ON PRINCIPAL EIGENVALUES FOR INDEFINITE-WEIGHT ELLIPTIC PROBLEMS

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1 INTRODUCTION

Consider the quantum mechanical system $H_\mu = -\Delta - \mu V$ in $\mathbb{R}^d$, where $\mu \in \mathbb{R}$ is a spectral parameter and $V \in C_0^\infty(\mathbb{R}^d)$. It is well known that for $d \geq 3$, the Schrödinger operator H_μ has no bound states provided that $|\mu|$ is sufficiently small. On the other hand, for $d = 1, 2$, B. Simon proved the following delicate result (see [37, 35], and also the discussion in the Notes of [35]).

Theorem 1.1 *Suppose that $d = 1, 2$. Then H_μ has a negative eigenvalue for all negative μ if and only if*

$$\int_{\mathbb{R}^d} V(x)dx \leq 0.$$

The explanation of the above phenomena is closely related to the following notions.

Let P be a linear, second order, elliptic operator defined in a subdomain Ω of a noncompact, connected, Riemannian manifold X of dimension d. Here P is an elliptic operator with real Hölder continuous coefficients which in any coordinate system $(U; x_1, \ldots, x_d)$ has the form

$$P(x, \partial_x) = -\sum_{i,j=1}^{d} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{d} b_i(x)\partial_i + c(x), \tag{1.1}$$

where $\partial_i = \partial/\partial x_i$. We assume that for every $x \in \Omega$ the real quadratic form

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j, \quad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d \tag{1.2}$$

is positive definite.

We denote the cone of all positive (classical) solutions of the elliptic equation $Pu = 0$ in Ω by $\mathcal{C}_P(\Omega)$. We denote by P^* the formal adjoint of P.

Let $\{\Omega_k\}_{k=1}^{\infty}$ be *an exhaustion of* Ω, i.e. a sequence of smooth, relatively compact domains such that $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\cup_{k=1}^{\infty}\Omega_k = \Omega$. Assume that $\mathcal{C}_P(\Omega) \neq \emptyset$. Then for every $k \geq 1$ the Dirichlet Green function $G_P^{\Omega_k}(x,y)$ exists and is positive. By the generalized maximum principle, $\{G_P^{\Omega_k}(x,y)\}_{k=1}^{\infty}$ is an increasing sequence which, by the elliptic Harnack inequality, converges uniformly in any compact subdomain of Ω either to $G_P^{\Omega}(x,y)$, the positive *minimal Green function* of P in Ω and P is said to be *subcritical operator* in Ω, or to infinity and in this case P is *critical* in Ω. The operator P is said to be *supercritical* if $\mathcal{C}_P(\Omega) = \emptyset$ [23, 26, 32].

These notions of criticality and subcriticality which were first introduced by B. Simon for Schrödinger operators [38] have been proved to be valuable tools for proving many results concerning positive solutions of linear (and sometimes nonlinear) elliptic and parabolic equations (see for example, [16, 19, 20, 23, 28, 30, 31, 32, 38, 40] and the references therein). The criticality theory approach has the advantage that it applies to general critical or subcritical operators in a general domain.

It follows that P is critical (resp. subcritical) in Ω if and only if P^* is critical (resp. subcritical) in Ω. Furthermore, if P is critical in Ω then $\mathcal{C}_P(\Omega)$ is a one-dimensional cone and any positive supersolution of the equation $Pu = 0$ in Ω is a solution. In this case $\phi \in \mathcal{C}_P(\Omega)$ is called a *ground state of P in Ω* [1].

Remark 1.2 We would like to point out that the criticality theory is also valid for the class of weak solutions of elliptic equations in divergence form and also for the class of strong solutions of strongly elliptic equations with locally bounded coefficients. Nevertheless, for the sake of clarity we prefer to present our results for the class of classical solutions.

Subcriticality is a stable property in the following sense. If P is subcritical in Ω and $V \in C_0^{\alpha}(\Omega)$ then there exists $\epsilon > 0$ such that $P - \mu V$ is subcritical for all $|\mu| < \epsilon$ [23, 26]. On the other hand, if P is critical in Ω and $V \in C^{\alpha}(\Omega)$ is a nonzero, nonnegative function then for any $\epsilon > 0$ the operator $P + \epsilon V$ is subcritical and $P - \epsilon V$ is supercritical in Ω.

Note that $P = -\Delta$ is critical in $\mathbb{R}^d$ if $d = 1, 2$, and subcritical for $d \geq 3$. Moreover, the Schrödinger operator H_μ considered in Theorem 1.1 has a negative eigenvalue if and only if H_μ is supercritical.

We turn now to the critical case. The following result demonstrates again the instability of criticality. The theorem which is the main result of [28] extends Theorem 1.1 to the case of a general (non-selfadjoint), linear, second order elliptic operator which is critical in a domain Ω.

Theorem 1.3 *Let P be a critical elliptic operator on Ω and let ϕ_0 (resp. $\tilde{\phi}_0$) be the ground state of the operator P (resp. P^*) in Ω. Assume that $W \in C_0^{\alpha}(\Omega)$. Then there exists $\mu < 0$ such that $P - \mu W(x)$ is subcritical in Ω if and only if*

$$\int_{\Omega} W(x)\phi_0(x)\tilde{\phi}_0(x)dx > 0. \tag{1.3}$$

This type of result appears first in a theorem of Picone for the following Neumann indefinite eigenvalue problem

$$\begin{aligned} -u'' &= tW(x)u \quad x \in (a,b) \\ u'(a) &= u'(b) = 0 \end{aligned}$$

([25], see also [8]). The indefinite character of the problem is due to the changing of sign of the function W.

In the last two decades theorems of this kind were proved for linear and nonlinear problems by many authors (see [2, 4, 5, 7, 9, 10, 13, 14, 15, 18, 21, 28, 36, 37, 39] and the references therein). Most of these results deal with either symmetric operators or problems in smooth, bounded domains. So, one can either use variational characterization of the principal eigenvalue or perturbation spectral theory of compact operators and the Krein-Rutman theorem.

The study of such problems has been motivated in part by the wish to understand related linear and nonlinear indefinite boundary value problems such as

$$Pu = \mu W(x) f(u(x)) \quad \text{for } x \in \Omega, \quad u = 0 \quad \text{for } x \in \partial\Omega. \tag{1.4}$$

Problems of this kind appear in mathematical physics and in many reaction diffusion problems like population genetics, ecological problems and superconductivity.

The aim of this paper is to extend Theorem 1.3 to a wider class of perturbations which we call *weak perturbations* (see Theorem 4.1). It turns out that small and semismall perturbations which were studied in [24, 27] are weak perturbations. Moreover, most of the results of the type of Theorem 1.3 which were mentioned above are special cases of Theorem 4.1.

The outline of this paper is as follows. In Section 2, we give some basic definitions, collect some results and fix notations. In Section 3, we define the notion of weak perturbation of subcritical and critical operators, study some of the properties and give some examples of such perturbations. Finally, our main result (Theorem 4.1) is proved in Section 4.

2 PRELIMINARIES

In this section we collect some terminology and results which we need in this paper.

Consider a noncompact, connected, C^3-smooth Riemannian manifold X of dimension d. We associate to any subdomain $\Omega \subseteq X$ an exhaustion $\{\Omega_n\}_{n=1}^{\infty}$ and a sequence $\{\chi_n(x)\}_{n=1}^{\infty}$ of smooth cutoff functions in Ω such that $\chi_n(x) \equiv 1$ in Ω_n, $\chi_n(x) \equiv 0$ in $\Omega \setminus \Omega_{n+1}$, and $0 \leq \chi_n(x) \leq 1$ in Ω. Let $W \in C^{\alpha}(\Omega)$, $0 < \alpha \leq 1$. We denote $W_n(x) = \chi_n(x)W(x)$ and $W_n^*(x) = W(x) - W_n(x)$. We say that a function $f \in C(\Omega)$ *vanishes at infinity of* Ω if for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|f(x)| < \epsilon$ in $\Omega \setminus \Omega_n$.

The next notion was introduced by S. Agmon in [1].

Definition 2.1 Let P be an elliptic operator defined in a domain $\Omega \subseteq X$. A function $u \in C(\Omega \setminus \Omega_n)$ is said to be a *positive solution of the operator P of minimal growth in a neighborhood of infinity in* Ω if u satisfies the following two conditions:

(i) There exists $n \in \mathbb{N}$ such that u is a positive solution of the equation $Pu = 0$ in $\Omega \setminus \overline{\Omega_n}$;

(ii) If v is a continuous function on $\Omega \setminus \Omega_k$ for some $k \geq n$ which is a positive solution of the equation $Pu = 0$ in $\Omega \setminus \overline{\Omega_k}$, and $u \leq v$ on $\partial\Omega_k$, then $u \leq v$ on $\Omega \setminus \Omega_k$.

Remark 2.2 If P is subcritical in Ω then $G_P^\Omega(x,x_0)$ is a positive solution of the equation $Pu=0$ of minimal growth in a neighborhood of infinity in Ω. On the other hand, if P is critical in Ω then the ground state is a positive solution of the equation $Pu=0$ in Ω which has minimal growth in a neighborhood of infinity in Ω.

Definition 2.3 Let $P_i,\ i=1,\ 2$ be two subcritical operators in $\Omega\subseteq X$. We say that the Green functions $G_{P_1}^\Omega(x,y)$ and $G_{P_2}^\Omega(x,y)$ are *equivalent* (resp. *semi-equivalent*) if there exists $C>0$ such that

$$C^{-1}G_{P_2}^\Omega(x,y)\leq G_{P_1}^\Omega(x,y)\leq CG_{P_2}^\Omega(x,y) \tag{2.1}$$

for all $x,\ y\in\Omega,\ x\neq y$ (resp. for some $x\in\Omega$ and all $y\in\Omega\setminus\{x\}$).

Many papers deal with sufficient conditions, in terms of proximity near infinity in Ω between two given subcritical operators, which imply that the Green functions are equivalent (see, [6, 23, 24, 26, 27, 29, and the references therein]). The following notion is closely related to this problem.

Definition 2.4 Let P be a subcritical operator in $\Omega\subseteq X$ and let $W\in C^\alpha(\Omega)$. We say that W is a *small perturbation* of P in Ω if

$$\lim_{k\to\infty}\left\{\sup_{x,y\in\Omega\setminus\Omega_k}\int_{\Omega\setminus\Omega_k}\frac{G_P^\Omega(x,z)|W(z)|G_P^\Omega(z,y)}{G_P^\Omega(x,y)}\,dz\right\}=0\,. \tag{2.2}$$

The following refinement of Definition 2.4 was recently introduced by M. Murata [24].

Definition 2.5 Let P be a subcritical operator in Ω and let $W\in C^\alpha(\Omega)$. Fix a reference point $x_0\in\Omega_1$. We say that W is a *semismall perturbation* of P in Ω if

$$\lim_{k\to\infty}\left\{\sup_{y\in\Omega\setminus\Omega_k}\int_{\Omega\setminus\Omega_k}\frac{G_P^\Omega(x_0,z)|W(z)|G_P^\Omega(z,y)}{G_P^\Omega(x_0,y)}\,dz\right\}=0\ . \tag{2.3}$$

A small perturbation is semismall [24]. It is known [24, 27] that if the operators P and $P+W$ are subcritical in Ω and W is a small (resp. semismall) perturbation of P in Ω then $G_P^\Omega(x,y)$ and $G_{P+W}^\Omega(x,y)$ are equivalent (resp. semi-equivalent). This follows from pointwise estimates of the iterated Green kernels corresponding to the Neumann series expansion of the Green function $G_{P+\epsilon W}^\Omega(x,y)$ in terms of $G_P^\Omega(x,y)$ and W. Moreover, the corresponding Martin boundaries are homeomorphic.

On the other hand, suppose that W is a nonzero function such that the Green functions $G_P^\Omega(x,y)$ and $G_{P+|W|}^\Omega(x,y)$ are (resp. semi-) equivalent and let $K\in C^\alpha(\Omega)$ be an arbitrary function which vanishes at infinity of Ω. Then the function $K(x)W(x)$ is a (resp. semi-) small perturbation of the operator P in Ω. Moreover, using the resolvent equation it follows that if the (best) equivalence constant of $G_P^\Omega(x,y)$ and $G_{P+|W_n^*|}^\Omega(x,y)$ tends to 1 as n tends to infinity then W is a (resp. semi-) small perturbation.

Remark 2.6 There are several examples where the semi-equivalence of the Green functions implies that the perturbation is small.

Another class of perturbations was recently studied by A. Ancona [6]. In this paper, Ancona investigates the question of the equivalence of the Green functions of two uniformly elliptic, weakly coercive operators with uniformly bounded coefficients. The perturbation is allowed to be not only in the zero order term but also in the higher order terms. It is proved that the Green functions and the Martin boundaries of such two operators are equivalent provided that the "distance near infinity" between the operators is sufficiently small. Moreover, the equivalence constant tends to 1 if the supports of the perturbations are concentrated at infinity in Ω. Therefore, a zero order perturbation of Ancona's type is a small perturbation. Note that unlike the notion of small and semismall perturbations the definition of Ancona's distance does not depend explicitly on the behavior of the Green functions at infinity but only on the difference between the coefficients of the operators.

The discussion above and in paprticular Remark 2.6 implies that the results in [6, 24, 27] concerning the equivalence of the Green functions and the Martin boundaries seems to be almost optimal. Nevertheless, it turns out that Theorem 1.3 can be extended to a wider class of perturbations, namely, *weak perturbations*.

Throughout this paper we use the following key lemma many times (see [28, Theorem 3.1]). For the completeness, we present here an outline of the proof.

Lemma 2.7 *Let P be an elliptic operator defined on a domain $\Omega \subseteq X$ and assume that P is of the form (1.1). Consider the one parameter family of operators $P_t = P+tW+(1-t)V$, where $V, W \in C^\alpha(\Omega)$ and $0 \le t \le 1$. Suppose that $\mathcal{C}_{P_i}(\Omega) \neq \emptyset, i = 0, 1$. Then $\mathcal{C}_{P_t}(\Omega) \neq \emptyset$ for all $0 < t < 1$. Moreover, P_t is subcritical for all $0 < t < 1$ unless $P_0 = P_1$ and P_0 is critical in Ω.*

Proof: Let v_i be positive supersolutions of the equations $P_i u = 0$ in Ω, $i = 0, 1$. One can check easily that if $0 < t < 1$ then

$$v_t(x) = (v_0(x))^{1-t}(v_1(x))^t$$

is a positive supersolution of the equation $P_t u = 0$ in Ω. Moreover, v_t is a solution of $P_t u = 0$ in Ω if and only if $P_0 = P_1$ and $v_0 = Cv_1 \in \mathcal{C}_{P_0}(\Omega)$. These properties easily imply the statements of the Lemma. □

3 WEAK PERTURBATION

In this section we introduce the notion of weak perturbation and study some properties of such kind of perturbations.

Definition 3.1 Let P be a subcritical operator in $\Omega \subseteq X$. A function $W \in C^\alpha(\Omega)$ is said to be *a weak perturbation* of the operator P in Ω if the following condition holds true.

(*) For every $\eta \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that the operator $P - \eta W_n^*(x)$ is subcritical in Ω for any $n \ge N$.

A function $W \in C^\alpha(\Omega)$ is said to be *a weak perturbation* of a critical operator P in Ω if there exists a nonzero, nonnegative function $V \in C_0^\alpha(\Omega)$ such that the function W is a weak perturbation of the subcritical operator $P + V$ in Ω.

Remarks 3.2 1. Using Lemma 2.7 it can be easily shown that $W \in C^\alpha(\Omega)$ is a weak perturbation of a critical operator P in Ω if and only if for *any* nonzero, nonnegative function $V \in C_0^\alpha(\Omega)$ the function W is a weak perturbation of the subcritical operator $P+V$ in Ω.

2. It follows immediately from Definition 3.1 that if W is a weak perturbation of P in Ω then it is also a weak perturbation of $P+V$ for any nonnegative function $V \in C^\alpha(\Omega)$. Also, the set of all functions W which are weak perturbations of P in Ω is a vector space.

3. If $|W|$ is a weak perturbation of a subcritical operator P in Ω then P and W satisfy $(*)$ with respect to any exhaustion of Ω and any associated sequence of cutoff functions.

4. Using the results in [24, 27], it follows that if W is a small or even semismall perturbation of an operator P in Ω then $|W|$ is a weak perturbation of P in Ω. In particular, if one deals only with zero order perturbations then the perturbations considered recently by A. Ancona [6] are also weak. For example, if W is a perturbation of a weakly coercive operator P such that the operators P and $P+W$ satisfy the assumptions of Remark 1.2 in [6] then $|W|$ is a weak perturbations of P in X.

5. One can use known integral estimates on the number of the negative eigenvalues of subcritical operators to show that a perturbation is weak. For example, let $d \geq 3$. By the Cwikel-Lieb-Rozenblum bound (see, [35]), if $W \in L^{d/2}(\mathbb{R}^d)$ then $|W|$ is a weak perturbation of $-\Delta$ in $\mathbb{R}^d$. Moreover, such a function W is a weak perturbation of a symmetric subcritical operator P with periodic coefficients in $\mathbb{R}^d$ [11]. For further results in this direction see also [12, 17]. Note that $W(x) = (1+|x|)^{-2}$ is not a weak perturbation of $-\Delta$ in $\mathbb{R}^d, d \geq 3$. On the other hand, for any $\epsilon > 0$ the function $W(x) = (1+|x|)^{-(2+\epsilon)}$ is a small perturbation of $-\Delta$ in $\mathbb{R}^d, d \geq 3$ [23, 26, 29, 33, 34]. For some other examples of weak perturbations, see [23, 24, 26, 27, 29, and the references therein].

The following example shows that a weak perurbation is a weaker notion than semismall perturbation.

Example 3.3 Let $P = -\Delta$ in $\mathbb{R}^d$, $d \geq 3$, and consider a nonnegative function $W \in L^{d/2}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} W(y)(1+|y|)^{2-d} dy = \infty \,.$$

By the Cwikel-Lieb-Rozenblum inequality W is a weak perturbation. Suppose that W is a semismall perturbation, then the ground state ϕ_0 of the operator $-\Delta - \lambda_0 W$ behaves near infinty like the Green function of the Laplacian and therefore,

$$\begin{aligned}
\phi_0(x) &= \lambda_0 \int_{\mathbb{R}^d} G_{-\Delta}^{\mathbb{R}^d}(x,y) W(y)\phi_0(y) dy \\
&\geq C \int_{\mathbb{R}^d} |x-y|^{2-d} W(y)(1+|y|)^{2-d} dy \\
&\geq C \int_{|x-y|<2|x|} |x|^{2-d} W(y)(1+|y|)^{2-d} dy \\
&\geq C|x|^{2-d} \int_{|y|<|x|} W(y)(1+|y|)^{2-d} dy \geq C\phi_0(x) F(|x|),
\end{aligned}$$

where $F(s)$ is a real function which tends to infinity as s tends to infinity. This is a contradiction.

Lemma 3.4 *(i) If W is a weak perturbation of a subcritical operator P in Ω then there exists $\epsilon > 0$ such that the operator $P - \mu W(x)$ is subcritical in Ω for any $|\mu| < \epsilon$. Moreover, if the operator $P - \mu W$ is subcritical in Ω then there exists $\delta > 0$ such that $P - (\mu + t)W$ is subcritical in Ω for all $|t| < \delta$.*

(ii) If W is a weak perturbation of a critical or subcritical operator P in Ω and for some $\mu \in \mathbb{R}$ the operator $P - \mu W(x)$ is subcritical or critical in Ω then W is also a weak perturbation of the operator $P - \mu W(x)$ in Ω.

Proof: (i). There exists $N \in \mathbb{N}$ such that $P \pm W_N^*$ is subcritical in Ω. Since W_N has compact support and P is subcritical there exists $\delta > 0$ such that $P \pm \delta W_N$ is subcritical in Ω. Set $\epsilon = \frac{\delta}{1+\delta}$. It follows from Lemma 2.7 that $P \pm \epsilon W_N^* \pm (1-\epsilon)\delta W_N = P \pm \epsilon W$ is subcritical in Ω. Thus, by Lemma 2.7, $P - \mu W$ is subcritical in Ω for any $|\mu| \leq \epsilon$

Suppose now that $P - \mu W$ is subcritical in Ω for some $\mu > 0$ (the case $\mu < 0$ can be treated similarly). It is enough to show that $P - \mu_1 W$ is subcritical in Ω for some $\mu_1 > \mu$. There exists $N \in \mathbb{N}$ such that $P - 2\mu W_N^*$ is subcritical in Ω. On the other hand, there exists $\epsilon > 0$ such that $P - \mu W - \epsilon W_N$ is subcritical in Ω. Set $\beta = \frac{2\mu}{2\mu+\epsilon}$. Then $P - \beta(\mu W + \epsilon W_N) - (1-\beta)2\mu W_N^* = P - \mu_1 W$ is subcritical in Ω, where $\mu_1 = \frac{2\mu(\mu+\epsilon)}{2\mu+\epsilon}$. Since $\mu_1 > \mu$ the first part of the lemma is proved.

(ii). Without loss of generality, we may assume that $\mu > 0$. Suppose first that P and $P - \mu W$ are subcritical in Ω. By the first part of the lemma, there exists $\delta = \delta(\mu) > 0$ such that $P - (\mu + \delta)W$ is subcritical. On the other hand, for every $\eta \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $P - \eta W_n^*$ is subcritical in Ω for all $n \geq N$. Set $\beta = \frac{\mu}{\mu+\delta}$. Then $P - \beta(\mu+\delta)W - (1-\beta)\eta W_N^* = P - \mu W - \tilde{\eta} W_n^*$ is subcritical in Ω for all $n \geq N$, where $\tilde{\eta} = \frac{\delta\eta}{\mu+\delta}$. Hence, we can make $|\tilde{\eta}|$ arbitrarily large provided we choose $|\eta|$ large enough.

Suppose now that P is subcritical or critical and $P - \mu W$ is critical in Ω. Then for any nonzero, nonnegative $V \in C_0^\alpha(\Omega)$, the operators $P + V$ and $P + V - \mu W$ are subcritical. Moreover, By Remark 3.2, W is a weak perturbation of $P + V$ in Ω. Therefore, we are again in the situation of the the previous case.

Finally, assume that P is critical and $P - \mu W$ is subcritical in Ω. Take a nonzero, nonnegative $V \in C_0^\alpha(\Omega)$. There exists $\epsilon = \epsilon(\mu) > 0$ such that $P - \mu W - \epsilon V$ is subcritical in Ω. On the other hand, W is a weak perturbation of $P + V$ in Ω and by the previous part, W is also a weak perturbation of the subcritical operator $P + V - \mu W$. Consequently, for every $\eta \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $P + V - \mu W - \eta W_n^*$ is subcritical for all $n \geq N$. Take $\beta = \frac{\epsilon}{1+\epsilon}$. It follows that $P + \beta(V - \mu W - \eta W_n^*) + (1 - \beta)(-\mu W - \epsilon V) = P - \mu W - \beta\eta W_n^*$ is subcritical in Ω for all $n \geq N$ and this completes the proof. $\square$

Remark 3.5 It follows from Lemma 3.4 that if $|W|$ is a weak perturbation of $-\Delta$ in $\mathbb{R}^d, d \geq 3$ and $-\Delta + W$ is subcritical then $|W|$ is strongly subcritical in the sense of [16]. Therefore, the results in [16] apply to such perturbations.

Let $V \in C^\alpha(\Omega)$ be a nonzero, nonnegative function and $W \in C^\alpha(\Omega)$. Define

$$\mu_+(\lambda, V) = sup\{\mu \in \mathbb{R} | \mathcal{C}_{P-\lambda V-\mu W}(\Omega) \neq \emptyset\} \tag{3.1}$$

$$\mu_-(\lambda, V) = inf\{\mu \in \mathbb{R} | \mathcal{C}_{P-\lambda V-\mu W}(\Omega) \neq \emptyset\} \tag{3.2}$$

and denote

$$S_+(\lambda, V) = \{\mu \in \mathbb{R} | P - \lambda V - \mu W \text{ is subcritical in } \Omega\}.$$

Note that if P is critical in Ω then either $\mu_-(0, V) = 0$ or $\mu_+(0, V) = 0$. Moreover, for weak perturbations it follows from Lemma 2.7 and Lemma 3.4 that

Lemma 3.6 *Let P be a subcritical or critical operator in Ω and $V \in C^\alpha(\Omega)$ be a nonzero, nonnegative function. Assume that W is a weak perturbation of P in Ω. If $|\mu_\pm(0, V)| < \infty$ then the operator $P - \lambda V - \mu_\pm(\lambda, V)W$ is critical in Ω for all $\lambda \leq 0$. In other words, we have $S_+(\lambda, V) = (\mu_-(\lambda, V), \mu_+(\lambda, V))$. Moreover, for every $\lambda < 0$, $\mu_-(\lambda, V) < 0 < \mu_+(\lambda, V)$ and if P is subcritical then we also have $\mu_-(0, V) < 0 < \mu_+(0, V)$.*

4 PROOF OF THE MAIN THEOREM

In this section we state and prove our main result which extends Theorem 1.3 (where the perturbation W is a function with compact support) to the case of a weak perturbation.

Theorem 4.1 *Let P be a critical operator in Ω and $W \in C^\alpha(\Omega)$ a weak perturbation of the operator P in Ω, where $0 < \alpha \leq 1$. Let $x_0 \in \Omega_1$ be a fixed reference point.*

Denote by ϕ_0 (resp. $\tilde{\phi}_0$) the ground state of the operator P (resp. P^*) in Ω such that $\phi_0(x_0) = 1$ (resp. $\tilde{\phi}_0(x_0) = 1$). Assume also that

$$\int_\Omega |W(x)| \phi_0(x) \tilde{\phi}_0(x) dx < \infty. \tag{4.1}$$

(i) If there exists $\mu < 0$ such that $P - \mu W(x)$ is subcritical in Ω then

$$\int_\Omega W(x) \phi_0(x) \tilde{\phi}_0(x) dx > 0. \tag{4.2}$$

(ii) Assume that for some nonnegative, nonzero function $V \in C_0^\alpha(\Omega)$ there exist $\mu_0 < 0$ and a positive constant C such that

$$G^\Omega_{P+V-\mu W}(x, x_0) \leq C\phi_0(x) \quad \text{and} \quad G^\Omega_{P+V-\mu W}(x_0, x) \leq C\tilde{\phi}_0(x) \tag{4.3}$$

for all $x \in \Omega \setminus \Omega_1$ and $\mu_0 \leq \mu < 0$. If the integral condition (4.2) holds true then there exists $\mu < 0$ such that $P - \mu W(x)$ is subcritical in Ω.

Remarks 4.2 1. If $W \in C^\alpha(\Omega)$ is a small perturbation of a subcritical operator P in Ω then the ground states $\phi_\pm(x)$ of the operator $P - \mu_\pm(0, V)W(x)$ are comparable with $G^\Omega_{P-\mu W}(x, x_0)$ for all μ such that $\mu_- < \mu < \mu_+$. ([27], see also [24], where this result is proved for a semismall perturbation of P^*). In particular, this assertion holds true for a perturbation W of the type considered by Ancona in [6] (see also [3, 14, 15, 20, 22, 23, 29, 36, 40]). Thus, small perturbations and even perturbations which are semismall of both P and P^* all satisfy assumption (4.3).

2. Recently, T. Weidl [39] proved an abstract result on the number of bound states appearing below the spectrum of a semi-bounded symmetric operator in Hilbert space in the case of a "weak" non-signdefined perturbation. As an application he considered a perturbation of a critical Schrödinger operator $H = -\Delta - \beta_0 V$ by a function W in $\mathbb{R}^d$. It is assumed that there exist $\beta > \beta_0 > 0$ and $\alpha > 0$ such that the negative spectra

of the operators $H = -\Delta - \beta V$ and $H = -\Delta - \beta_0 V - \alpha|W|$ are finite. Under some additional assumptions it is shown ([39, Theorem 8.1]) that the statement of Theorem 1.3 holds true. Note that in Theorem 4.1 we have no assumption on the critical operator P but only on the perturbation W. On the other hand, if W is a weak perturbation of a subcritical or critical Schrödinger operator H then the negative spectrum of $H - \alpha W$ is finite for all $\alpha \in \mathbb{R}$.

3. We are not aware of any example of a weak perturbation W of a critical operator P in Ω such that the integrability condition (4.1) is not satisfied.

Proof of Theorem 4.1: (i) Suppose first that $P - \mu W$ is subcritical for some $\mu < 0$. By Lemma 3.4 (ii), W is also a weak perturbation of the operator $P - \mu W$ in Ω. Consequently, there exists N_0 such that for every $n \geq N_0$ the operator $P - \mu W + \mu W_n^* = P - \mu W_n$ is subcritical in Ω. Since the function W_n has compact support, Theorem 1.3 implies that for all $n \geq N$

$$\int_\Omega W_n(x)\phi_0(x)\tilde{\phi}_0(x)dx > 0. \tag{4.4}$$

By our assumption, the function $W\phi_0\tilde{\phi}_0$ is integrable in Ω. Thus, using the Lebesgue dominated convergence theorem we obtain

$$\int_\Omega W(x)\phi_0(x)\tilde{\phi}_0(x)dx \geq 0. \tag{4.5}$$

In order to show that we have strict inequality in (4.5) we proceed as in the proof of Theorem 1.4 in [28]. Suppose that we have equality in (4.5). Let $V \in C_0^\infty(\Omega)$ be a nonnegative, nonzero function. Since the operator $P - \mu W$ is subcritical in Ω there exists $\epsilon > 0$ such that the operator $P - \mu W - \epsilon V$ is subcritical in Ω. Therefore, we can use (4.5) with the function $W(x) + \frac{\epsilon}{\mu}V(x)$ which is a weak perturbation of P in Ω. Thus,

$$\int_\Omega (W(x) + \frac{\epsilon}{\mu}V(x))\phi_0(x)\tilde{\phi}_0(x)dx \geq 0. \tag{4.6}$$

Since $\mu < 0$ and

$$\int_\Omega V(x)\phi_0(x)\tilde{\phi}_0(x)dx > 0,$$

we see that (4.6) contradicts our assumption of an equality in (4.5).

(ii) Assume now that

$$\int_\Omega W(x)\phi_0(x)\tilde{\phi}_0(x)dx > 0 \tag{4.7}$$

and suppose that $\mu_-(0, V) = 0$, where $V \in C_0^\alpha(\Omega)$ is a nonnegative, nonzero function. In particular, W changes sign in Ω and therefore, $\mu_-(\lambda, V) > -\infty$ for any $\lambda \leq 0$. Lemma 3.6 implies that if $\lambda < 0$ then $\mu_-(\lambda, V) < 0$ and furthermore, the operator $P - \lambda V - \mu_-(\lambda, V)W$ is critical in Ω. Denote by $\phi_{(\lambda,-)}(x)$ (resp. $\tilde{\phi}_{(\lambda,-)}(x)$) the ground state of the operator $P - \lambda V - \mu_-(\lambda, V)W$ (resp. $P^* - \lambda V - \mu_-(\lambda, V)W$) such that $\phi_{(\lambda,-)}(x_0) = 1$ (resp. $\tilde{\phi}_{(\lambda,-)}(x_0) = 1$).

Note that, $\lim_{\lambda\to 0_-} \mu_-(\lambda, V) = \mu_-(0, V) = 0$. It follows from remark 2.2 that if $\lambda < 0$ and $|\lambda|$ is sufficiently small, then both $\phi_{(\lambda,-)}(x)$ and $G^\Omega_{P+V-\mu_-(\lambda,V)W}(x, x_0)$ are positive solutions of minimal growth in a neighborhood of infinity in Ω of the equation

$(P-\mu_-(\lambda, V)W(x))u = 0$. Using our assumption (4.3), the Harnack inequality and the generalized maximum principle we infer that there exist $\lambda_1 < 0$ and a constant $C_1 > 0$ such that

$$\phi_{(\lambda,-)}(x) \leq C_1\phi_0(x) \quad \text{and} \quad \tilde{\phi}_{(\lambda,-)}(x) \leq C_1\tilde{\phi}_0(x) \tag{4.8}$$

for every $\lambda_1 \leq \lambda < 0$ and all $x \in \Omega$. In particular, $W\phi_{(\lambda,-)}\tilde{\phi}_{(\lambda,-)} \in L^1(\Omega)$. Recall that for every $\lambda < 0$ the operator $P - \lambda V$ is subcritical while $P - \lambda V - \mu_-(\lambda, V)W$ is critical in Ω. Moreover, W is a weak perturbation of the operator $P-\lambda V-\mu_-(\lambda, V)W$. Consequently, by first part of the theorem for all $\lambda_1 \leq \lambda < 0$ we have

$$\int_\Omega W(x)\phi_{(\lambda,-)}(x)\tilde{\phi}_{(\lambda,-)}(x)dx < 0. \tag{4.9}$$

Recall that, $\lim_{\lambda\to 0_-} \mu_-(\lambda, V) = \mu_-(0, V) = 0$. Moreover, ϕ_0 (resp. $\tilde{\phi}_0$) is the unique positive normalized solution of the equation $Pu = 0$ (resp. $P^*u = 0$) in Ω. Using the Harnack inequality and elliptic Schauder estimates it follows that as $\lambda \nearrow 0$ the functions $\phi_{(\lambda,-)}$ (resp. $\tilde{\phi}_{(\lambda,-)}$) converge uniformly in any compact subset of Ω to ϕ_0 (resp. $\tilde{\phi}_0$).

Consequently, inequalities (4.8), (4.9) and the Lebesgue dominated convergence theorem imply that

$$\int_\Omega W(x)\phi_0(x)\tilde{\phi}_0(x)dx \leq 0$$

which contradicts our assumption (4.7). □

Remark 4.3 Consider a critical operator P defined on Ω and a class of functions W such that the second part of Theorem 4.1 holds true. It follows that for any nonzero, nonnegative function V with compact support there exists $\epsilon > 0$ such that $P+V-\mu W$ is subcritical in Ω for all $|\mu| < \epsilon$. Thus, for the second part of the theorem one should assume that the perturbation is in some sense "small".

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