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Multi-scale, multi-profile asymptotic limits of global solutions of evolution equations

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Plan of this talk

- Self-similar solutions (forward)
- Asymptotically self-similar solutions
- More general asymptotic limits
  - multi-profile limits
  - chaotic behavior of solutions
  - multi-timescale limits
  - non-parabolic asymptotic limits
- Other equations and related work

Forward self-similar solutions of heat equations on $\mathbb{R}^N$

We consider the parabolic equation:

$$u_t - \Delta u - \kappa |u|^\alpha u = 0. \qquad (H)$$

Here $\alpha > 0$, and $\kappa = 1, -1$ or 0.
Also $u = u(t,x), t > 0, x \in \mathbb{R}^N$.

We denote $u(t) = u(t, \cdot)$.
$(\alpha = p - 1)$

As is well-known, u is a solution of (H), if and only if u_λ is also a solution of (H), for all $\lambda > 0$, where

$$u_\lambda(t,x) = \lambda^\sigma u(\lambda^2 t, \lambda x).$$

We will only consider the case $\sigma > 0$, and we must have

$$\sigma = 2/\alpha, \text{ if } \kappa \neq 0.$$

The solution u of (1) is self-similar if

$$u = u_\lambda, \text{ all } \lambda > 0.$$

Equivalently, u is self-similar if there exists $\theta : \mathbb{R}^N \to \mathbb{R}$ such that

$$u(t,x) = t^{-\sigma/2}\theta(\frac{x}{\sqrt{t}}).$$

The function $\theta = u(1)$ is called the profile of the self-similar solution u.

If the self-similar solution u has an intial value :

$$u_0(x) = \lim_{t \to 0} u(t, x),$$

it follows from $u = u_\lambda$ that u_0 is a homogeneous function on $\mathbb{R}^N$ of degree $-\sigma$ i.e. $u_0(\lambda x) = \lambda^{-\sigma} u_0(x)$.

The initial value and the profile formally satisfy the following relationship:

$$u_0(x) = |x|^{-\sigma} \lim_{r \to \infty} r^\sigma \theta(\frac{x}{|x|} r).$$

Asymptotically self-similar solutions

Let us fix a specific self-similar solution U with profile θ :

$$U(t,x) = t^{-\sigma/2}\theta(\frac{x}{\sqrt{t}}).$$

There are three formally equivalent ways of saying that a general solution u is asymptotically self-similar, to the self-similar solution U.

1. $u_\lambda \to U$ as $\lambda \to \infty$,
2. $u(t) - U(t) \to 0$ as $t \to \infty$, faster than either solution separately goes to 0,
3. $t^{\sigma/2}u(t, \cdot\sqrt{t}) \to \theta = U(1)$, as $t \to \infty$.

Example: the case $\kappa = 0$ (the heat semigroup):

If $0 < \sigma < N$, then $U(t) = e^{t\Delta}(c|\cdot|^{-\sigma})$ is a self-similar solution with profile $\theta = e^{\Delta}(c|\cdot|^{-\sigma})$.
If $u_0 \in C_0(\mathbb{R}^N)$ is such that $u_0(x) \approx c|x|^{-\sigma}$ for large $|x|$, i.e.

$$|x|^\sigma u_0(x) \to c \neq 0, \quad \text{as} \quad |\mathrm{x}| \to \infty,$$

then $u(t) = e^{t\Delta}u_0$ is asymptotically self-similar to

$$U(t) = e^{t\Delta}(c|\cdot|^{-\sigma}).$$

Also,

$$U(t,x) = (4\pi t)^{-N/2} e^{-\frac{|x|^2}{4t}}$$

is a self-similar solution (with $\sigma = N$). If $u_0 \in L^1(\mathbb{R}^N)$, then $u(t) = e^{t\Delta}u_0$ is asymptotically self-similar to

$$U(t) = (4\pi t)^{-N/2} e^{-\frac{|x|^2}{4t}}.$$

A sampling of articles on self-similar and asymptotically self-similar solutions

$\kappa = 1$: (absorption)
Gmira and Véron, 1984
Kamin and Peletier, 1985
W., 1986
Brezis, Terman and Peletier, 1986
Escobedo and Kavian, 1987
Escobedo, Kavian and Matano, 1995
Bricmont and Kupiainen, 1996
Wayne, 1997
Herraiz, 1999
Cazenave, Dickstein, Escobedo, W., 2001

$\kappa = -1$: (source)

Haraux and W., 1982
W., 1986
Peletier, Terman and W., 1986
Kavian, 1987
Bricmont, Kupiainen and Lin, 1994
Yanagida, 1996
Kawanago, 1996
Galaktionov and Vazquez, 1997
Wayne, 1997
Kwak, 1998
Dolman and Hirose, 1998
Ribaud, 1998
Cazenave and W., 1998
Snoussi, Tayachi and W., 1999, 2001
Naito and Suzuki ; Souplet and W. ; Naito ;
Fila, Winkler and Yanagida ; Mizoguchi ;
Bidaut-Véron (p-Laplacian) ; ...

Basic result, true in many circumstances, with $\kappa = \pm 1$ or 0

If $u_0 \in C_0(\mathbb{R}^N)$ is such that

$$|x|^\sigma u_0(x) \to c \neq 0, \quad \text{as} \quad |\mathrm{x}| \to \infty, \tag{SL}$$

where $\sigma = 2/\alpha$ if $\kappa = \pm 1$ (i.e. $u_0(x) \approx c|x|^{-\sigma}$ for large $|x|$, so u_0 asymptotically homogeneous for large $|x|$),
then the solution $u(t)$ of the equation

$$u_t - \Delta u - \kappa |u|^\alpha u = 0. \tag{H}$$

with intial value u_0 is asymptotically self-similar to the self-similar solution $U(t)$ whose initial value is

$$z(x) = c|x|^{-\sigma}.$$

The condition (SL) can be extended by allowing c to depend on the angle at which $|x| \to \infty$. In this case, the intial value $z(x)$ is an arbitrary homogeneous function of degree $-\sigma$.

Reformulation of basic result

The condition (SL) can be (at least formally) re-written as

$$\lambda^\sigma u_0(\lambda x) \to z(x), \quad \text{as} \quad \lambda \to \infty. \qquad (DL)$$

Also, let $\mathcal{S}(t)$ denote the semiflow generated by the (linear or nonlinear) heat equation (H). We don't worry for the moment if this is well-defined.

The basic result can therefore be reformulated to say that if u_0 and z satisfy (DL), i.e. if the dilations of u_0 converge to z, and if $u(t) = \mathcal{S}(t)u_0$ and $U(t) = \mathcal{S}(t)z$ are the corresponding solutions of (H) with intial values u_0 and z, respectively (U being self-similar since z is homogeneous), then

$$u_\lambda \to U \quad \text{as} \quad \lambda \to \infty,$$

and in particular,

$$t^{\sigma/2}u(t, \cdot\sqrt{t}) \to \theta = \mathcal{S}(1)z \quad \text{as} \quad \text{t} \to \infty. \qquad (AL)$$

Multi-profile limits

A natural weakening of (DL) is to consider the possibility that the limit exists only along a sequence $\lambda_k \to \infty$.

$$\lambda_k^{\sigma} u_0(\lambda_k x) \to z(x), \quad \text{as} \quad \text{k} \to \infty. \tag{L}$$

In order to get some control over the limit process, it is reasonable to impose the condition

$$\sup_{x\in\mathbb{R}^N} |x|^{\sigma} |u_0(x)| < \infty. \tag{M}$$

It turns out that (M) and (L) are enough to recover something similar to (AL).

Theorem

Let $u_0 \in C_0(\mathbb{R}^N)$ satisfy (M). Let $z \in \mathcal{D}'(\mathbb{R}^N \setminus \{0\})$ Let $\lambda_k \to \infty$ and set $t_k = \lambda_k^2$. It follows that the condition (L) is equivalent to

$$t_k^{\sigma/2} u(t_k, \cdot\sqrt{t}_k) \to \theta = \mathcal{S}(1) z \quad \text{as} \quad \text{k} \to \infty. \tag{A}$$

Remarks

- The convergence in (L) is in $\mathcal{D}'(\mathbb{R}^N \setminus \{0\})$, and in $\mathcal{D}'(\mathbb{R}^N)$ if $0 < \sigma < N$.
- The convergence in (A) is in $C_0(\mathbb{R}^N)$.
- If $\kappa = 0$, then we need $0 < \sigma < N$.
- If $\kappa = -1$, then we need also need $0 < \sigma < N$, i.e. $\alpha > 2/N$, since otherwise all positive solutions of (H) blowup in finite time. In addition, a smallness condition is required with (M).
- If $\kappa = 1$ and $\alpha < 2/N$, so $\sigma > N$, we need to restrict outselves to positive solutions only. Also, the semiflow $\mathcal{S}(t)$ needs to be carefully defined, since the Cauchy problem for (H) is not well-posed. (Marcus and Véron 1999.)

- One key technical fact used in the proof is that a closed ball in the Banach space

$$W_\sigma \equiv \{z \in L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}) : |\cdot|^\sigma z \in L^\infty\}$$

  is weak* compact, and metrizable. So the convergence in (L) in fact takes place in a compact metric space.
- It follows from the scaling properties of (H) that if $t_k = \lambda_k^2$, then

$$t_k^{\sigma/2} u(t_k, \cdot\sqrt{t}_k) = \mathcal{S}(1)[\lambda_k^\sigma u_0(\lambda_k \cdot)],$$

  and so the relation (A) can be interpreted as continuous dependence on initial data at time $t = 1$, with respect to the above topology.
- The function $z \in W_\sigma$ might or might not be homogeneous (necessarily of degree $-\sigma$). If it is homogeneous, the solution $U(t) = \mathcal{S}(t)z$ is self-similar with profile $\theta = \mathcal{S}(1)z$. In this case, the solution $u(t) = \mathcal{S}(t)u_0$ is asymptotically self-similar along the sequence $t_k \to \infty$ to the self-similar solution $U(t) = \mathcal{S}(t)z$.

Consequences

The above theorem shows, at least in a certain context, that the long time asymptotics of a solution of (H) are determined by the spatial asymptotics of the initial value. The limit along a specific sequence of dilations of the initial value determines the limit of the resulting solution (appropriately rescaled) along a corresponding time sequence.

If we wish to produce a solution of (H) which has different asymptotic properties along different time sequences, it suffices to construct an initial value u_0 with different spatially asymptotic properties under different sequences of dilations. We accomplish this by specifiying u_0 on thick shells in R^N which are spaced further and further apart, such that under certains dilations, the behavior of u_0 is essentially determined by its values on one particular shell.

If $u_0 \in C_0(\mathbb{R}^N)$ we denote respectively by $\Omega_\sigma(u_0)$ and $\omega_\sigma(u_0)$ all the possible limits attainable in (L) and (A), then the above theorem implies that if $u_0 \in W_\sigma$, then

$$\omega_\sigma(u_0) = \mathcal{S}(1)\Omega_\sigma(u_0).$$

How big can these objects be?
In fact, there exists $u_0 \in C_0(\mathbb{R}^N) \cap W_\sigma$ such that $\Omega_\sigma(u_0)$ is a closed ball in W_σ.

Theorem

There exists a solution of (H) which is asymptotic along an appropriate time sequence to every self-similar solution whose initial value is in a fixed ball of W_σ.

Chaotic behavior of solutions

For any fixed $\lambda > 1$ we define the operator

$$Fg = F_\lambda g = \lambda^\sigma [\mathcal{S}(\lambda^2 - 1)g](\lambda \cdot).$$

("Renormalization group map" of Bricmont and Kupiainen).

Theorem

The discrete dynamical system generated by F_λ is chaotic on the compact set $\mathcal{S}(1)B_M \subset C_0(\mathbb{R}^N)$, where B_M is the closed ball of radius M in W_σ.

A discrete dynamical system is chaotic if periodic orbits are dense, it is topologically transitive, and there is sensitive dependence on intial data.

The basic idea of the proof is to show first that for any fixed $\lambda > 1$ the dilation operator

$$g \to \lambda^\sigma g(\lambda \cdot)$$

generates a chaotic discrete dynamical system on the compact metric space B_M. Then F_λ is simply the induced action on $C_0(\mathbb{R}^N)$.

Multi-timescale asymptotic limits.

For the rest of the talk, I suppose $\kappa = 0$, and so $\mathcal{S}(t) = e^{t\Delta}$. In this case, there is no "intrinsic" value of σ, and so it is natural to wonder if there exists a solution $u(t) = e^{t\Delta} u_0$ to the heat equation which has non-trivial asymptotic limits with respect to different time scalings, along different time sequences. In particular, can it be that one solution $u(t) = e^{t\Delta} u_0$ has different decay rates along different time sequences?

Theorem

There exists $u_0 \in C_0(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$, $u_0 \geq 0$ *such that*

1. *for all* $0 < \sigma < N$ *and* $0 \leq c < \infty$, *there exists a sequence* $t_k \to \infty$ *such that* $t_k^{\sigma/2} \|\mathcal{S}(t_k) u_0\|_{L^\infty} \to c$ *as* $k \to \infty$,
2. $cG \in \omega_\sigma(u_0)$, $\forall \sigma$ *with* $0 < \sigma < N$, *and* $\forall c \geq 0$, *where* $G(x) = (4\pi)^{-N/2} e^{-|x|^2/4}$.

Recall that $\omega_\sigma(u_0) =$

$$\{\theta \in C_0(R^N), \exists t_k \to \infty \, \mathrm{s.t.} \, ||t_k^{\sigma/2} u(t_k, \cdot\sqrt{t}_k) - \theta||_\infty \to 0 \text{ as k} \to \infty\}.$$

Idea of proof (in the case $N = 1$): It suffices to construct $u_0 \in C_0(\mathbb{R})$, $u_0 \geq 0$, such that for every σ with $0 < \sigma < 1$, and every $c \geq 0$, there exists $\lambda_k \to \infty$ such that $\lambda_k^\sigma u_0(\lambda_k \cdot) \to c\delta$ as measures. Then apply e^Δ.

Theorem

There exists $u_0 \in C_0(\mathbb{R}) \cap C^\infty(\mathbb{R})$ such that

1. *if $\sigma \in (0,1)$ is rational, then $\omega_\sigma(u_0) = C_0(\mathbb{R})$;*
2. *for all $\sigma \in (0,1)$, we have $\mathcal{N} \subset \omega_\sigma(u_0)$, where*

$$\mathcal{N} = \{cG^{(k)}, c \in \mathbb{R}, k \geq 0\};$$

3. *for almost all $\sigma \in (0,1), \omega_\sigma(u_0) = \mathcal{N}$.*

Non-parabolic asymptotic limits of solutions of heat equations

Consider $u(t,x) = [e^{t\Delta}u_0](x)$, and set

$$\Gamma_\lambda^{\mu,\beta}u(t,x) = \lambda^\mu u(\lambda^2 t, \lambda^{2\beta}x).$$

If $w = \Gamma_\lambda^{\mu,\beta}u$, then

$$\partial_t w = \lambda^{2-4\beta}\Delta w.$$

So if there is some sort of limit h as $\lambda \to \infty$, then we can expect

- $\Delta h = 0$, if $\beta < 1/2$,
- $h_t = 0$, if $\beta > 1/2$.

Since the only harmonic function in $C_0(\mathbb{R}^N)$ is 0, it is the second case, $\beta > 1/2$, which turns out to be interesting.

Theorem

Fix $\beta > 1/2$ and $0 < \mu < N$. Given $f \in \mathcal{S}$, $f \geq 0$, there exists $u_0 \in C_0(R^N) \cap C^\infty(R^N)$, $u_0 \geq 0$ with the following properties.

1. *There exists a sequence $\lambda_k \to \infty$ such that $\Gamma_\lambda^{\mu,\beta} u \to f$ uniformly on $[\epsilon, T] \times \mathbb{R}^N$, where $u = e^{t\Delta} u_0$.*
2. *There exists a sequence $t_k \to \infty$ such that $t_k^{\mu/2} u(t_k, \cdot t_k^{2\beta}) \to f$ uniformly, where $u = e^{t\Delta} u_0$.*
3. $\sup_{x \in \mathbb{R}^N} |x|^\sigma |u_0(x)| < \infty$, *where $\sigma = \mu/2\beta$.*
4. *If f is radially symmetric, and radially decreasing, then u_0 can be chosen to have these same properties.*

Remark: Along the time sequence t_k the solution $u(t) = e^{t\Delta} u_0$ can not have any nontrivial parabolic limit in $C_0(R^N)$. Thus, the non-parabolic limit is essential to a complete understanding of the asymptotic behavior of $e^{t\Delta} u_0$.

Other equations and related work

- Navier-Stokes system (Cazenave, Dickstein and W)
- Schrödinger equations (Cazenave and W, Cazenave, Xie and Zhang)
- porous medium equation, p-Laplacian (Vázquez and Zuazua)

References

1. T. Cazenave, F. Dickstein, W., Universal solutions of the heat equation in $\mathbb{R}^N$, Discrete and Cont. Dyn. Sys. **9** (2003), 1105-1132.
2. T. Cazenave, F. Dickstein, W., Universal solutions of the nonlinear heat equation in $\mathbb{R}^N$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), Vol. **2**, Fasc. 1 (2003), 77-117.
3. T. Cazenave, F. Dickstein, W., A solution of the heat equation with a continuum of decay rates., *Elliptic and Parabolic Problems: a special tribute to Haim Brezis*, Progress in Nonlinear Differential Equations and their Applications, Vol. 63, C. Bandle and H. Berestycki (eds.), Birkhäuser, Basel, 2005, 135-138.

References (continued)

4. T. Cazenave, F. Dickstein, W., Multiscale asymptotic behavior of a solution of the heat equation on $\mathbb{R}^N$, *Contributions to Nonlinear Analysis: a tribute to D.G. de Figueiredo on the occasion of his 70th birthday*, Progress in Nonlinear Differential Equations and their Applications, Vol. 66, Birkhäuser, Basel, 2005, 185-194.
5. T. Cazenave, F. Dickstein, W., A solution of the constant coefficient heat equation on $\mathbb{R}$ with exceptional asymptotic properties: an explicit construction, Jnl. Maths. Pures Appl. **85** (2006), 119-150.
6. T. Cazenave, F. Dickstein, W., Non-parabolic asymptotic limits of solutions of the heat equation on $\mathbb{R}^N$, Jnl. Dynam. Diff. Eqns. **19** (2007), 789-818.