\lambda}^R(x)dx, \\
I_5 &= (1-\alpha_1(\lambda))\int_{\mathbb{R}^4} V_1(x)\left(v_{1,\lambda}(x)\right)^2 dx, \\
I_6 &= (1-\alpha_2(\lambda))\int_{\mathbb{R}^4} V_2^R(x)\left(v_{2,\lambda}^R(x)\right)^2 dx, \\
I_7 &= \int_{\mathbb{R}^4} V_2^R(x)\left(v_{1,\lambda}(x)\right)^2 dx, \\
I_8 &= \int_{\mathbb{R}^4} V_1(x)\left(v_{2,\lambda}^R(x)\right)^2 dx. \qquad (2.31)
\end{aligned}
$$

Using (2.8) we see that there exists a positive constant C such that

$$
I_1 + I_2 \leq -C^{-1}\langle R\rangle^{-2}. \qquad (2.32)
$$

The asymptotic behavior of $\alpha_i(\lambda)$ near zero (see (2.4)) and (2.5) imply that

$$
|I_3| + |I_4| \leq C\langle R\rangle^{-4}(\log\langle R\rangle)^{-1}. \qquad (2.33)
$$

Also, by (2.4) we see that

$$|I_5| + |I_6| \leq C\langle R\rangle^{-2}(\log\langle R\rangle)^{-1}. \tag{2.34}$$

By (2.9) we have

$$|I_7| + |I_8| \leq C\langle R\rangle^{-4}. \tag{2.35}$$

On the other hand, (2.10) and (2.11) imply that

$$\|u_{R,\lambda}\|^2_{L^2(\mathbb{R}^4)} \geq C\log\langle R\rangle. \tag{2.36}$$

Therefore, using (2.7) and estimates (2.32)–(2.36) in (2.30) we see that

$$E(R) \leq \frac{(P_R u_{R,\lambda}, u_{R,\lambda})}{\|u_{R,\lambda}\|^2_{L^2(\mathbb{R}^4)}} \leq -C|R|^{-2}(\log|R|)^{-1}. \tag{2.37}$$

Note that actually we have shown that

$$\frac{(P_R u_{R,\lambda}, u_{R,\lambda})}{\|u_{R,\lambda}\|^2_{L^2(\mathbb{R}^4)}} \sim -C|R|^{-2}(\log|R|)^{-1}. \tag{2.38}$$

□

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