

MANIFOLDS WITH GROUP ACTIONS AND ELLIPTIC OPERATORS

VLADIMIR YA. LIN AND YEHUDA PINCHOVER

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Abstract

Let P be an invariant linear second order elliptic operator on a connected noncompact manifold X with a given action of a group G . The action is supposed to be compactly generating, that is, $GV = X$ for some compact subset $V \subset X$.

We study classical positive global solutions of the equation $Pu = 0$.

It turns out that the set of all normalized positive solutions which are also eigenfunctions of the given G -action can be realized as a real analytic submanifold Γ_0 in an appropriate topological vector space \mathcal{H} . For a finitely generated group G this space \mathcal{H} is of finite dimension, and in nontrivial cases Γ_0 is the boundary of a strictly convex body in \mathcal{H} .

If G is nilpotent, then a positive solution u can be represented as an integral with respect to some uniquely defined positive Borel measure over the set Γ_0 .

We also discuss related results for parabolic equations and noncompact manifolds with boundary.

Key words and phrases: Cauchy problem, eigenfunctions of a group action, extreme point, invariant elliptic operator, Liouville property, nilpotent groups, positive solutions.

1. INTRODUCTION

In this paper we consider a triple (X, G, P) , where X is a smooth noncompact connected countable at infinity manifold of dimension $d \geq 1$, G is a group acting on X by diffeomorphisms, and P is a linear second order elliptic operator which is defined on X and satisfies some invariance condition with respect to the given G -action. The action is assumed to be *compactly generating*, that is, $GV = X$ for some compact subset $V \subset X$.

The main aim of this paper is to study the structure of the convex cone \mathcal{P}_+ of all positive classical solutions of the equation

$$Pu = 0 \quad \text{in } X.$$

Fix some point $x_0 \in V$. The set

$$\mathcal{K} = \left\{ u \in \mathcal{P}_+ \mid u(x_0) = 1 \right\}$$

is a convex base of \mathcal{P}_+ ; it is well known that \mathcal{K} is compact with respect to the compact open topology in \mathcal{P}_+ . Therefore, by the Krein-Milman theorem, it is important (and in some sense, enough) to study the set $\text{ex } \mathcal{K}$, the set of all extreme points of \mathcal{K} .

It turns out (Theorem 6.8) that *if G is nilpotent then a positive solution k is an extreme point of \mathcal{K} if and only if k is an eigenfunction of the given G -action (such a function we also call G -multiplicative; see Section 2.5).*

We would like to point out that the above characterization of $\text{ex } \mathcal{K}$ is not true in general neither for solvable nor for virtually nilpotent groups.

It should be mentioned that our motivation and inspiration in studying this question came from the following two sources.

The first one is the work of S. Agmon [2], who has made a complete analysis of all positive solutions of the equation $Pu = 0$ in \mathbb{R}^d if the coefficients of P are periodic functions with respect to each variable. Agmon also has extended his results to the case of a noncompact *abelian* covering X of a compact Riemannian manifold Y and an elliptic operator P on X which is the lifting of a second order elliptic operator defined on Y . (For related results in this direction see also [15] and the references in [2].)

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The second source is the work of G. A. Margulis [30] on positive harmonic functions on (discrete) nilpotent groups. In this paper it was shown that a positive harmonic function on a nilpotent group is constant on the commutator subgroup. Roughly speaking, the proof contains three ingredients: a) using the Krein-Milman theorem; b) invoking the Schauder-Tychonoff fixed-point theorem; c) induction arguments. It is interesting to mention that in Agmon's paper [2] (which was done independently of the Margulis' work) there are two different proofs of the existence of positive solutions which are also eigenfunctions of the action of the (abelian) group of covering transformations. One of these proofs uses argument (a), and the other uses (b). In our paper, although in a different setting, we use all three ingredients of Margulis' technique; this enables us to obtain the characterization of extreme points of \mathcal{K} for the case of a nilpotent group G (see also [18]).

Suppose that the operator P satisfies $P1 = 0$, where 1 is the constant function on X taking at any point $x \in X$ the same value 1 . We say that P has the *Liouville property* if there is no nonconstant bounded solution of the equation $Pu = 0$ in X . The Liouville property of the Laplace-Beltrami operator of a Riemannian manifold was studied very extensively (see for example [12, 14, 20, 22, 29, 43] and the references therein). In [28] the Liouville property for invariant elliptic operators on a smooth manifold (and also for holomorphic functions on a complex space) with a group action was considered. The study of the structure of the cone \mathcal{Y}_+ of all positive solutions of the equation $Pu = 0$ in X is closely related to the Liouville property. For example, if one knows that the cone \mathcal{Y}_+ is of dimension 1 or at least that $1 \in \text{ex } \mathcal{K}$ then one deduces that P has the Liouville property. In this paper we obtain some new results concerning the Liouville property and related problems and give new proofs for some known results (see Sections 4, 5 and 6).

The outline of this paper is as follows.

In Section 2 we discuss some notions and some simple results related to a group action. Section 3 is devoted to some results concerning second order elliptic equations $Pu = f$ in X , concentrating, in particular, on various aspects of the maximum principle. We apply these results to prove that if the operator P satisfies the condition $P1 \geq \beta$ with some positive constant β , then the equation $Pu = f$ admits a bounded solution u_f for every bounded and Hölder continuous function f (Theorem 3.9).

In Section 4 we suppose that our elliptic operator P satisfies some invariance condition with respect to a given compactly generating G -action. We show that P admits a principal eigenvalue λ_{inv} with a positive G -invariant formal eigenfunction u_0 . This result hinges on the fact that the resolvent $R(\lambda)$ of the operator P can be defined in the ordered Banach space $C^G(X)$, the space of all G -invariant continuous functions on X with the uniform norm. In this space the resolvent $R(\lambda)$ is a compact operator which is u_0 -positive for suitable values of λ .

Section 5 deals with a G -invariant elliptic operator P on a manifold with a given compactly generating action of an arbitrary group G . We discuss some properties of the set $\mathcal{M}_{\mathcal{K}}$ of all normalized positive G -multiplicative solutions of the equation $Pu = 0$.

The set $\mathcal{M}_{\mathcal{K}}$ is compact (maybe empty), and $\overline{\text{conv}}\mathcal{M}_{\mathcal{K}}$, the closure of its convex hull, is a convex compact set. As was mentioned above, if G is nilpotent then $\overline{\text{conv}}\mathcal{M}_{\mathcal{K}} = \mathcal{K}$. In general, this is not true for a nonnilpotent group G . However, we shall see in Section 5 that even for general G , the set $\overline{\text{conv}}\mathcal{M}_{\mathcal{K}}$ has some interesting properties. In particular, we prove that $\overline{\text{conv}}\mathcal{M}_{\mathcal{K}}$ is a *simplex*, and either the set $\mathcal{M}_{\mathcal{K}}$ is empty, or $\mathcal{M}_{\mathcal{K}}$ is one point set (these two cases we consider as trivial) or $\mathcal{M}_{\mathcal{K}}$ is homeomorphic to a compact set Γ_0 (in some topological vector space \mathcal{H}) which is the boundary of a strictly convex open set. Moreover, for nontrivial $\mathcal{M}_{\mathcal{K}}$, the intersection of the set Γ_0 with a vector subspace $\mathcal{L} \subseteq \mathcal{H}$ of a finite dimension m is a compact real analytic submanifold in \mathcal{L} of dimension $m - 1$ (provided that this intersection is nontrivial). The above space \mathcal{H} is an appropriate subspace of the vector space of all homomorphisms $G \rightarrow \mathbb{R}$. If the given G -action on X is free, properly discontinuous and compactly generating, then X is a regular covering over a compact manifold X/G . This case was considered by S. Agmon in [2]. Agmon obtained a realization of the set $\mathcal{M}_{\mathcal{K}}$ (if it is nontrivial) as a real analytic boundary of a strictly convex body in the finite dimensional de Rham cohomology space $H_{\text{DR}}^1(X/G, \mathbb{R})$. In the more general case which is considered in our paper the de Rham realization is irrelevant. However, for operators on a covering of a compact manifold our approach gives indeed the same realization of the set $\mathcal{M}_{\mathcal{K}}$ which was obtained in [2].

We conclude Section 5 with some results concerning the L^2 -spectrum of the operator P .

Section 6 is devoted to the case of a compactly generating *nilpotent* group action. The main result of this section (and indeed one of the main results of the whole paper) is the description of the set $\text{ex } \mathcal{K}$ (see Theorem 6.8 cited above). Note that Section 6 almost does not depend on other parts of the paper.

In Section 7 we study nonnegative solutions of an invariant *parabolic* operator on a manifold X with a compactly generating G -action. Namely, let T be either the real line \mathbb{R} or the half line $(-\infty, 0) \subset \mathbb{R}$. Consider the parabolic operator $L = \partial_t + P(t, x, \partial_x)$ on $T \times X$, ($\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$), where P is a G -invariant second order elliptic operator on X with coefficients which are periodic in t (of a period 1) or time independent. It turns out, that a *nonnegative solution of the equation* $Lu = 0$ *in* $T \times X$ *is strictly positive*. In the time independent case, this fact implies *the uniqueness of the positive Cauchy problem for the operator* L *in* $[0, T] \times X$. These problems were discussed in [24,34], and we use some arguments of these papers. On the other hand, here we consider more general situations than were considered in [24,34].

In Section 8 we describe briefly the structure of the set of positive solutions (with zero Dirichlet boundary condition) of invariant elliptic operators on a *manifold with boundary*, provided a compactly generating group action is given. We show that our previous methods and results apply also to this case (see also [36]).

Section 9 contains examples and problems that stress the strictness and the limitation of our results. We conclude the paper with an appendix where we prove some analytic property of the the principal eigenvalue.

Basic notation

We denote by $C(X)$ the algebra of all real continuous functions on a given Hausdorff topological space X , and endow this algebra with the compact open topology (the topology of the uniform convergence on compact subsets $K \subset X$). If E is a linear subspace of $C(X)$, we denote by BE the subspace which consists of all bounded functions in E .

For a subset S of a topological vector space E , we denote by $\text{int } S$ its interior, by $\text{conv } S$ its convex hull, and by $\overline{\text{conv } S}$ we denote the closure of this hull in E . If E is locally convex and the subset S is compact, then the set $\overline{\text{conv } S}$ is compact. If $B \subseteq E$ is a convex subset, we denote by $\text{ex } B$ the set of all its extreme points.

$C = C(\cdot, \dots, \cdot)$ denotes a constant depending on the quantities appearing in parentheses. In a given context the same letter C may be used to denote different constants depending on the same set of arguments.

We denote by δ_{ij} the Kronecker's symbol.

2. SOME NOTIONS CONNECTED WITH GROUP ACTION

2.1. Notations and definitions. Let G be a group (which is considered without any topology or, alternatively, with the discrete one). We denote by e the unit of G . If $R \subseteq G$ is a normal subgroup we write $R \triangleleft G$. By $\text{Tors}(G)$ we denote the smallest normal subgroup in G which contains all the elements of a finite order. For any two subsets $Q, S \subseteq G$ we denote by (Q, S) the set of all commutators $(q, s) = qsq^{-1}s^{-1}$, where $q \in Q$ and $s \in S$. By $[Q, S]$ we denote the smallest subgroup in G containing the set (Q, S) . If Q and S are both normal subgroups of the group G then $[Q, S]$ is also a normal subgroup. The normal subgroup $[G, G]$ is, by definition, the commutator subgroup of the group G ; it consists of all finite products of the form $(q_1, s_1) \cdots (q_m, s_m)$, where $q_i, s_i \in G$. If the subset $Q = \{q\}$ consists of a single element q , we write (q, S) instead of $(\{q\}, S)$. When an element $h \in G$ is considered as fixed and g runs over G , we write also h_g instead of (h, g) :

$$h_g = (h, g) = hgh^{-1}g^{-1}. \quad (2.1)$$

Let $Q \triangleleft G$, and let $\pi: G \rightarrow G/Q$ be the natural projection. Fix an element $h \in G$ and denote by $Q(h)$ the π -preimage of the centralizer of the element $\pi(h)$ in G/Q . So, $Q(h)$ consists of all elements $g \in G$ such that $h_g \in Q$. The following lemma is almost evident.

2.2. Lemma. *The set $Q(h)$ is a subgroup in G . In particular, if $s \in Q_h$, then $s^n \in Q(h)$ for every integer n .*

Proof. First, $e \in Q(h)$, because $h_e = e$. It is easy to check that for all elements $s, t \in G$

$$h_{(st)} = h_s \cdot (sh_t s^{-1}) \quad \text{and} \quad h_{(s^{-1})} = s^{-1} \cdot (h_s)^{-1} \cdot s. \quad (2.2)$$

Since Q is a normal subgroup, the lemma follows from (2.2). ■

2.3. Group actions. Let X be a connected countable at infinity noncompact d -dimensional smooth manifold of class C^3 . Such X may be considered as an imbedded manifold in \mathbb{R}^N , for some $N \geq d$. Hence, the Euclidean metric and the Lebesgue measure on \mathbb{R}^N induce a Riemannian metric and a measure on X . For an open set $\Omega \subseteq X$ and for $m \in \mathbb{N}$, $0 < \alpha \leq 1$, $p \geq 1$, we shall consider the Hölder spaces $C^{m, \alpha}(\Omega)$ and the Sobolev spaces $W^{m, p}(\Omega)$ with respect to the above metric and measure.

Suppose that some left G -action T on the manifold X is given:

$$T: G \times X \rightarrow X, \quad T(g, x) = T_g x = gx.$$

We assume that any transformation $T_g: X \rightarrow X$ is a C^3 -diffeomorphism.

Recall that such an action is called *properly discontinuous*, if for every compact subset $K \subset X$ the set $\{g \in G \mid K \cap gK \neq \emptyset\}$ is finite. For any point $x \in X$, denote by G_x the isotropy subgroup $\{g \in G \mid gx = x\}$. The action is said to be *free*, if $G_x = \{e\}$ for each $x \in X$.

A connected unramified covering $p: X \rightarrow Y$ over a linearly connected locally contractible topological space Y is said to be *regular*, if for some (and then also for every) point $x_0 \in X$ the image $p_*(\pi_1(X, x_0))$ of the fundamental group $\pi_1(X, x_0)$ is a normal subgroup of the fundamental group $\pi_1(Y, p(x_0))$.

It is well known that

(RC) *If a G -action is properly discontinuous and free, then the quotient space X/G has a natural structure of a smooth manifold, and the natural projection $p: X \rightarrow X/G = Y$ is a smooth regular unramified covering. Moreover, the group G is isomorphic to the quotient group $\pi_1(Y, p(x_0))/p_*(\pi_1(X, x_0))$.*

A given G -action on X is said to be *compactly generating*, if

(CG) $GV = X$ for some compact subset $V \subset X$.

Such a compact subset $V \subset X$ (if it exists!) is said to be a *compact fundamental set* of the action T .

Remark. Properly discontinuous action satisfying **(CG)** is usually called *cocompact*. ■

From now on, we suppose that the given G -action T is compactly generating and we fix some fundamental compact set $V \subset X$.

2.4. Lemma. *Let H be a subgroup of a finite index in G . Then the induced H -action of H on X is compactly generating.*

Proof. Let V be a fundamental compact set of the G -action T . Since H is of a finite index in G , there exists a finite subset $\{g_1, \dots, g_m\} \subset G$ such that $G = \bigcup_{i=1}^m Hg_i$. It is evident that the set $V' = \bigcup_{i=1}^m g_i V \subset X$ is compact. On the other hand, $HV' = \bigcup_{i=1}^m Hg_i V = GV = X$. ■

