On the Localization of Binding for Schrödinger Operators and its Extension to Elliptic Operators

Yehuda Pinchover[*]
Department of Mathematics, Technion-Israel Institute of Technology
32000 Haifa, ISRAEL
e-mail: pincho@tx.technion.ac.il

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Abstract

In this paper we study the asymptotic behavior of the ground state energy $E(R)$ of the Schrödinger operator

$$P_R = -\Delta + V_1(x) + V_2(x-R), \quad x, R \in \mathbb{R}^n,$$

where the potentials V_i are small perturbations of the Laplacian in $\mathbb{R}^n, n \geq 3$. The methods presented here apply also in the investigation of the ground state energy $E(g)$ of the operator

$$P_g = P + V_1(x) + V_2(gx), \quad x \in X, g \in G,$$

where P_g is an elliptic operator which is defined on a noncompact manifold X, G is a discrete group acting on X by diffeomorphisms $G \times X \ni (g, x) \mapsto gx \in X$, and P is a G-invariant elliptic operator which is subcritical in X.

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1 Introduction

In this paper we study the ground state energy $E(R)$ of the Schrödinger operator

$$P_R = -\Delta + V_1(x) + V_2(x-R) \ \ x, R \in \mathbb{R}^n, \tag{1.1}$$

where the potentials V_i are small perturbations of the Laplacian in $\mathbb{R}^n, n \geq 3$ and the operators $P_i = -\Delta + V_i(x)$ are nonnegative. Recall [1] that a Schrödinger operator P defined in a domain $\Omega \subseteq \mathbb{R}^n$ is nonnegative if and only if there exists a positive function u which is a solution of the equation $Pu = 0$ in Ω.

Definition 1.1 *Let P be a second order elliptic operator defined in a domain $\Omega \subseteq \mathbb{R}^n$. A function u is said to be a ground state in the sense of Agmon of the operator P in the domain Ω with an eigenvalue zero if*

(i) u is a positive solution of the equation $Pu = 0$ in Ω.

(ii) If v is any positive solution of the equation $Pu = 0$ in some subdomain $\Omega_1 = \Omega \setminus K_1, K_1 \subset\subset \Omega$ then there exist a positive constant C and a subdomain $\Omega_2 = \Omega \setminus K_2, K_2 \subset\subset \Omega$ such that $u \leq Cv$ in Ω_2.

In other words, a ground state is a positive global solution of the equation $Pu = 0$ which has minimal growth at a neighborhood of infinity in Ω.

It is well known that if the operator P admits a positive solution in Ω then either P admits *a minimal (Dirichlet) Green function* $G_P^\Omega(x,y)$ and in this case we say that P is *subcritical* in Ω, or P admits a ground state in the sense of Agmon with an eigenvalue zero and in this case P is said to be *critical* in Ω. (For more details see Section 2 and also [7, 8].)

The motivation for studying the ground state energy $E(R)$ comes from a remarkable phenomenon known as the Efimov effect for a three-body Schrödinger operator. Such an operator H takes (in the center-of-mass frame) the following form

$$H = H_0 + \sum_{1 \leq j < k \leq 3} V_{jk}(r_j - r_k),$$

where the operator H_0 (the free Hamiltonian) and the operator H act on the space $L^2(\mathbb{R}^6)$, V_{jk} are short-range potentials in $\mathbb{R}^3$ and r_j denotes the position vector of the j-th particle. Suppose that all the three two-body subsystems admit ground states in the sense of Agmon with eigenvalues zero (so, V_{jk} are *critical potentials*) then by the Efimov effect the three-particle system has an infinite number of negative L^2-eigenvalues accumulating to zero. Note that this effect holds true even if each pair potential V_{jk} is a function with compact support.

A variational method for proving the Efimov effect was given in [6] by Yu. N. Ovchinnikov and I. M. Sigal and was generalized by H. Tamura in [11, 12] (see also [3, 10] and related results in [2, 13] and in the references therein). The proof relies on the following

(*i*) It is well known that the Schrödinger operator $-\Delta + C(1+|x|)^{-2}$ on $\mathbb{R}^n$ has an infinite number of negative eigenvalues provided that $C < -1/4$.

(*ii*) Suppose that the functions $V_i, i = 1,2$ are short range critical potentials in $\mathbb{R}^3$. Then the ground state energy $E(R)$ of the operator P_R in $\mathbb{R}^3$ satisfies

$$\lim_{R\to\infty} R^2 E(R) = -\beta < -1/4. \tag{1.2}$$

(*iii*) The number of the negative eigenvalues of the three-body Hamiltonian H is not less than the number of the negative eigenvalues of a certain Schrödinger operator on $\mathbb{R}^3$ with a potential the leading order term of which is $E(R)$ (hence, the effective interaction is long-range).

So, the study of the ground state energy $E(R)$ of such a type of perturbation is a natural problem in the spectral theory of Schrödinger operators.

We show that (*ii*) is a very special three-dimensional fact (and therefore, the Efimov effect is conceivably also a special three-dimensional effect). More precisely, suppose that the operators P_i admit (Agmon) ground states v_i, in $\mathbb{R}^n$, $n > 4$ (in other words, P_i are critical in $\mathbb{R}^n$). We prove that there exists a positive constant C such that the ground state energy $E(R)$ satisfies

$$-C|R|^{2-n} \leq E(R) \leq -C^{-1}|R|^{2-n} \tag{1.3}$$

provided that $|R|$ is large enough (see Theorem 4.3 and compare it with the remarks in pages 84 and 87 in [3]). Note that the ground states v_i are in L^2 if and only if $n > 4$.

Assume now that the operators P_i are nonnegative in $\mathbb{R}^n, n \geq 3$ but do not admit (Agmon) ground states (so, P_i are *subcritical* in $\mathbb{R}^n$). We prove that P_R is also nonnegative (actually, subcritical) if $|R|$ is large enough (Theorem 3.1, see also [3, 6, 10] where potentials with compact supports were considered). These results demonstrate again that subcriticality is a stable property while criticality is not (for further examples see [7, 8, 9]).

The proof for the critical (respectively, subcritical) case in the above cited papers applies only to the Laplacian in $\mathbb{R}^3$ (respectively, $\mathbb{R}^n, n \geq 3$). The authors there use the explicit form of the positive Green functions of $-\Delta + k^2$ and thus the proofs do not apply, for example, in the case of a perturbation of a periodic Schrödinger operator. The proofs presented here are elementary and also more general. They are based on the general theory of positive solutions of elliptic operators and therefore, can be applied to more general situations (see Section 5). Nevertheless, in the case of the Laplace operator in $\mathbb{R}^n$ our results are optimal in the sense that the decay assumptions on the given potentials V_i can not significantly be weakened.

The outline of this paper is as follows. In Section 2 we give some basic definitions, fix notations and collect and develop some general results. In Section 3 we prove Theorem 3.1 concerning the subcritical case while in Section 4 we prove Theorem 4.3 for the critical case.

In Section 5 we extend our results to a wider class of elliptic operators. We consider a noncompact manifold X and a discrete group G acting on X by diffeomorphisms $G \times X \ni (g, x) \mapsto gx \in X$. We suppose that a G-invariant (more precisely, equivariant) subcritical elliptic operator P is defined on X. We deal with perturbations given by potentials V_i with compact supports.

In Theorem 5.2 we generalize Theorem 3.1 to the operator $P_g = P + V_1(x) + V_2(gx)$. The generalization holds even in the nonsymmetric case. Theorem 5.4 is the extension of Theorem 4.3 to the case of a subcritical G-invariant operator P which is formally selfadjoint. We show that if the operators P_i are critical and admit L^2-ground states then under some additional assumptions the ground state energy $E(g)$ of the operator P_g is negative if gx_0 is outside a certain compact set and $E(g)$ behaves like the decay rate of the Green function of the operator P.

2 Preliminaries

In this section we collect and develop some general results concerning positivity properties of elliptic operators (see also [7, 8, 9, and the references therein]).

We consider a linear elliptic operator P of second order which is defined on some noncompact connected countable at infinity n-dimensional smooth manifold X such that P acts on functions $u \in C^2(X)$. So, we assume that in any coordinate system $(U; x_1, \ldots, x_n)$ the operator P is of the form

$$P(x, \partial_x) = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(x)\partial_i + c(x), \tag{2.1}$$

here $\partial_i = \partial/\partial x_i$.

We assume that for every $x \in X$ the quadratic form

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathrm{I\!R}^n \tag{2.2}$$

is positive definite. The matrix $[a^{ij}]$, the inverse of $[a_{ij}]$, defines a Riemannian metric g on the manifold X

$$g(\partial/\partial x_i, \partial/\partial x_j) = a^{ij}. \tag{2.3}$$

We denote by ρ and dx the distance and volume element induced by the Riemannian metric (2.3). By $B(x, r)$ we denote the geodesic ball of radius r centered at x.

We assume that the coefficients a_{ij}, b_i and c are *real* and locally Hölder continuous.

Let $\Omega \subset X$ be a domain and let $\{\Omega_k\}_{k=1}^{\infty}$ be a sequence of smooth relatively compact domains such that $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\cup_{k=1}^{\infty}\Omega_k = \Omega$. Denote by $\mathcal{C}_P(\Omega)$ the convex cone of all classical positive solutions of the equation

$$Pu = 0 \;\; in\, \Omega. \tag{2.4}$$

If $\mathcal{C}_P(\Omega) \neq \emptyset$ then for every $k \geq 1$ the Dirichlet Green function $G_P^{\Omega_k}(x, y)$ exists and is positive. By the generalized maximum principle $\{G_P^{\Omega_k}(x, y)\}_{k=1}^{\infty}$ is an increasing sequence which, by the Harnack inequality, converges uniformly in every compact subdomain either to $G_P^{\Omega}(x, y)$, the positive *minimal Green function* of P in Ω and P is said to be *subcritical operator* in Ω, or to infinity and in this case P is *critical* in Ω. The operator P is said to be *supercritical* if $\mathcal{C}_P(\Omega) = \emptyset$. It turns out that P is critical in Ω if and only if P admits a *ground state* (in the sense of Agmon) with an eigenvalue zero. Moreover, in the critical case $\mathcal{C}_P(\Omega)$ is a one dimensional cone. On the other hand, P is subcritical in Ω if and only if there exists a positive function $u \in C^2(\Omega)$ such that $Pu \geq 0$ in Ω but $Pu \neq 0$.

A solution $u \in \mathcal{C}_P(\Omega)$ is said to be *minimal* if for any $v \in \mathcal{C}_P(\Omega)$ with $v(x) \leq u(x)$ in Ω there exists $c > 0$ such that $v = cu$ (in other words, u belongs to an extreme ray of $\mathcal{C}_P(\Omega)$).

Suppose that P is subcritical in Ω and let $x_0 \in \Omega$ be a fixed reference point. Define the *Martin quotient*

$$K(x,y) = \frac{G_P^\Omega(x,y)}{G_P^\Omega(x_0,y)}. \tag{2.5}$$

A sequence $\{y_j\}_{j=1}^\infty \subset \Omega$ is called *a fundamental sequence* if $\{y_j\}_{j=1}^\infty$ has no point of accumulation in Ω and the sequence $\{K(x,y_j)\}$ converges to a function $K(x,\{y_j\}) \in \mathcal{C}_P(\Omega)$ which is said to be *a Martin function.* We identify two fundamental sequences if their Martin functions are equal. By the Martin theorem any minimal solution is (up to a multiplicative constant) a Martin function.

The *generalized principal eigenvalue* λ_0 is defined by

$$\lambda_0 = \lambda_0(P,\Omega) = \sup\{\lambda \in \mathbb{R}^n | \mathcal{C}_{P-\lambda}(\Omega) \neq \emptyset\} \tag{2.6}$$

(see [5]). In the classical case, of a smooth relatively compact domain Ω and elliptic operator P with (up to the boundary) smooth coefficients, the generalized principal eigenvalue λ_0 is just the first (Dirichlet) eigenfunction. Moreover, if P is defined on a general domain Ω and has a selfadjoint realization in $L^2(\Omega)$ then λ_0 is the bottom of the L^2 spectrum of P (see for example [1, 5]).

Let $P_i,\ i = 1,2$ be two subcritical operators in $\Omega \subset X$. We say that the Green functions $G_{P_1}^\Omega(x,y)$ and $G_{P_2}^\Omega(x,y)$ are *equivalent* ($G_{P_1}^\Omega(x,y) \sim G_{P_2}^\Omega(x,y)$) if there exists $C > 0$ such that

$$C^{-1}G_{P_2}^\Omega(x,y) \leq G_{P_1}^\Omega(x,y) \leq CG_{P_2}^\Omega(x,y) \tag{2.7}$$

for all $x,y \in \Omega,\ x \neq y$.

Remark 2.1 *Throughout the paper, we usually do not indicate explicitly the quantities that a given constant C depends on. Moreover, in a given context, the same letter C may be used to denote different constants depending on the same set of arguments.*

Definition 2.2 *Let P be a subcritical operator in Ω and let $V \in C^\alpha(\Omega)$. We say that V is a small perturbation of P in Ω if*

$$\lim_{k\to\infty}\left\{\sup_{x,y\in\Omega\setminus\Omega_k}\int_{\Omega\setminus\Omega_k}\frac{G_P^\Omega(x,z)|V(z)|G_P^\Omega(z,y)}{G_P^\Omega(x,y)}\,dz\right\} = 0. \tag{2.8}$$

Note that a Hölder continuous function with a compact support in Ω is a small perturbation of a subcritical operator P in Ω. In Sections 3 and 4 we study certain kinds of small perturbations of the Laplacian in $\mathbb{R}^n$, $n \geq 3$. To this end we need the following definition.

Definition 2.3 *Let $n \geq 3$, the space of functions*

$$K_n^\infty = \left\{ V \in C^\alpha(\mathbb{R}^n) \middle| \lim_{M\to\infty} \sup_{x\in\mathbb{R}^n} \int_{|y|>M} \frac{|V(y)|}{|x-y|^{n-2}}\, dy = 0 \right\} \tag{2.9}$$

is called the Kato class at infinity (see also [9, and the references therein]).

Remarks 2.4 *(a) It is easy to see that if $V \in K_n^\infty$ then V is a small perturbation of the Laplacian in $\mathbb{R}^n$, $n \geq 3$.*
(b) If $V(x) \in K_n^\infty$ then there exists a smooth positive function w such that

- *(i)* $wV \in K_n^\infty$
- *(ii)* $\lim_{|x|\to\infty} w(x) = \infty$.

For the completeness we present here the idea of the proof of the latter fact. Let

$$a_k(x) = \int_{k\leq|y|\leq k+1} \frac{|V(y)|}{|x-y|^{n-2}} dy, \quad k = 1, 2, \ldots\ .$$

By the definition of K_n^∞ the series $\sum a_k(x)$ converges uniformly in x. Therefore, there exists a nondecreasing sequence b_k such that $\lim_{k\to\infty} b_k = \infty$ and $\sum b_k a_k(x)$ is a uniformly convergent series. Define a smooth nondecreasing function w such that

$$w(x) = b_k \qquad \text{if } k + 1/4 \leq |x| \leq k + 1,$$

and

$$b_{k-1} \leq w(x) \leq b_k \quad \text{if } k \leq |x| \leq k + 1/4.$$

It is clear that such a function w satisfies the above desired properties. We shall use these simple facts on K_n^∞ in the proof of Theorem 3.1.

The following lemma shows that many positivity properties of a given operator P in a domain Ω are stable under small perturbations.

Lemma 2.5 *Let P be a subcritical operator in a domain $\Omega \subset X$, $n \geq 2$, and let $G_P^\Omega(x, y)$ be its minimal positive Green function. Let V be a Hölder continuous function in Ω such that V is a small perturbation of P in Ω. Suppose that the operator $P + V(x)$ is critical in Ω and let v be the ground state of the operator $P + V$ with an eigenvalue 0.*

(i) The operator $P + tV$ is supercritical in Ω for all $t > 1$ and subcritical in Ω for all $0 \leq t < 1$. Moreover,

$$G^{\Omega}_{P+tV}(x, y) \sim G^{\Omega}_{P}(x, y), \; 0 \leq t < 1. \tag{2.10}$$

(ii) Let $y_0 \in \Omega$ be a fixed reference point and let $\delta > 0$ be such that $B(y_0, \delta) \subset\subset \Omega$. There exists a constant $C > 0$ such that

$$C^{-1} G^{\Omega}_{P}(x, y_0) \leq v(x) \leq C G^{\Omega}_{P}(x, y_0) \tag{2.11}$$

for all $x \in \Omega \setminus B(y_0, \delta)$.

(iii) The ground state v satisfies the integral equation

$$v(x) = -\int_{\Omega} G^{\Omega}_{P}(x, y) V(y) v(y) dy. \tag{2.12}$$

Remark 2.6 *Part (iii) of Lemma* 2.5 *was previously proven only when V is a nonnegative function or a function with a compact support* (see [8, Theorem 2.1]).

Proof of Lemma 2.5: Part (*i*) follows from [7, Lemma 2.4] and [8, Theorem 3.1] while part (*ii*) follows from [7, Lemma 3.6].

So, there remains to prove part (*iii*). To this end, let $\{V_k(x)\}_{k=1}^{\infty}$ be a sequence of Hölder continuous functions such that $|V_k(x)| \leq |V(x)|$ and

$$V_k(x) = \begin{cases} V(x) & \text{if } x \in \Omega_{k-1} \\ 0 & \text{if } x \notin \Omega_k, \end{cases} \tag{2.13}$$

and denote also $W_k = V - V_k$. It follows from the results in [7] that there exist positive constants k_0 and C such that for every $k \geq k_0$:

(a) The operator $P + W_k$ is subcritical in Ω.

(b)
$$C^{-1} G^{\Omega}_{P}(x, y) \leq G^{\Omega}_{P+W_k}(x, y) \leq C G^{\Omega}_{P}(x, y). \tag{2.14}$$

Since the functions V_k have compact supports in Ω it follows from [8, Theorem 2.1] that

$$v(x) = -\int_{\Omega} G^{\Omega}_{P+W_k}(x, y) V_k(y) v(y) dy. \tag{2.15}$$

On the other hand, (2.13) and (2.14) imply that for all $x, y \in \mathbb{R}^n$

$$G^{\Omega}_{P+W_k}(x,y)|V_k(y)| \leq C G^{\Omega}_{P}(x,y)|V(y)|. \tag{2.16}$$

Moreover,

$$\lim_{k\to\infty} G^{\Omega}_{P+W_k}(x,y)V_k(y) = G^{\Omega}_{P}(x,y)V(y). \tag{2.17}$$

Using (2.11) and the definition of small perturbation we deduce that the function $G^{\Omega}_{P}(x,y)|V(y)|v(y)$ is integrable in $\mathbb{R}^n$. Now, the Lebesgue dominated convergence theorem, (2.16) and (2.17) imply (2.12). $\square$

Lemma 2.7 *Let $V \in K^{\infty}_{n}$ and suppose that the operator $P = -\Delta + V(x)$ is critical in $\mathbb{R}^n, n \geq 3$. Let v be the ground state of P. Then there exists a positive constant C such that*

$$\lim_{|x|\to\infty} \frac{v(x)}{G^{\mathbb{R}^n}_{-\Delta}(x,0)} = C. \tag{2.18}$$

Proof: It follows from Lemma 2.5 (*ii*) that there exists a positive number C such that

$$C^{-1} \leq \liminf_{|x|\to\infty} \frac{v(x)}{G^{\mathbb{R}^n}_{-\Delta}(x,0)} \leq \limsup_{|x|\to\infty} \frac{v(x)}{G^{\mathbb{R}^n}_{-\Delta}(x,0)} \leq C.$$

So, we need only to prove that the limit in (2.18) does exist. Let $W \in C^{\infty}_{0}(\mathbb{R}^n)$ be a nonzero nonnegative function. Then the operator $P + W(x)$ is subcritical in $\mathbb{R}^n$ and by Lemma 2.5 the ground state v satisfies the equation

$$v(x) = \int_{\mathbb{R}^n} G^{\mathbb{R}^n}_{P+W}(x,y)W(y)v(y)dy. \tag{2.19}$$

Since the Martin boundary of the operator $P+W(x)$ consists of exactly one positive normalized solution u (see [9]) and $G^{\mathbb{R}^n}_{P+W}(x,y)$ is symmetric in x and y it follows that

$$\lim_{|x|\to\infty} \frac{G^{\mathbb{R}^n}_{P+W}(x,y)}{G^{\mathbb{R}^n}_{P+W}(x,0)} = u(y), \tag{2.20}$$

where $u \in \mathcal{C}_{P+W}(\mathbb{R}^n)$. Divide Equation (2.19) by $G^{\mathbb{R}^n}_{P+W}(x,0)$ and let $|x| \to \infty$. Recall that the support of W is contained in some ball B, and the convergence in (2.20) is uniformly in B. Consequently

$$\lim_{|x|\to\infty} \frac{v(x)}{G^{\mathbb{R}^n}_{P+W}(x,0)} = \int_{\mathbb{R}^n} u(y)W(y)v(y)dy = C_1 > 0. \tag{2.21}$$

On the other hand, it follow from Lemma 2.5 and the results in [7] that the Green function $G^{\mathbb{R}^n}_{P+W}(x,y)$ is equivalent to $G^{\mathbb{R}^n}_{-\Delta}(x,y)$ and satisfies the resolvent equation (at $y=0$):

$$G^{\mathbb{R}^n}_{P+W}(x,0) = G^{\mathbb{R}^n}_{-\Delta}(x,0) - \int_{\mathbb{R}^n} G^{\mathbb{R}^n}_{-\Delta}(x,z)(V(z)+W(z))G^{\mathbb{R}^n}_{P+W}(z,0)dz. \tag{2.22}$$

Divide (2.22) by $G^{\mathbb{R}^n}_{-\Delta}(x,0)$. We have

$$\frac{G^{\mathbb{R}^n}_{P+W}(x,0)}{G^{\mathbb{R}^n}_{-\Delta}(x,0)} = 1 - \int_{\mathbb{R}^n} \frac{G^{\mathbb{R}^n}_{-\Delta}(x,z)(V(z)+W(z))G^{\mathbb{R}^n}_{P+W}(z,0)dz}{G^{\mathbb{R}^n}_{-\Delta}(x,0)}. \tag{2.23}$$

For every $|x|>1$ define the function $f_x(z)$ by

$$f_x(z) = \frac{G^{\mathbb{R}^n}_{-\Delta}(x,z)(V(z)+W(z))G^{\mathbb{R}^n}_{P+W}(z,0)}{G^{\mathbb{R}^n}_{-\Delta}(x,0)}. \tag{2.24}$$

We claim that the family of functions $\{f_x(z)\}_{|x|>1}$ is a set of uniformly integrable functions. Suppose that $|z| \le |x-z|$ then by the triangle inequality $|x| \le 2|x-z|$. So, for such z

$$|f_x(z)| \le C(|V(z)| + W(z))|z|^{2-n}. \tag{2.25}$$

Therefore, by the definition of K_n^∞ we have for such z that the integral

$$\int_{\{|z|>M\}\cap\{|z|\le|x-z|\}} |f_x(z)|dz \le C\int_{|z|>M} (|V(z)|+W(z))|z|^{2-n}dz$$

is arbitrarily small provided that M is large enough. On the other hand, if $|x-z| \le |z|$ then $|x| \le 2|z|$ and therefore,

$$|f_x(z)| \le C(|V(z)| + W(z))|x-z|^{2-n} \tag{2.26}$$

and again if M is large then the integral

$$\int_{\{|z|>M\}\cap\{|z|\ge|x-z|\}} |f_x(z)|dz \le C\int_{|z|>M} (|V(z)|+W(z))|x-z|^{2-n}dz$$

is small. It is also clear from (2.25), (2.26) and the definition of K_n^∞ that if M is large enough then

$$\int_{\{z|f_x(z)>M\}} |f_x(z)|dz$$

is arbitrarily small. So, the family $\{f_x(z)\}_{|x|>1}$ is uniformly integrable. Now, let $|x| \to \infty$ we obtain from (2.23) that

$$\lim_{|x|\to\infty} \frac{G^{\mathbb{R}^n}_{P+W}(x,0)}{G^{\mathbb{R}^n}_{-\Delta}(x,0)} = 1 - \int_{\mathbb{R}^n} (V(z)+W(z))G^{\mathbb{R}^n}_{P+W}(z,0)dz = C_2. \quad (2.27)$$

Combining (2.21) and (2.27) we see that the limit in (2.18) exists. □

Remark 2.8 *Using the same argument as in the proof of Lemma* 2.7 *it can be shown that in fact,*

$$\lim_{|x|\to\infty} \frac{G^{\mathbb{R}^n}_{P+W}(x,y)}{G^{\mathbb{R}^n}_{-\Delta}(x,0)}$$

exists for every $y \in \mathbb{R}^n$ *(and not only for* $y = 0$*) and equals to* $C_2 u(y)$*, where* C_2 *is the positive constant in* (2.27) *and* u *is the normalized positive solution* $u \in \mathcal{C}_{P+W}(\mathbb{R}^n)$*. Moreover, the solution* u *satisfies the equation*

$$C_2 u(y) = 1 - \int_{\mathbb{R}^n} G^{\mathbb{R}^n}_{P+W}(y,z)(V(z)+W(z))dz.$$

Lemma 2.9 *Let* $W, V \in K_n^\infty$ *and suppose that* $P = -\Delta + W(x)$ *is subcritical in* $\mathbb{R}^n$*. Define the function*

$$v(x) = \int_{\mathbb{R}^n} G^{\mathbb{R}^n}_P(x,y)|V(y)|dy. \quad (2.28)$$

Then

(i)

$$\lim_{|x|\to\infty} v(x) = 0 \quad (2.29)$$

(ii) the function v *satisfies the equation*

$$Pv(x) = |V(x)|,\ x \in \mathbb{R}^n. \quad (2.30)$$

Proof: (i) It follows from Lemma 2.5 and Remark 2.4 that $G^{\mathbb{R}^n}_P(x,y) \sim G^{\mathbb{R}^n}_{-\Delta}(x,y)$. Therefore, there exists a positive constant C such that

$$C^{-1}|x-y|^{2-n} \le G^{\mathbb{R}^n}_P(x,y) \le C|x-y|^{2-n}. \quad (2.31)$$

The definition of K_n^∞ and (2.31) easily imply (2.29).

(*ii*) Let $V_k(x)$ be a sequence of Hölder continuous functions such that $V_k(x) \leq |V(x)|$ and

$$V_k(x) = \begin{cases} |V(x)| & \text{if } |x| < k \\ 0 & \text{if } |x| > k+1 \end{cases}$$

and let

$$v_k(x) = \int_{\mathbb{R}^n} G_P^{\mathbb{R}^n}(x,y) V_k(y) dy.$$

Then

$$Pv_k(x) = |V(x)| \quad \text{for all } x \in B(0,K) \text{ and } k > K. \tag{2.32}$$

Moreover, by the Lebesgue dominated convergence theorem

$$\lim_{k \to \infty} v_k(x) = v(x). \tag{2.33}$$

Using standard Schauder estimates, (2.32) and (2.33) we obtain (2.30). □

3 The subcritical case

In this section we consider the subcritical case for Schrödinger operators in $\mathbb{R}^n$, $n \geq 3$. Let $u \in C(\mathbb{R}^n)$ and $R \in \mathbb{R}^n$, we denote

$$u^R(x) = u(x - R).$$

Our goal here is to prove the following theorem

Theorem 3.1 *Let* $V_i(x) \in K_n^\infty$, $i = 1, 2$, *be two functions such that the operators* $P_i = -\Delta + V_i(x)$, $i = 1, 2$, *are subcritical in* $\mathbb{R}^n$, $n \geq 3$. *There exists* $R_0 > 0$ *such that the operator*

$$P_R = -\Delta + V_R(x) = -\Delta + V_1(x) + V_2^R(x)$$

is subcritical for all vectors $R \in \mathbb{R}^n \setminus B(0, R_0)$.

Proof: We shall show that for $|R|$ sufficiently large P_R admits a positive supersolution u_R in $\mathbb{R}^n$ such that $u_R \notin \mathcal{C}_{P_R}(\mathbb{R}^n)$. Let $G_{P_i}^{\mathbb{R}^n}(x,y)$, $i = 1, 2$, be the Green functions of the operators P_i in $\mathbb{R}^n$. Define

$$v_i(x) = \int_{\mathbb{R}^n} G_{P_i}^{\mathbb{R}^n}(x,y) |V_i(y)| dy. \tag{3.1}$$

Since $V_i \in K_n^\infty$ it follows from Remark 2.4 (b) that one can find two positive functions $W_i \in K_n^\infty$ such that

$$\lim_{|x|\to\infty} \frac{W_i(x)}{|V_i(x)|} = \infty. \tag{3.2}$$

Define also the functions

$$w_i(x) = \int_{\mathrm{I\!R}^n} G_{P_i}^{\mathrm{I\!R}^n}(x,y)W_i(y)dy. \tag{3.3}$$

It follows from Lemma 2.9 that for every $R \in \mathrm{I\!R}^n$

$$\left(-\Delta + V_i^R(x)\right)\left(v_i^R(x) + w_i^R(x)\right) = |V_i^R(x)| + W_i^R(x) \tag{3.4}$$

and

$$\lim_{|x|\to\infty}(v_i^R(x) + w_i^R(x)) = 0.$$

We denote

$$M_i = \|v_i + w_i\|_\infty. \tag{3.5}$$

Consider the function

$$u_R(x) = v_1(x) + w_1(x) + v_2^R(x) + w_2^R(x). \tag{3.6}$$

Then by (3.4) we have

$$\begin{aligned} P_R u_R(x) &= (-\Delta + V_1(x) + V_2^R(x))u_R(x) \\ &= |V_1(x)| + V_1(x)(v_2^R(x) + w_2^R(x)) + W_1(x) + \\ &\quad |V_2^R(x)| + V_2^R(x)(v_1(x) + w_1(x)) + W_2^R(x). \end{aligned} \tag{3.7}$$

Now, if $M_i \le 1,\ i = 1,2$, then it is clear that $P_R u_R(x) > 0$ for every $R \in \mathrm{I\!R}^n$. We shall show that the positive potentials W_i keep $P_R u_R(x)$ positive even at the points, where $v_i(x) + w_i(x) \ge 1$ provided that R is sufficiently large.

So, it is natural to consider the compact sets

$$A_i = \{x | v_i(x) + w_i(x) \ge 1\},\ i = 1,2 \tag{3.8}$$

and let K_0 be a positive number such that $A_i \subset B(0,K_0)$. Denote

$$A_i^R = \{x | x = y + R,\ y \in A_i\},\ i = 1,2. \tag{3.9}$$

It is clear that outside A_2^R

$$|V_1(x)| + V_1(x)(v_2^R(x) + w_2^R(x)) \geq 0,$$

and hence,

$$|V_1(x)| + V_1(x)(v_2^R(x) + w_2^R(x)) + W_1(x) > 0, \ x \notin A_2^R. \tag{3.10}$$

On the other hand, using (3.2) we deduce that there exists R_0 such that if $|x| \geq R_0$ then

$$\frac{W_1(x)}{|V_1(x)|} > M_2,$$

where M_2 is defined by (3.5). Therefore, for such points $|x| \geq R_0$ we have

$$W_1(x) + V_1(x)(v_2^R(x) + w_2^R(x)) \geq W_1(x) - M_2|V_1(x)| > 0. \tag{3.11}$$

Note that if $|R| > R_0 + K_0$ then $A_2^R \bigcap B(0, R_0) = \emptyset$ and thus Inequality (3.11) implies that

$$|V_1(x)| + V_1(x)(v_2^R(x) + w_2^R(x)) + W_1(x) > 0, \ x \in A_2^R. \tag{3.12}$$

Combining (3.10) and (3.12) we see that if $|R| > R_0 + K_0$

$$|V_1(x)| + V_1(x)(v_2^R(x) + w_2^R(x)) + W_1(x) > 0, \ x \in \mathbb{R}^n. \tag{3.13}$$

Similarly, one proves that if $|R|$ is large enough then

$$|V_2^R(x)| + V_2^R(x)(v_1(x) + w_1(x)) + W_2^R(x) > 0, \ x \in \mathbb{R}^n. \tag{3.14}$$

Now, (3.13), (3.14) and (3.7) imply that

$$P_R u_R(x) > 0.$$

Thus, P_R is subcritical in $\mathbb{R}^n$. □

Remark 3.2 *For potentials with compact support Theorem 3.1 was previously proven in* [3]. *In* [10] *B. Simon gave an alternative proof for potentials with compact support by relating subcriticality to Brownian motion.*

4 The critical case

In this section we assume that the operators $P_i = -\Delta + V_i(x), i = 1, 2$, are *critical* in $\mathbb{R}^n, n \geq 5$. We consider again the family of operators

$$P_R = -\Delta + V_R(x) = -\Delta + V_1(x) + V_2^R(x).$$

Our main aim is to study the ground state energy $E(R) = \lambda_0(P_R, \mathbb{R}^n)$ of the operator P_R (λ_0 is defined by (2.6)). We shall prove in Theorem 4.3 that (under some mild assumptions) the operator P_R is supercritical in $\mathbb{R}^n$ for $|R|$ large enough and that $E(R)$ behaves like $-C|R|^{2-n}$ as $|R| \to \infty$.

Throughout this section we assume that the "critical" potentials $V_i(x)$, $i = 1, 2$ are two Hölder continuous functions which satisfy the following inequalities

$$|V_i(x)| \leq \frac{C}{\langle x \rangle^{\beta}}, \ x \in \mathbb{R}^n, \tag{4.1}$$

where $\langle x \rangle = 1 + |x|$, $\beta > n - 2$ and C is some positive constant. Denote by v_i the ground states of the operator P_i in $\mathbb{R}^n$. It follows from Lemma 2.5 that there exists $C > 0$ so that

$$C^{-1}\langle x \rangle^{2-n} \leq v_i(x) \leq C\langle x \rangle^{2-n}, \quad x \in \mathbb{R}^n. \tag{4.2}$$

By our assumption $n \geq 5$, so, (4.2) implies that $v_i \in L^2(\mathbb{R}^n), i = 1, 2$. Without loss of generality we assume that $\|v_i\|_{L^2(\mathbb{R}^n)} = 1$.

Given the differential operator P_R one can define the maximal realization H_R of P_R in $L^2(\mathbb{R}^n)$ as follows (see [1]):

$$\begin{aligned} D(H_R) &= \left\{u | u \in L^2(\mathbb{R}^n) \cap H^1_{loc}(\mathbb{R}^n), V_R u \in L^1_{loc}(\mathbb{R}^n), P_R u \in L^2(\mathbb{R}^n)\right\} \\ H_R u &= P_R u \ \text{ for } u \in D(H_R), \end{aligned} \tag{4.3}$$

where P_R acts on u in the distribution sense. The operator H_R is the *only* self-adjoint realization of P in $L^2(\mathbb{R}^n)$. It follows [1] that $E(R)$ is equal to the bottom of the spectrum of H_R and therefore,

$$E(R) = \inf_{u \in D(H_R)} \left\{ \frac{(P_R u, u)}{\|u\|^2_{L^2(\mathbb{R}^n)}} \right\}. \tag{4.4}$$

The key estimates for studying the asymptotic behavior of $E(R)$ are given in the following lemma.

Lemma 4.1 *Let* $V_i(x) \in C^{\alpha}(\mathbb{R}^n)$, $i = 1, 2$ *be two functions which satisfy* (4.1)*. Suppose that the operators* $P_i = -\Delta + V_i(x), i = 1, 2,$ *are critical in* $\mathbb{R}^n, n \geq 5$ *and let* v_i *be their ground states. Then there exist positive constants* C *and* R_0 *such that for all* $R \in \mathbb{R}^n \backslash B(0, R_0)$ *the following inequalities hold:*

(i)

$$\int_{\mathbb{R}^n} v_2^R(x) V_1(x) v_1(x) dx \leq -C^{-1} \langle R \rangle^{2-n}, \tag{4.5}$$

(ii)

$$\int_{\mathbb{R}^n} (v_1(x))^2 |V_2^R(x)| dx \leq C \langle R \rangle^{-\beta}, \tag{4.6}$$

(iii)

$$\inf_{x \in \mathbb{R}^n} \left\{ \frac{V_1(x) v_2^R(x)}{v_1(x) + v_2^R(x)} \right\} \geq -C \langle R \rangle^{2-n}. \tag{4.7}$$

Proof: (i) It follows from Lemma 2.7 and the invariance of $G_{-\Delta}^{\mathbb{R}^n}(R, x)$ under translations that the limit

$$\lim_{|R| \to \infty} \frac{v_2^R(x)}{G_{-\Delta}^{\mathbb{R}^n}(R, x)} = C > 0 \tag{4.8}$$

exists and is positive. Thus

$$\begin{aligned} &\int_{\mathbb{R}^n} v_2^R(x) V_1(x) v_1(x) dx = C \int_{\mathbb{R}^n} G_{-\Delta}^{\mathbb{R}^n}(R, x) V_1(x) v_1(x) dx + \\ &\int_{\mathbb{R}^n} \left(\frac{v_2^R(x)}{G_{-\Delta}^{\mathbb{R}^n}(R, x)} - C \right) G_{-\Delta}^{\mathbb{R}^n}(R, x) V_1(x) v_1(x) dx = \\ &-C v_1(R) + \int_{|x| > M} \left(\frac{v_2^R(x)}{G_{-\Delta}^{\mathbb{R}^n}(R, x)} - C \right) G_{-\Delta}^{\mathbb{R}^n}(R, x) V_1(x) v_1(x) dx + \\ &\int_{|x| \leq M} \left(\frac{v_2^R(x)}{G_{-\Delta}^{\mathbb{R}^n}(R, x)} - C \right) G_{-\Delta}^{\mathbb{R}^n}(R, x) V_1(x) v_1(x) dx. \end{aligned} \tag{4.9}$$

It follows from Lemma 2.5 (ii) that there exists $C_1 > 0$ such that

$$-C v_1(R) < -C_1 \langle R \rangle^{2-n}, \quad R \in \mathbb{R}^n, \tag{4.10}$$

and for all $x, R \in \mathbb{R}^n$

$$\left| \frac{v_2^R(x)}{G_{-\Delta}^{\mathbb{R}^n}(R,x)} - C \right| \le C_1^{-1}. \tag{4.11}$$

On the other hand, for every negative number $\gamma \le \delta < 0$

$$\langle x\rangle^{\gamma}\langle R-x\rangle^{\delta} \le \begin{cases} C\langle R\rangle^{\delta}\langle x\rangle^{\gamma} & \text{if } |x| \le |R-x| \\ C\langle R\rangle^{\delta}\langle R-x\rangle^{\gamma} & \text{if } |R-x| \le |x|. \end{cases} \tag{4.12}$$

Hence

$$\langle R-x\rangle^{2-n}|V_1(x)|\langle x\rangle^{2-n} \le C\langle R\rangle^{2-n}|V_1(x)|\left\{\langle x\rangle^{2-n} + \langle R-x\rangle^{2-n}\right\}. \tag{4.13}$$

Therefore, by definition of K_n^{∞}, estimates (4.2), (4.11) and (4.13), one sees that if M is large enough then

$$\int_{|x|>M} \left| \frac{v_2^R(x)}{G_{-\Delta}^{\mathbb{R}^n}(R,x)} - C \right| G_{-\Delta}^{\mathbb{R}^n}(R,x)|V_1(x)|v_1(x)dx \le \frac{C_1}{3\langle R\rangle^{n-2}} \tag{4.14}$$

for all $R \in \mathbb{R}^n$. Now, it follows from (4.1), (4.8) and (4.13) that we can choose $|R|$ large enough such that $|\frac{v_2^R(x)}{G_{-\Delta}^{\mathbb{R}^n}(R,x)} - C|$ is so small for all $|x| \le M$ to guarantee that

$$\int_{|x|\le M} \left| \frac{v_2^R(x)}{G_{-\Delta}^{\mathbb{R}^n}(R,x)} - C \right| G_{-\Delta}^{\mathbb{R}^n}(R,x)|V_1(x)|v_1(x)dx \le \frac{C_1}{3\langle R\rangle^{n-2}}. \tag{4.15}$$

Using estimates (4.10), (4.14) and (4.15) in (4.9) we obtain

$$\int_{\mathbb{R}^n} v_2^R(x)V_1(x)v_1(x)dx \le -\frac{C_1}{3\langle R\rangle^{n-2}}.$$

(*ii*) According to the decay assumption (4.1) on the function V_2 we deduce that $v_1^2(x)|V_2^R(x)|$ is bounded by $C\langle x\rangle^{4-2n}\langle R-x\rangle^{-\beta}$. Without loss of generality we may assume that $\beta \le 2n-4$. Now, use (4.12) with $\gamma = 4-2n$ and $\delta = -\beta$ and integrate over $\mathbb{R}^n$ to obtain (4.6).

(*iii*) A simple computation shows that for every $x \in \mathbb{R}^n$

$$\frac{|V_1(x)|v_2^R(x)}{v_1(x) + v_2^R(x)} \le C\frac{\langle x\rangle^{-\beta}\langle R-x\rangle^{2-n}}{\langle x\rangle^{2-n} + \langle R-x\rangle^{2-n}} \le C\langle R\rangle^{2-n},$$

which implies (4.7). Recall that P_1 is critical and therefore, V_1 is negative somewhere in $\mathbb{R}^n$. So, the infimum in (4.7) is indeed negative. $\square$

Remark 4.2 *As can be seen from our proof the assumptions in Lemma* 4.1 *can be slightly weakened. In part (i) of Lemma* 4.1 *it is enough to assume that* $V_1 \in K_n^\infty$ *and* $n \geq 3$ *while part (iii) holds true if* $n \geq 3$ *and* $\beta \geq n-2$.

Now, we prove the theorem.

Theorem 4.3 *Let* $V_i \in C^\alpha(\mathbb{R}^n)$, $i = 1, 2$ *be two functions which satisfy Inequality* (4.1). *Assume that the operators* $P_i = -\Delta + V_i(x)$, $i = 1, 2$, *are critical in* $\mathbb{R}^n$, $n \geq 5$. *Then*

- *(i) there exists* $R_0 > 0$ *such that the operator* P_R *is supercritical for all vectors* $R \in \mathbb{R}^n \backslash B(0, R_0)$,
- *(ii) there exists a positive constant* C *such that the ground state energy* $E(R)$ *of the operator* P_R *satisfies*

$$-C|R|^{2-n} \leq E(R) \leq -C^{-1}|R|^{2-n} \quad \textit{for all } |R| \geq R_0. \tag{4.16}$$

Proof: The operator P_R is supercritical if and only if $E(R) < 0$. Therefore, in order to prove the theorem it is enough to prove Inequality (4.16). We prove first the upper bound in (4.16).

Upper bound: Let v_i be the ground states of the operators $P_i, i = 1, 2$ and consider the function

$$u_R(x) = v_1(x) + v_2^R(x) \tag{4.17}$$

as a test function in the Rayleigh-Ritz quotient for the operator H_R (see (4.4)). Note that since $v_i, i = 1, 2$ are positive functions in the unit sphere of $L^2(\mathbb{R}^n)$ it follows that

$$2 \leq \|u_R\|_{L^2(\mathbb{R}^n)} \leq 4. \tag{4.18}$$

In fact, $\lim_{|R|\to\infty} \|u_R\|_{L^2(\mathbb{R}^n)} = 2$.

Furthermore,

$$\begin{aligned} P_R u_R(x) &= (-\Delta + V_1(x) + V_2^R(x))u_R(x) \\ &= P_1 v_1(x) + V_2^R(x) v_1(x) + (-\Delta + V_2^R(x)) v_2^R(x) + V_1(x) v_2^R(x) \\ &= V_2^R(x) v_1(x) + V_1(x) v_2^R(x). \end{aligned} \tag{4.19}$$

Hence, using (4.4) we have

$$\begin{aligned} E(R) &\leq \frac{(P_R u_R, u_R)}{\|u_R\|^2_{L^2(\mathbb{R}^n)}} \\ &= \frac{\int_{\mathbb{R}^n} \left\{ v_1(x) V_2^R(x) v_2^R(x) + v_1(x) V_1(x) v_2^R(x) \right\} dx}{\|u_R\|^2_{L^2(\mathbb{R}^n)}} + \\ &\quad \frac{\int_{\mathbb{R}^n} \left\{ (v_1(x))^2 V_2^R(x) + (v_2^R(x))^2 V_1(x) \right\} dx}{\|u_R\|^2_{L^2(\mathbb{R}^n)}}. \end{aligned} \tag{4.20}$$

Using (4.5) and (4.18) we find that

$$\frac{\int_{\mathbb{R}^n} \left\{ v_1(x) V_2^R(x) v_2^R(x) + v_1(x) V_1(x) v_2^R(x) \right\} dx}{\|u_R\|^2_{L^2(\mathbb{R}^n)}} \leq -C \langle R \rangle^{2-n} \tag{4.21}$$

for all $|R| \geq R_0$. On the other hand, (4.6) and (4.18) imply that

$$\frac{\int_{\mathbb{R}^n} \left\{ (v_1(x))^2 |V_2^R(x)| + (v_2^R(x))^2 |V_1(x)| \right\} dx}{\|u_R\|^2_{L^2(\mathbb{R}^n)}} \leq C \langle R \rangle^{-\beta}. \tag{4.22}$$

Combining (4.20)–(4.22) we obtain that

$$E(R) \leq -C^{-1} |R|^{2-n} \text{ for all } |R| \geq R_0. \tag{4.23}$$

Lower bound: Recall the Protter-Weinberger variational principle for $E(R)$ (see [5]);

$$E(R) = \sup_{u>0} \inf_{x \in \mathbb{R}^n} \left\{ \frac{P_R u(x)}{u(x)} \right\}. \tag{4.24}$$

We use the same function u_R as a test function for the Protter-Weinberger variational principle.

$$\begin{aligned} E(R) &\geq \inf_{x \in \mathbb{R}^n} \left\{ \frac{(-\Delta + V_1(x) + V_2^R(x)) u_R(x)}{u_R(x)} \right\} \\ &= \inf_{x \in \mathbb{R}^n} \left\{ \frac{V_1(x) v_2^R(x) + V_2^R(x) v_1(x)}{u_R(x)} \right\} \\ &\geq \inf_{x \in \mathbb{R}^n} \left\{ \frac{V_1(x) v_2^R(x)}{u_R(x)} \right\} + \inf_{x \in \mathbb{R}^n} \left\{ \frac{V_2^R(x) v_1(x)}{u_R(x)} \right\}. \end{aligned} \tag{4.25}$$

Using estimate (4.7) twice in (4.25), we obtain that

$$-C|R|^{2-n} \leq E(R),$$

which together with (4.23) imply the second part of the theorem and hence also the first part. $\square$

Remark 4.4 *(a) The case of critical potentials in* $\mathbb{R}^3$ *was treated in* [3, 6, 11, 12]*. The proof for the case* $n = 4$ *should be similar to the three dimensional case but the asymptotic behavior of* $E(R)$ *is different.*

(b) One can check that actually $\frac{(P_R u_R, u_R)}{\|u_R\|^2_{L^2(\mathbb{R}^n)}} \sim -C|R|^{2-n}$ *as* $|R| \to \infty$*.*

Let $R \in \mathbb{R}^n$, denote by

$$S_R = \left\{(t_1, t_2) | \mathcal{C}_{P+t_1V_1+t_2V_2^R}(\mathbb{R}^n) \neq \emptyset, \ \ t_1, t_2 \geq 0\right\}.$$

It follows from Theorem 3.1 in [8] that S_R is a convex compact set. Moreover, since the operators $P_i, i = 1, 2$ are critical, the points $(1, 0)$ and $(0, 1)$ are extreme points of S_R. In the following remark we discuss the asymptotic behavior of S_R which is a geometric interpretation of our results.

Remark 4.5 *The potentials* $V_i, i = 1, 2$ *tend to zero at infinity and the operators* P_i *are critical, therefore*

$$\limsup_{|R|\to\infty} S_R \subset [0, 1] \times [0, 1].$$

Theorem 3.1 *implies that*

$$\liminf_{|R|\to\infty} S_R \supset [0, 1) \times [0, 1).$$

On the other hand, Theorem 4.3 *implies that*

$$(1, 1) \notin \limsup_{|R|\to\infty} S_R.$$

5 Extensions to manifolds with group actions

The results which were described in the previous sections for small perturbations of the Laplacian on $\mathbb{R}^n$ can be generalized to a wider class of elliptic operators on certain noncompact manifolds. We now discuss these extensions. We consider a triple (X, G, P). Here

(i) X is a smooth noncompact connected countable at infinity manifold of dimension $n \geq 2$. We fix a sequence $\{X_k\}_{k=1}^{\infty}$ of smooth relatively compact subdomains which exhausts X and fix a reference point $x_0 \in X_1$. The point *infinity* (which by some abuse of the notations will be denoted by ∞) corresponds to the one-point compactification of the manifold X. So, a neighborhood of infinity in X is any open set of the form $X \setminus K$, where K is a compact set in X.

(ii) G is a group (with the discrete topology) acting on the manifold X by C^3-diffeomorphisms:

$$T : G \times X \to X, \quad T(g, x) = T_g x = gx.$$

Note that such an action induces a natural action on continuous functions. Namely, for any function u on X and any $g \in G$ one defines the function u^g by

$$u^g(x) = u(gx).$$

(iii) P is a linear second order elliptic operator which is defined on X, has the form (2.1) in any coordinate system and satisfies the following invariance condition

$$P\left(u^g\right) = (Pu)^g.$$

Such an operator is said to be a *G-invariant operator*. It is clear that if P is G-invariant then the Riemannian structure (metric, distance, etc.) of X which was introduced in Section 2 is also G-invariant (for more details see [4]).

The following are some important examples of elliptic operators which are invariant under some group actions.

Examples 5.1 *(a) The operator P is an elliptic operator on $\mathbb{R}^n$ with constant (respectively, one-periodic with respect to each variable) coefficients. Here the additive group $\mathbb{R}^n$ (respectively, $\mathbb{Z}^n$) acts on $\mathbb{R}^n$ by shifts. More generally, we may assume that the coefficients of P are independent on (respectively, periodic in) the variables $x_1, \ldots, x_k$, $1 \leq k \leq n$.*

(b) X is a noncompact Riemannian manifold, P is the Laplace-Beltrami operator on X and G is a subgroup of the group of all isometries of X. A particular case is when X is a noncompact covering of a compact Riemannian manifold Y, or X is a homogeneous space of a Lie group.

(c) The manifold X is $\mathbb{R}^n \setminus \{0\}$, $n \geq 2$ and G is the multiplicative group $\mathbb{R}^*$ of all positive numbers acting on X by homotheties: $x \longmapsto sx$, where $x \in X$ and $s \in \mathbb{R}^*$. An invariant operator for this action is of the form

$$P(x, \partial_x) = -|x|^2 \sum_{i,j=1}^{n} a_{ij}(x/|x|)\partial_i\partial_j + |x|\sum_{i=1}^{n} b_i(x/|x|)\partial_i + c(x/|x|),$$

where a_{ij}, b_i, c are Hölder continuous functions on the unit sphere.

In theorems 5.2 and 5.4 we study the asymptotic behavior of the ground state energy of the operator

$$P_g = P + V_g(x) = P + V_1(x) + V_2^g(x), \tag{5.1}$$

where P is a subcritical G-invariant elliptic operator and V_i, $i = 1, 2$, have *compact supports*. Theorem 5.2 is the generalization of Theorem 3.1.

Theorem 5.2 *Let P be a subcritical G-invariant elliptic operator on a non-compact connected manifold X with a given group action. Let $V_i(x) \in C_0^\alpha(X)$, $i = 1, 2$, $0 < \alpha \leq 1$, be two functions such that the operators $P_i = P + V_i(x)$, $i = 1, 2$, are subcritical in X.*

(i) Let $\{g_k\}_{k=1}^\infty \subset G$ be a sequence of elements of the group G such that

$$\lim_{k\to\infty} g_k x_0 = \lim_{k\to\infty} g_k^{-1} x_0 = \infty,$$

and

$$\lim_{k\to\infty} G_P^X(g_k x_0, x_0) = \lim_{k\to\infty} G_P^X(g_k^{-1} x_0, x_0) = 0. \tag{5.2}$$

Then there exists $k_0 \in \mathbb{N}$ such that the operator P_{g_k} is subcritical for all $k \geq k_0$ (P_g is defined by (5.1)*).*

(ii) If

$$\lim_{x\to\infty} G_P^X(x, x_0) = 0. \tag{5.3}$$

Then there exists $k_0 \in \mathbb{N}$ such that the operator P_g is subcritical for all $g \in G$ such that $gx_0, g^{-1}x_0 \in X \setminus X_{k_0}$.

Sketch of the proof: (i) We shall show that P_{g_k} admits a positive supersolution in X for k sufficiently large. Let $G_{P_i}^X(x, y), i = 1, 2$ be the Green functions of the operators P_i in X. Define

$$v_i(x) = \int_X G_{P_i}^X(x, y)|V_i(y)|dy. \tag{5.4}$$

Since P is G-invariant and V_i have compact supports it follows that for every $g \in G$

$$(P + V_i^g(x))\, v_i^g(x) = \{(P + V_i(x))(v_i(x))\}^g = |V_i^g(x)|. \tag{5.5}$$

By our assumption (5.2) and the Harnack inequality

$$\lim_{k\to\infty} v_i^{g_k}(x) = \lim_{k\to\infty} v_i^{g_k^{-1}}(x) = 0 \tag{5.6}$$

uniformly in any compact set in X. Consider the function

$$u_k(x) = v_1(x) + v_2^{g_k}(x). \tag{5.7}$$

Then by (5.5) and (5.6) we have for k sufficiently large

$$P_{g_k} u_k(x) = (P + V_1(x) + V_2^{g_k}(x))u_k(x) =$$

$$|V_1(x)| + V_1(x)v_2^{g_k}(x) + |V_2^{g_k}(x)| + V_2^{g_k}(x)v_1(x) \geq 0.$$

Thus, P_{g_k} is subcritical in X.

Part (*ii*) is proved similarly. □

For the critical case we have to assume that the subcritical G-invariant operator P is *formally self-adjoint* on X. That is, in any coordinate system $(U; x_1, \ldots, x_n)$ the operator P has the form

$$P(x, \partial_x) = -\sum_{i,j=1}^{n} a^{-1/2}(x)\partial_i(a^{1/2}(x)a_{ij}(x))\partial_j + c(x),$$

where $a^{-1}(x) = det[a_{ij}(x)]$. We assume that (2.2) is satisfied and that the coefficients a_{ij}, $\partial_i(a^{1/2}a_{ij})$ and c are real and locally Hölder continuous. For simplicity, we also assume that the *critical* potentials V_i, $i = 1, 2$ have compact supports.

The following lemma is essential for studying the asymptotic behavior of $E(g) = \lambda_0(P_g, X)$.

Lemma 5.3 *Assume that the G-invariant operator P is subcritical and formally self-adjoint on X. Let $V_i \in C_0^\alpha(X)$,$i = 1, 2$ be two functions such that the operators $P_i = P + V_i(x)$, $i = 1, 2,$ are critical in X and denote by v_i the corresponding ground states.*

Let $\{g_k\}_{k=1}^\infty \subset G$ be a sequence of elements of the group G such that the sequence $\{g_k x_0\}_{k=1}^\infty$ is a fundamental sequence with respect to the Martin topology corresponding to a minimal point of the Martin boundary and such that

$$\lim_{k\to\infty} G_P^X(g_k x_0, x_0) = 0. \tag{5.8}$$

(a) There exist positive constants C_i such that

$$\lim_{k\to\infty} \frac{v_i(g_k x)}{G_P^X(g_k x, x_0)} = C_i, \quad i = 1, 2, \quad x \in X. \tag{5.9}$$

(b) There exist $k_0 \in \mathbb{N}$ and $C > 0$ such that for all $k \geq k_0$ the following inequalities hold:

(b.i)

$$\int_X v_2^{g_k}(x) V_1(x) v_1(x) dx \leq -C^{-1} G_P^X(g_k x_0, x_0), \tag{5.10}$$

(b.ii)

$$\int_X v_1^2(x) |V_2^{g_k}(x)| dx \leq C \left(G_P^X(g_k x_0, x_0) \right)^2, \tag{5.11}$$

(b.iii)

$$\inf_{x\in X} \left\{ \frac{V_1(x) v_2^{g_k}(x)}{v_1(x) + v_2^{g_k}(x)} \right\} \geq -C G_P^X(g_k x_0, x_0). \tag{5.12}$$

(c) Suppose further that the Martin boundary of X with respect to P is a one point set and

$$\lim_{x\to\infty} G_P^X(x, x_0) = 0. \tag{5.13}$$

Then the limits

$$\lim_{x\to\infty} \frac{v_i(x)}{G_P^X(x, x_0)} = C_i, \quad i = 1, 2, \tag{5.14}$$

exist and there exist positive constants C and k_0 such that inequalities (5.10)–(5.12) *hold for all $g \in G$ such that $g x_0 \in X \setminus X_{k_0}$.*

Proof: (a) It follows from Lemma 2.5 that if $g_k x \in X \setminus X_1$ then the quotient in (5.9) is bounded and bounded away from zero. Therefore, we only need to prove that the limit in (5.9) does exist. By Lemma 2.5 the ground states v_i satisfy the equation

$$v_i(x) = -\int_X G_P^X(x, y) V_i(y) v_i(y) dy. \tag{5.15}$$

Using the symmetricity of the Green function $G_P^X(x, y)$ with respect to x and y, the Harnack inequality and our assumption that the sequence $\{g_k x_0\}$ is a fundamental sequence corresponding to a *minimal* Martin function $u \in$

$\mathcal{C}_P(X)$ we conclude that for every $x \in X$ the sequence $\{g_k x\}$ is also a Martin sequence which is equivalent to $\{g_k x_0\}$ and

$$\lim_{k\to\infty} \frac{G_P^X(g_k x, y)}{G_P^X(g_k x, x_0)} = u(y)$$

uniformly in any compact set in X. Dividing Equation (5.15) by $G_P^X(g_k x, x_0)$ and letting $k \to \infty$ we see that

$$\lim_{k\to\infty} \frac{v_i(g_k x)}{G_P^X(g_k x, x_0)} = -\int_X u(y) V_i(y) v_i(y) dy = C_i > 0. \tag{5.16}$$

$(b.i)$ It follows from part (a) and (5.15) that

$$\int_X v_2^{g_k}(x) V_1(x) v_1(x) dx = C_2 \int_X G_P^X(g_k x, x_0) V_1(x) v_1(x) dx +$$

$$\int_X \left(\frac{v_2^{g_k}(x)}{G_P^X(g_k x, x_0)} - C_2 \right) G_P^X(g_k x, x_0) V_1(x) v_1(x) dx = -C_2 v_1(g_k^{-1} x_0) +$$

$$\int_X \left(\frac{v_2^{g_k}(x)}{G_P^X(g_k x, x_0)} - C_2 \right) G_P^X(g_k x, x_0) V_1(x) v_1(x) dx. \tag{5.17}$$

Lemma 2.5 (ii) implies that there exists $C_3 > 0$ such that for all $x \in X$ and $g_k \in G$ so that $g_k x \in X \setminus X_1$

$$-C_2 v_1(g_k x) < -C_3 G_P^X(g_k x, x_0), \tag{5.18}$$

and

$$\left| \frac{v_2^{g_k}(x)}{G_P^X(g_k x, x_0)} - C_2 \right| \le C_3^{-1}. \tag{5.19}$$

Now, using (5.9) and Harnack inequality one sees that if $g_k x_0 \in X \setminus X_{k_0}$ then

$$\int_X \left| \frac{v_2^{g_k}(x)}{G_P^X(g_k x, x_0)} - C_2 \right| G_P^X(g_k x, x_0) |V_1(x)| v_1(x) dx \le 0.3 C_3 G_P^X(g_k x_0, x_0). \tag{5.20}$$

Applying estimates (5.18) and (5.20) in (5.17) we obtain

$$\int_X v_2^{g_k}(x) V_1(x) v_1(x) dx \le -0.3 C_3 G_P^X(g_k x_0, x_0).$$

($b.ii$) Using the Harnack inequality and Lemma 2.5 we obtain

$$\int_X v_1^2(x)|V_2^{g_k}(x)|dx = \int_X \left(v_1^{g_k^{-1}}(x)\right)^2 |V_2(x)|dx \leq$$

$$C\left(G_P^X(g_k^{-1}x_0, x_0)\right)^2 = C\left(G_P^X(g_k x_0, x_0)\right)^2.$$

($b.iii$) Recall that P_1 is critical and therefore, V_1 is negative somewhere in X. So, the infimum in (5.12) is indeed negative. Estimate (5.12) follows easily from assumption (5.8), the Harnack inequality and Lemma 2.5.

(c) The proof is similar to the proofs of parts (a) and (b). □

We shall use Lemma 5.3 to prove the following extension of Theorem 4.3.

Theorem 5.4 *Assume that P is a subcritical G-invariant formally self-adjoint operator on X. Let $V_i \in C_0^\alpha(X)$, $i = 1, 2$ be two functions so that the operators $P_i = P + V_i(x)$, $i = 1, 2$, are critical in X and admit $L^2(X)$ normalized ground states v_i with eigenvalues zero.*

(i) Let $\{g_k\}_{k=1}^\infty \subset G$ be a sequence of elements of the group G such that the sequences $\{g_k x_0\}_{k=1}^\infty$ and $\{g_k^{-1}x_0\}_{k=1}^\infty$ are fundamental sequences with respect to the Martin topology corresponding to minimal points of the Martin boundary and such that

$$\lim_{k\to\infty} G_P^X(g_k x_0, x_0) = 0. \tag{5.21}$$

Then there exists $k_0 \in \mathbb{N}$ so that the operator P_{g_k} is supercritical for all $k \geq k_0$.

(ii) If the Martin boundary of X with respect to P is a one point set and

$$\lim_{x\to\infty} G_P^X(x, x_0) = 0, \tag{5.22}$$

then there exists $k_0 \in \mathbb{N}$ such that the operator P_g is supercritical for all $g \in G$ with $gx_0, g^{-1}x_0 \in X \setminus X_{k_0}$. Moreover, there exists a positive constant C so that the ground state energy $E(g) = \lambda_0(P_g, X)$ of the operator P_g satisfies

$$-CG_P^X(gx_0, x_0) \leq E(g) \leq -C^{-1}G_P^X(gx_0, x_0) \tag{5.23}$$

for all $g \in G$ such that $gx_0, g^{-1}x_0 \in X \setminus X_{k_0}$.

Proof: The proof of the theorem is quite similar to the proof of Theorem 4.3. So, we prove only part (*ii*). Since the operator P_g is supercritical if and only if $E(g) < 0$ it is enough to prove Inequality (5.23).

Upper bound: Consider the operator H_g the maximal realization of P_g which is defined in a similar way as in (4.3). Take the function

$$u_g(x) = v_1(x) + v_2^g(x) \tag{5.24}$$

as a test function in the Rayleigh-Ritz quotient for the operator H_g. Note that since $v_i, i = 1, 2$ are positive functions in the unit sphere of $L^2(X)$ it follows that

$$2 \leq \|u_g\|_{L^2(X)} \leq 4. \tag{5.25}$$

Furthermore,

$$P_g u_g(x) = V_2^g(x) v_1(x) + V_1(x) v_2^g(x).$$

Hence,

$$\begin{aligned} E(g) &= \inf_{u \in D(H_g)} \left\{ \frac{(P_g u, u)}{\|u\|^2_{L^2(X)}} \right\} \leq \frac{(P_g u_g, u_g)}{\|u_g\|^2_{L^2(X)}} \\ &= \frac{\int_X \{v_1(x)V_2^g(x)v_2^g(x) + v_1(x)V_1(x)v_2^g(x)\}\, dx}{\|u_g\|^2_{L^2(X)}} + \\ &\quad \frac{\int_X \{v_1^2(x)V_2^g(x) + (v_2^g(x))^2 V_1(x)\}\, dx}{\|u_g\|^2_{L^2(X)}}. \end{aligned} \tag{5.26}$$

Since the Green function $G_P^X(x, y)$ is symmetric in x and y and satisfies $G_P^X(gx, gy) = G_P^X(x, y)$ for all $g \in G$ we have

$$G_P^X(gx_0, x_0) = G_P^X(g^{-1}x_0, x_0). \tag{5.27}$$

Using (5.10), (5.25) and (5.27) we find that

$$\frac{\int_X \{v_1(x)V_2^g(x)v_2^g(x) + v_1(x)V_1(x)v_2^g(x)\}\, dx}{\|u_g\|^2_{L^2(X)}} \leq -C G_P^X(gx_0, x_0) \tag{5.28}$$

for all $g \in G$ such that $gx_0, g^{-1}x_0 \in X \setminus X_{k_0}$. On the other hand, (5.11), (5.25) and (5.27) imply that

$$\frac{\int_X \{v_1^2(x)|V_2^g(x)| + (v_2^g(x))^2 |V_1(x)|\}\, dx}{\|u_g\|^2_{L^2(X)}} \leq C \left(G_P^X(gx_0, x_0)\right)^2. \tag{5.29}$$

Combining (5.22) and (5.26)–(5.29) we obtain that

$$E(g) \leq -CG_P^X(gx_0, x_0) \tag{5.30}$$

for all $g \in G$ such that $gx_0, g^{-1}x_0 \in X \setminus X_{k_0}$.

Lower bound: Applying the Protter-Weinberger variational principle for $E(g)$ (see (4.24)) with the test function u_g we obtain

$$\begin{aligned} E(g) &\geq \inf_{x\in X}\left\{\frac{(P+V_1(x)+V_2^g(x))u_g(x)}{u_g(x)}\right\} \\ &= \inf_{x\in X}\left\{\frac{V_1(x)v_2^g(x)+V_2^g(x)v_1(x)}{u_g(x)}\right\} \\ &\geq \inf_{x\in X}\left\{\frac{V_1(x)v_2^g(x)}{u_g(x)}\right\}+\inf_{x\in X}\left\{\frac{V_2^g(x)v_1(x)}{u_g(x)}\right\}. \end{aligned} \tag{5.31}$$

Using estimate (5.12) twice in (5.31), we obtain that

$$-CG_P^X(gx_0, x_0) \leq E(g),$$

which together with (5.30) imply (5.23) and hence part (*ii*) of the theorem. □

Remarks 5.5 *(a) As it can be seen from our proofs the lower bounds in* (5.23) *and also in* (4.16) *hold true even when $v_i \notin L^2$.*
(b) In the general case, when P is not formally self adjoint in X (and v_i are not necessarily in $L^2(X)$) one obtains the following lower bound

$$E(g) \geq -C\left(G_P^X(gx_0, x_0) + G_P^X(g^{-1}x_0, x_0)\right).$$

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