Green's Function Estimates for Some Linear and Nonlinear Operators

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Abstract

Global bilateral estimates will be presented for Green's function of the fractional Schrödinger operator $\mathbf{Lu = (-\Delta)^{\alpha/2}u - qu}$ $(\mathbf{0} < \alpha \leq \mathbf{2})$. Here $\mathbf{q}$ is a general nonnegative measurable function (or measure). These results will be deduced from sharp estimates of the kernel of the Neumann series associated with a general integral operator with positive kernel satisfying a quasimetric property.

Analogous estimates will be discussed for the $\mathbf{p}$-Laplacian with a natural growth term, $\mathbf{-\Delta_p u - qu^{p-1}}$, $\mathbf{1 < p < \infty}$, as well as more general quasilinear and fully nonlinear elliptic operators, in particular $\mathbf{F_k[-u] - qu^k}$ where $\mathbf{F_k}$ is the $\mathbf{k}$-Hessian operator, for general $\mathbf{q \geq 0}$.

Applications to: isolated singularities, equations with natural growth in the gradient, ground states.

This talk is based on joint work with Michael Frazier, Benjamin Jaye, Fedor Nazarov, and Nguyen Cong Phuc.

Basic Linear and Nonlinear Operators

Elliptic equations involving the following operators will be considered:

1. The fractional Laplacian
$$(-\Delta)^{\alpha/2}u$$
on the entire Euclidean space R^n for $0 < \alpha < n$, or a bounded NTA domain $\Omega \subseteq R^n$ for $0 < \alpha \leq 2$.
2. The **p**-Laplacian operator Δ_p $(1 < p < \infty)$
$$\Delta_p u = \mathrm{div}\,(\nabla u |\nabla u|^{p-2})$$
on $\Omega \subseteq R^n$.
3. The **k**-Hessian operator F_k $(k = 1, 2, \ldots, n)$
$$F_k[u] = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$
Here $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hessian matrix $D^2 u$ on $\Omega \subseteq R^n$.

Typical global results

Fundamental solution for quasilinear equations with natural growth terms

(Serrin, local estimates; Véron, isolated singularities; Tintarev-Pinchover, ground state alternative and criticality)

Theorem (Jaye-Verbitsky, 2010)

Let $\mathbf{1 < p < n}$ *and* $\omega \geq \mathbf{0}$.
(i) If there exists a fundamental solution $\mathbf{u = u(x, x_0)}$ *to the equation*

$$-\Delta_p u = \omega u^{p-1} + \delta_{x_0}, \quad u \geq 0,$$

on $\mathbf{R^n}$, $\mathbf{p}$*-superharmonic,* $\inf_{\mathbf{R^n}} \mathbf{u = 0}$, *then*

$$u(x, x_0) \geq c_1 \, |x - x_0|^{\frac{p-n}{p-1}} \, \exp\left\{ c_2 \int_0^{|x-x_0|} \left(\frac{|B_r(x)|_\omega}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right\}$$

$$\times \exp\left\{ c_2 \int_0^{|x-x_0|} \frac{|B_r(x_0)|_\omega}{r^{n-p}} \frac{dr}{r} \right\}.$$

Theorem (Jaye-Verbitsky, 2010)

(ii) Conversely, if $\mathbf{1 < p < n}$*, then under a natural assumption on* ω *discussed below there exists a minimal fundamental solution* $\mathbf{u}$*,*

$$\mathbf{u(x,x_0) \le c_3\,|x-x_0|^{\frac{p-n}{p-1}}\ \exp\left\{c_4 \int_0^{|x-x_0|} \left(\frac{|B_r(x)|_\omega}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right\}}$$

$$\mathbf{\times \exp\left\{c_4 \int_0^{|x-x_0|} \frac{|B_r(x_0)|_\omega}{r^{n-p}} \frac{dr}{r}\right\}.} \qquad (1)$$

(iii) If $\mathbf{p \ge n}$*, there is no positive solution to* $\mathbf{-\Delta_p u = \omega u^{p-1} + \delta_{x_0}}$ *unless* $\mathbf{\omega = 0}$*.*

Here $\mathbf{|B_r(x_0)|_\omega = \int_{|x-x_0|<r} d\omega}$.

Global equivalence of fundamental solutions

Corollary (Jaye-Verbitsky, 2010)

Suppose, for all $\mathbf{x}, \mathbf{x_0} \in \mathbf{R^n}$*,*

$$C_1 \, |x - x_0|^{\frac{p-n}{p-1}} \le u(x, x_0) \le C_2 \, |x - x_0|^{\frac{p-n}{p-1}}.$$

Then necessarily

$$\sup_{x \in R^n} \int_0^\infty \frac{\omega(B(x,r))}{r^{n-p}} \, \frac{dr}{r} < \infty, \quad 1 < p \le 2,$$

$$\sup_{x \in R^n} \int_0^\infty \left(\frac{\omega(B(x,r))}{r^{n-p}} \right)^{1/(p-1)} \frac{dr}{r} < \infty, \quad p > 2.$$

Conversely, these conditions are sufficient for the equivalence, under a natural assumption on ω *discussed below.*

Neumann Series and Fractional Schrödinger Equations

We obtain global bilateral bounds for Green's functions and kernels of Neumann series for a broad class of differential and integral equations with possibly singular coefficients, data, and boundaries of the domains.

Linear operators: the Schrödinger operator with potential $\mathbf{q}$, defined by $\mathbf{H} = -\boldsymbol{\Delta} - \mathbf{q}$, on a domain $\boldsymbol{\Omega} \subseteq \mathbf{R}^{\mathbf{n}}$, for $\mathbf{n} \geq 3$. More generally, we consider the non-local operator $\mathbf{H}_{\alpha} = (-\boldsymbol{\Delta})^{\alpha/2} - \mathbf{q}$ and the associated Green's function on $\boldsymbol{\Omega}$. Here $\mathbf{q}$ is a locally integrable function (or measure).

We consider domains $\boldsymbol{\Omega}$ with a Green's operator $\mathbf{G}^{(\alpha)}$ for $(-\boldsymbol{\Delta})^{\alpha/2}$ with a positive Green's function. Our theory is applicable to any bounded domain $\boldsymbol{\Omega}$ with the boundary Harnack principle (Jerison and Kenig for NTA domains). This principle holds for a large class of domains in $\mathbf{R}^{\mathbf{n}}$, $\mathbf{n} \geq 2$, including Lipschitz and uniform domains in the classical case $\alpha = 2$, and even more general domains with the interior corkscrew condition if $0 < \alpha < 2$.

Global estimates of Green's function, conditional gauge

Let $\mathbf{V}$ denote the minimal Green's function:

$$\mathbf{V(x,y)} = \sum_{j=1}^{\infty} (\mathbf{G}^{(\alpha)})^{j}(\mathbf{x},\mathbf{y}),$$

$(\mathbf{G}^{\alpha})^{\mathbf{j}}$ is the $\mathbf{j}$-th iteration of Green's kernel $\mathbf{G}^{\alpha}\left(\mathbf{q}\cdot(\mathbf{G}^{\alpha})^{\mathbf{j}-1}\right)$. Then

$$\mathbf{C}_1\mathbf{G}^{(\alpha)}(\mathbf{x},\mathbf{y})e^{c_1\Phi(x,y)} \le \mathbf{V}(\mathbf{x},\mathbf{y}) \le \mathbf{C}_2\mathbf{G}^{(\alpha)}(\mathbf{x},\mathbf{y})e^{c_2\Phi(x,y)},$$

$$\Phi(x,y) = \frac{1}{G^{\alpha}(x,y)}\int_{\Omega} G^{\alpha}(x,z)\,G^{\alpha}(z,y)\,q(z)\,dz.$$

This gives global bilateral estimates of the conditional gauge:

$$C_1\,e^{c_1\,\Phi(x,y)} \le \mathbb{E}_y^x\left[e^{\int_0^{\zeta} q(X_s)\,ds}\right] \le C_2\,e^{c_2\,\Phi(x,y)},$$

where $\mathbf{X_t}$ is a $\mathbf{y}$-conditioned Brownian motion if $\alpha = 2$, or an α-stable symmetric process if $0 < \alpha < 2$ (starts at $\mathbf{x}$ and stops at $\mathbf{y}$, lifetime ζ).

Linear differential equations

Classical time-independent Schrödinger equations

Let $\mathbf{\Omega} \subseteq \mathbf{R^n}$ be a bounded domain. We want to find conditions for the existence of a solution $\mathbf{u}$ to the following problems:

$$\begin{cases} -\mathbf{\Delta u} = \mathbf{q} \cdot \mathbf{u} + \varphi \text{ on } \mathbf{\Omega}, \\ \mathbf{u} = \mathbf{g} \text{ on } \partial\mathbf{\Omega}, \end{cases}$$

where $\mathbf{q}, \varphi$, and $\mathbf{g}$ are given. We are especially interested in the existence of *nonnegative* solutions $\mathbf{u}$ for general nonnegative $\mathbf{q}, \varphi$, and $\mathbf{g}$ (measurable functions, or possibly measures).

Applying the Green's function operator $\mathbf{G} = (-\mathbf{\Delta})^{-1}$ to both we obtain $\mathbf{u} = \mathbf{G}(\mathbf{q} \cdot \mathbf{u}) + \mathbf{G}(\varphi) + \mathbf{P}(\mathbf{g})$, where $\mathbf{P}$ is the Poisson integral. This formulation is of the form

$$\mathbf{u} = \mathbf{G}(\mathbf{u}\, d\omega) + \alpha = \alpha + \sum_{j=1}^{\infty} \int_{\mathbf{\Omega}} \mathbf{G^j}(\mathbf{x}, \mathbf{y})\, \alpha(\mathbf{y})\, d\omega,$$

where $\alpha = \mathbf{G}(\varphi) + \mathbf{P}(\mathbf{g})$, $\omega(\mathbf{y}) = \mathbf{q}(\mathbf{y})\, d\mathbf{y}$, $\mathbf{G^j} = \mathbf{G}\left((\mathbf{G})^{\mathbf{j}-1} d\omega\right)$ iterated kernel, $\mathbf{G^0} = \mathbf{G}$.

Discrete Model

Discrete Schrödinger equation (Frazier–Verbitsky)

We follow C. Fefferman, Hedberg and Wolff, and Chang, Wilson, and Wolff in forming a discrete model for this problem. Let $\mathbf{\Omega = R^n}$, $\mathbf{n \geq 3}$. Let $\mathcal{Q}$ denote the set of all dyadic cubes $\mathbf{Q = 2^{-\nu}([0,1)^n + j)}$ ($\nu \in \mathbf{Z}, \mathbf{j} \in \mathbf{Z^n}$) in $\mathbf{R^n}$, and $\mathcal{Q}_\nu = \{\mathbf{Q} \in \mathcal{Q} : \ell(\mathbf{Q}) = \mathbf{2}^{-\nu}\}$, where $\ell(\mathbf{Q})$ is the side length of the cube $\mathbf{Q}$. For an appropriate constant $\mathbf{c} > \mathbf{1}$, define the dyadic Green's operator

$$\mathbf{G^d(u\,d\omega)} = \sum_{\nu \in \mathbf{Z}} \sum_{\mathbf{Q} \in \mathcal{Q}_\nu} \frac{|\mathbf{Q}|_\omega}{|\mathbf{Q}|^{1-\frac{2}{n}}} \frac{\mathbf{1}}{|\mathbf{Q}|_\omega} \int_{\mathbf{cQ}} \mathbf{u(y)\,d\omega(y)} \;\; \chi_{\mathbf{Q}}(\mathbf{x}),$$

where $|\mathbf{Q}|$ is Lebesgue measure of $\mathbf{Q}$ and $|\mathbf{Q}|_\omega = \int_{\mathbf{Q}} \mathbf{d}\omega$.

Dyadic model

We write

$$\mathbf{s_Q} = \frac{|\mathbf{Q}|_\omega}{|\mathbf{Q}|^{1-\frac{2}{n}}}.$$

For our model, define

$$\mathbf{Tu(x)} = \sum_{\mathbf{Q}\in\mathcal{Q}} \mathbf{s_Q} \frac{1}{|\mathbf{Q}|_\omega} \int_{\mathbf{cQ}} \mathbf{u(y)}\, \mathbf{d}\omega\mathbf{(y)}\ \chi_{\mathbf{Q}}\mathbf{(x)}.$$

Note that $\mathbf{T}$ is determined by ω and the sequence $\mathbf{s} = \{\mathbf{s_Q}\}_{\mathbf{Q}\in\mathcal{Q}}$, $\mathbf{s_Q} \geq \mathbf{0}$. Our model problem is to find conditions for the existence of a solution $\mathbf{u}$ to the equation

$$\mathbf{u} = \mathbf{Tu} + \alpha.$$

The formal solution is $\mathbf{u} = \sum_{\mathbf{k=0}}^{\infty} \mathbf{T^k}\alpha$. Our concern is to determine conditions under which this sum converges $\mathbf{d}\omega$-a.e., and estimate the kernel of $\mathbf{(I - T)^{-1}}$.

Dyadic Schrödinger equation

Observe that we can write

$$\mathbf{Tu(x)} = \int_{\mathbf{R^n}} \mathbf{K(x,y)u(y)\,d}\omega\mathbf{(y)}$$

for

$$\mathbf{K(x,y)} = \sum_{\mathbf{Q}\in\mathcal{Q}} \frac{\mathbf{s_Q}}{|\mathbf{Q}|_\omega} \chi_{\mathbf{Q}}\mathbf{(x)}\chi_{\mathbf{Q}}\mathbf{(y)}.$$

Operators of this form have been studied by Nazarov, Treil, and Volberg, among others.

Our dyadic model of the Schrödinger equation is

$$\mathbf{u(x)} = \int_{\mathbf{R^n}} \mathbf{K(x,y)u(y)\,d}\omega\mathbf{(y)} + \alpha,$$

for $\mathbf{u}$, $\alpha \in \mathbf{L^1_{loc}(d}\omega\mathbf{)}$, with arbitrary $\mathbf{s_Q} \geq \mathbf{0}$ in the definition of $\mathbf{K}$.

Carleson measures and solvability

Define a discrete Carleson norm

$$\|\mathbf{s}\|_\omega = \sup_{Q} |Q|_\omega^{-1} \sum_{P \subseteq Q} |s_P|\, |P|_\omega .$$

Let $A_Q(x) = \sum_{P\subseteq Q} s_P \chi_P(x)$. If $\|\mathbf{s}\|_\omega < \frac{1}{12}$, and

$$\sum_Q \frac{s_Q}{|Q|_\omega} e^{6A_Q(x)} \chi_Q(x) \int_Q e^{6A_Q(y)} |\alpha(y)|\, d\omega(y) < +\infty$$

$d\omega$-a.e., then there exists u satisfying $u = Tu + \alpha$. Conversely, if $\alpha \geq 0$ and the equation $u = Tu + \alpha$ has a solution $u \geq 0$, then $\|\mathbf{s}\|_\omega \leq 1$ and

$$\sum_Q \frac{s_Q}{|Q|_\omega} e^{\frac{1}{2}A_Q(x)} \chi_Q(x) \int_Q e^{\frac{1}{2}A_Q(y)} |\alpha(y)|\, d\omega(y) < +\infty$$

$d\omega$-a.e. These results are deduced from bilateral estimates for the kernel of the Neumann series $\sum_{j=0}^{\infty} T^j$.

Discrete Neumann series

Theorem (Frazier-Verbitsky, $\mathbf{1/12}$; Frazier-Nazarov-Verbitsky, $\mathbf{1/4}$)

Let $\mathbf{K_j}$ *be the kernel of* $\mathbf{T^j}$. *Then*

$$\mathbf{V(x,y)} = \sum_{j=1}^{\infty} \mathbf{K_j(x,y)} \geq \sum_{Q} \frac{s_Q}{|Q|_\omega} e^{\frac{1}{2}(A_Q(x)+A_Q(y))} \chi_Q(x)\chi_Q(y),$$

for all $\mathbf{x, y} \in \mathbf{R^n}$. *Moreover,* $\|s\|_\omega \leq 1$ *if* $\mathbf{K(x,\cdot)} \neq \infty$ $\mathbf{d}\omega$*-a.e.*
Conversely, if $\|s\|_\omega < \frac{1}{12}$, *then*

$$\mathbf{V(x,y)} = \sum_{j=1}^{\infty} \mathbf{K_j(x,y)} \leq c \sum_{Q} \frac{s_Q}{|Q|_\omega} e^{6(A_Q(x)+A_Q(y))} \chi_Q(x)\chi_Q(y),$$

for all $\mathbf{x, y} \in \mathbf{R^n}$.

General quasimetric kernels

Let (Ω, ω) be a measure space. Suppose K is a map from $\Omega \times \Omega$ into $(0, +\infty]$ such that $K(\cdot, y)$ is ω-measurable for all $y \in \Omega$. Define $d(x, y) = 1/K(x, y)$. We say that K is a *quasimetric* kernel on Ω (with quasimetric constant $\kappa > 1/2$) if:
(i): K is symmetric: $K(x, y) = K(y, x)$ for all $x, y \in \Omega$;
(ii): $K(x, y) < +\infty$ if $x \neq y$;
(iii): d satisfies the quasitriangle inequality with constant κ:

$$d(x, y) \leq \kappa \left(d(x, z) + d(z, y)\right)$$

for all $x, y, z \in \Omega$.
Geometry of quasimetric balls $B_r(x) = \{z \in \Omega : d(z, x) < r\}$.

Example

Suppose $\mathbf{\Omega} \subset \mathbf{R^n}$ is a bounded domain which supports the boundary Harnack principle for $\mathbf{(-\Delta)^{\frac{\alpha}{2}}}$, $\mathbf{0 < \alpha \leq 2}$. For Green's function $\mathbf{G^\alpha}$, let $\boldsymbol{\rho}\mathbf{(x) = \min(1, G^\alpha(x, x_0))}$, where $\mathbf{x_0}$ is a fixed pole in $\mathbf{\Omega}$.
Then (Ancona, 1999; Hansen, 2005)

$$\mathbf{K(x,y) = \frac{G^\alpha(x,y)}{\rho(x) \cdot \rho(y)}}$$

is a quasimetric kernel (sharp form of the 3-G inequality).

Inductively define $\mathbf{K_j}$ for $\mathbf{j \leq 1}$ by letting $\mathbf{K_1 = K}$ and, for $\mathbf{j \geq 2}$,

$$\mathbf{K_j(x,y) = \int_\Omega K(x,z) K_{j-1}(z,y)\, d\omega(z).}$$

Define the minimal Green's function

$$\mathbf{V(x,y) = \sum_{j=1}^{\infty} K_j(x,y).}$$

Quasimetric Neumann series

Theorem (Frazier-Verbitsky, 2009)

$$\mathbf{V(x,y) \geq K(x,y) e^{\frac{1}{16\kappa^2} K_2(x,y)/K(x,y)}}$$

for all $\mathbf{x, y \in \Omega}$*. Conversely, there exist* $\mathbf{c, C > 0}$ *such that*

$$\mathbf{V(x,y) \leq C\, K(x,y) e^{c\, K_2(x,y)/K(x,y)}},$$

under the weak boundedness condition $\mathbf{\|\omega\|_{wb} \leq c(\kappa)}$*:*

$$\mathbf{\|\omega\|_{wb} = \sup \frac{1}{|E|_\omega} \iint_{E \times E} K(x,y)\, d\omega(x)\, d\omega(y), \quad E \subset \Omega.}$$

Equivalent to boundedness of $\mathbf{T}$ (necessarily $\mathbf{||T||_{L^2(\omega) \to L^2(\omega)} \leq 1}$).
Sharp condition: $\mathbf{||T||_{L^2(\omega) \to L^2(\omega)} < 1}$ [Frazier-Nazarov-Verbitsky, 2010].

Nonlinear equations with natural growth in the gradient

Suppose $\Omega \subset \mathbf{R}^n$ is a bounded smooth domain and $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. Consider

$$-\triangle u = |\nabla u|^2 + \omega \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega,$$

ω is a positive measure on Ω.

Theorem (Frazier-Nazarov-Verbitsky)

A.) Suppose there is a weak solution u. *Then* $||T||_{L^2(\omega) \to L^2(\omega)} \leq 1$ *and*

$$\int_{\partial\Omega} e^{c_1 \, P[\delta(\, d\omega]} \, dm_{n-1}(z) < \infty,$$

$d\, m_{n-1}$ *is the surface measure on* $\partial\Omega$, $P[\delta \, d\omega]$ *is the balayage.*

Exponential integrability of the balayage

Theorem (Frazier-Nazarov-Verbitsky)

B.) Conversely, suppose $||\mathbf{T}||_{L^2(\omega)\to L^2(\omega)} < 1$ *and*

$$\int_{\partial\Omega} e^{c_2\, P[\delta\, d\omega](z)}\, dm_{n-1}(z) < \infty.$$

Then there is a weak solution **u***.*

Equivalent to the existence of a *positive* solution to Schrödinger's equation

$$-\triangle v = \omega\, v, \quad v > 0, \quad v = 1, \text{ on } \partial\Omega.$$

Exponential integrability of the balayage $\mathbf{P}[\delta\, d\omega]$ on $\partial\Omega$ is the key boundary condition. Weaker than $\delta\, d\omega$ a Carleson measure; related to

$$P[\delta\, d\omega] \in BMO(\partial\Omega).$$

Th. Wolff's potentials

For a positive measure μ on $\mathbb{R}^n$, $p > 1$, $\alpha > 0$, Wolff's potential ($p = 2$, $\alpha = 1$ Newtonian potential):

$$W_{\alpha,p}\mu(x) = \int_0^\infty \left[\frac{\mu(B_r(x))}{r^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dr}{r}, \quad x \in \mathbb{R}^n.$$

On bounded domains, for $0 < R \le 2\operatorname{diam}(\Omega)$:

$$W^R_{\alpha,p}\mu(x) = \int_0^R \left[\frac{\mu(B_r(x))}{r^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{dr}{r}, \quad x \in \Omega,$$

Wolff's inequality:

$$\int_{R^n} |(-\Delta)^{-\frac{\alpha}{2}}\mu|^{p'} dx \simeq \int_{R^n} W_{\alpha,p}\mu \, d\mu.$$

For p-Laplacian: $\alpha = 1$, $p > 1$. For k-Hessian: $\alpha = \frac{2k}{k+1}$, $p = k + 1$.

Local Wolff's potential estimates for $\mathbf{\Delta_p}$: [Kilpeläinen-Malý] (Acta Math., 1994): If $-\mathbf{\Delta_p u} = \mu$, $\mu \geq 0$, $\mathbf{u} \geq 0$ in $\mathbf{B_{3R}(x)} \subset \mathbf{\Omega}$, then

$$\mathbf{C_1 W^R_{1,p}}\mu \leq \mathbf{u} \leq \mathbf{C_2} \inf_{\mathbf{B(x,R)}} \mathbf{u} + \mathbf{C_3 W^{2R}_{1,p}}\mu,$$

(constants $\mathbf{C_i} > 0$ depend on $\mathbf{n}$, $\mathbf{p}$).

Global Wolff potential estimates for $\mathbf{\Delta_p}$:

Theorem (Nguyen-Verbitsky, Ann. Math., 2008)

$\mathbf{r} = \mathrm{dist}(\mathbf{x}, \partial\Omega)$, $\mathbf{R} = \mathrm{diam}(\Omega)$, $-\mathbf{\Delta_p u} = \mu$ *in* $\mathbf{\Omega}$, $\mathbf{u} = 0$ *on* $\partial\Omega$*:*

$$\mathbf{K_1 \, W^{\frac{r}{3}}_{1,p}}\mu \leq \mathbf{u} \leq \mathbf{K_2 \, W^{2R}_{1,p}}\mu,$$

($\mathbf{x} \in \mathbf{\Omega}$*;* $\mathbf{K_i} > 0$ *depend on* $\mathbf{n}$, $\mathbf{p}$*).*

Inverting the $\mathbf{p}$-Laplacian: If $-\mathbf{\Delta_p u} = \mu$ on $\mathbf{R^n}$, $\mathbf{u} = 0$ at ∞,
$\mathbf{u} \simeq \mathbf{W_{1,p}}\mu$.

Discrete models of nonlinear equations

Quasilinear equations with nonlinear source terms ($1 < p < \infty$):

$$-\Delta_p u = \omega\, u^q + \mu, \quad u = 0 \text{ on } \partial\Omega,$$

ω, μ are positive measures on $\Omega \subseteq \mathbf{R}^n$.
"Supernatural growth": $q > p - 1$ [Nguyen-Verbitsky, 2008, 2009].
"Natural growth": $q = p - 1$ [Jaye-Verbitsky, 2009] (end-point case).
Dyadic Wolff's potentials ($1 < p < \infty$, $\alpha > 0$):

$$\mathcal{W}^d_{\alpha,\, p}\omega(x) = \sum_{Q \in \mathcal{D}} \left[\frac{|Q|_\omega}{|Q|^{1-\frac{\alpha p}{n}}} \right]^{\frac{1}{p-1}} \chi_Q(x).$$

A dyadic model:

$$u = \mathcal{W}^d_{\alpha,\, p}(u^q\, d\omega) + f, \qquad u = 0 \text{ at } \infty,$$

$u \in L^q_{\text{loc}}(\mathbf{R}^n)$, $f = \mathcal{W}^d_{\alpha,\, p}\mu$ on $\mathbf{R}^n$.

Theorem (Nguyen-Verbitsky, J. Funct. Anal., 2009)

Let ω *and* μ *be nonnegative locally finite measures on* $\mathbf{R}^n$ *and let* $q > p-1 > 0$. *The equation*

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = \omega u^q + \mu$$

has a solution $u \in L^q_{\omega,\,\mathrm{loc}}(\mathbf{R}^n)$, $u \geq 0$, *if and only if, for all* $x \in \mathbf{R}^n$, $r > 0$,

$$\int_0^r \Big(\frac{\omega(B_t(x))}{t^{n-p}}\Big)^{\frac{1}{p-1}} \frac{dt}{t} \cdot \Big[\int_r^\infty \Big(\frac{\mu(B_t(x))}{t^{n-p}}\Big)^{\frac{1}{p-1}}\Big]^{\frac{q}{p-1}-1} \frac{dt}{t} \leq C$$

and, for all balls $B \subset \mathbf{R}^n$,

$$\int_B [W_{1,\,p}\mu_B(y)]^q \, d\omega(y) \leq C\,|B|_\omega.$$

Equivalently, $W_{1,\,p}[(W_{1,\,p}\mu)^q d\omega](x) \leq C\,W_{1,\,p}\mu(x)$ *(with* $C \leq C(p,n)$*). Moreover,* $c_1 W_{1,\,p}\mu(x) \leq u(x) \leq c_2 W_{1,\,p}\mu(x)$.

Quasilinear equations with natural growth terms

Theorem (Jaye-Verbitsky, 2010)

Let $1 < p < n$ *and* $\omega \in M^+(R^n)$.
(i) If there exists a fundamental solution to the equation $-\triangle_p = \omega u^{p-1} + \delta_{x_0}$, $x_0 \in R^n$, *then*

$$|E|_\omega \leq C\,\mathrm{cap}_{1,p}(E),$$

for every compact set E. *Here* $\mathrm{cap}_{1,p}(\cdot)$ *is the usual* p*-capacity associated with the Sobolev space* $L^{1,p}(R^n)$. *Moreover,*

$$u(x) \geq c_1\,|x - x_0|^{\frac{p-n}{p-1}} \exp\left\{c_2 \int_0^{|x-x_0|} \left(\frac{|B_r(x)|_\omega}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right\}$$

$$\times \exp\left\{c_2 \int_0^{|x-x_0|} \frac{|B_r(x_0)|_\omega}{r^{n-p}} \frac{dr}{r}\right\}.$$

Theorem (Jaye-Verbitsky, 2010)

(ii) Conversely, if the above capacitary condition holds with $\mathbf{C} = \mathbf{C}_1(\mathbf{p}, \mathbf{n})$*, then there exists a fundamental solution* $\mathbf{u}$*,*

$$\mathbf{u}(x) \leq c_3 \, |x - x_0|^{\frac{p-n}{p-1}} \exp\left\{ c_4 \int_0^{|x-x_0|} \left(\frac{|B_r(x)|_\omega}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right\}$$

$$\times \exp\left\{ c_4 \int_0^{|x-x_0|} \frac{|B_r(x_0)|_\omega}{r^{n-p}} \frac{dr}{r} \right\}.$$

If $\mathbf{p} \geq \mathbf{n}$*, there is no positive solution to* $-\triangle_p = \omega u^{p-1} + \delta_{x_0}$ *unless* $\omega = 0$*.*

Hessian operators

(Caffarelli, Nirenberg, and Spruck; Ivochkina; Krylov)

Let $\Omega \subset \mathbf{R}^n$. Let F_k ($k = 1, 2, \ldots, n$) be the k-Hessian operator

$$F_k[u] = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

Here $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hessian matrix $D^2 u$.

In other words: $F_k[u]$ is the sum of the $k \times k$ principal minors of $D^2 u$.

Recent progress:

The notion of the **k**-Hessian measure, weak continuity, nonlinear potential estimates, Hessian Sobolev and trace inequalities [Trudinger-Wang] (Ann. Math., 1999; Amer. Math. J., 2002), [Labutin] (Duke Math. J., 2003), [Nguyen-Verbitsky] (Ann. Math., 2008), [Verbitsky], (2010).

Local Wolff's potential estimates for $\mathbf{F_k}$: [Labutin] (Duke Math. J., 2003):
If $\mathbf{F_k[u] = \mu}$, $\mu \geq 0$, $\mathbf{u \leq 0}$ **k**-convex in $\mathbf{B_{3R}(x) \subset \Omega}$,

$$C_1 W^{R}_{\frac{2k}{k+1}, k+1}\mu \leq |u| \leq C_2 \inf_{B(x,R)} |u| + C_3 W^{2R}_{\frac{2k}{k+1}, k+1}\mu,$$

(constants $\mathbf{C_i > 0}$ depend on **n**, **p**).

Global Wolff potential estimates for $\mathbf{F_k}$:

Theorem (Nguyen-Verbitsky, Ann. Math., 2008)

$\mathbf{r = \mathrm{dist}(x, \partial\Omega)}$, $\mathbf{R = \mathrm{diam}(\Omega)}$, $\mathbf{F_k[u] = \mu}$ *in* $\mathbf{\Omega}$, $\mathbf{u = 0}$ *on* $\mathbf{\partial\Omega}$*:*

$$K_1\, W^{\frac{r}{3}}_{\frac{2k}{k+1}, k+1}\mu \leq |u| \leq K_2\, W^{2R}_{\frac{2k}{k+1}, k+1}\mu,$$

($\mathbf{x \in \Omega}$*;* $\mathbf{K_i > 0}$ *depend on* **n**, **p***).*

Inverting the **k**-Hessian: If $\mathbf{F_k[u] = \mu}$ on $\mathbf{R^n}$, $\mathbf{u = 0}$ at ∞,
$\mathbf{u \simeq W_{\frac{2k}{k+1}, k+1}\mu}$.

Hessian operators and k-convexity

In terms of viscosity solutions [Trudinger-Wang] (Ann. Math., 1999)

An upper semicontinuous function $\mathbf{u : \Omega \to [-\infty, \infty)}$ is **k-convex** in $\mathbf{\Omega}$ if $\mathbf{F_k[q] \geq 0}$ for any quadratic polynomial $\mathbf{q}$ such that $\mathbf{u - q}$ has a local finite maximum in $\mathbf{\Omega}$ ($\mathbf{1 \leq k \leq n}$). A function $\mathbf{u \in C^2_{loc}(\Omega)}$ is **k**-convex iff

$$\mathbf{F_j[u] \geq 0 \text{ in } \Omega, \quad j = 1, \ldots, k.}$$

Denote by $\mathbf{\Phi^k(\Omega)}$ the class of all **k**-convex functions in $\mathbf{\Omega}$ (not identically equal to $-\infty$ in each component of $\mathbf{\Omega}$).

$$\mathbf{\Phi^n(\Omega) \subset \Phi^{n-1}(\Omega) \cdots \subset \Phi^1(\Omega).}$$

$\mathbf{\Phi^1(\Omega)}$ classical subharmonic functions in $\mathbf{\Omega}$, $\mathbf{\Phi^n(\Omega)}$ convex functions.

Weak continuity

Theorem (Trudinger-Wang, Ann. Math., 1999)

For each $\mathbf{u} \in \mathbf{\Phi^k(\Omega)}$*, there exists a nonnegative Borel measure* $\mu_\mathbf{k}[\mathbf{u}]$ *in* $\mathbf{\Omega}$ *such that*

1. $\mu_\mathbf{k}[\mathbf{u}] = \mathbf{F_k[u]}$ *for* $\mathbf{u} \in \mathbf{C^2(\Omega)}$,
2. *if* $\{\mathbf{u_m}\}$ *is a sequence in* $\mathbf{\Phi^k(\Omega)}$ *converging in* $\mathbf{L^1_{loc}(\Omega)}$ *to* $\mathbf{u} \in \mathbf{\Phi^k(\Omega)}$*, then the corresponding measures* $\mu_\mathbf{k}[\mathbf{u_m}]$ *converge weakly to* $\mu_\mathbf{k}[\mathbf{u}]$.

Hessian measure: $\mu_\mathbf{k}[\mathbf{u}]$ is called the $\mathbf{k}$-Hessian measure associated with $\mathbf{u} \in \mathbf{\Phi^k(\Omega)}$.

Assumption

$\mathbf{\Omega}$ is a bounded uniformly $\mathbf{(k-1)}$-convex domain in $\mathbb{R}^\mathbf{n}$, $\mathbf{H_j(\partial\Omega) > 0}$, $\mathbf{j = 1, ..., k-1}$; $\mathbf{H_j(\partial\Omega)}$ denotes the $\mathbf{j}$-mean curvature of the boundary $\mathbf{\partial\Omega}$.

Hessian capacity

Trudinger-Wang defined $\mathbf{k}$-Hessian capacity analogously to complex Monge-Ampère:

$$\mathrm{cap}_{\mathbf{k}}(\mathbf{E}, \boldsymbol{\Omega}) = \sup \Big\{ \int_{\mathbf{E}} \mathbf{d}\mu_{\mathbf{k}}[\mathbf{u}] \Big\},$$

supremum is taken over $\mathbf{k}$-convex functions $\mathbf{u}$ in $\boldsymbol{\Omega}$ such that $-\mathbf{1} < \mathbf{u} < \mathbf{0}$, $\mu_{\mathbf{k}}[\mathbf{u}]$ is the $\mathbf{k}$-Hessian measure associated with $\mathbf{u}$.

Labutin (2003): Relation to Hausdorff measure $\mathbf{H}^{\mathbf{n}-\mathbf{2k}}(\cdot)$.

Nguyen-Verbitsky (2008): $\mathbf{k}$-Hessian capacity is equivalent to classical fractional capacity $\mathrm{cap}_{\frac{2k}{k+1}, \mathbf{k}+\mathbf{1}}(\cdot)$ for Sobolev space $\mathbf{W}^{\frac{2k}{k+1}, \mathbf{k}+\mathbf{1}}(\boldsymbol{\Omega})$.

Equations of Lane-Emden type

Removable singularities for

$$(-1)^k F_k[u] = u^q \quad \text{in } \Omega.$$

Dirichlet problem (viscosity solutions) for

$$(-1)^k F_k[u] = u^q + \mu \quad \text{in } \Omega,$$

$q > k$; $u, \mu \geq 0$. (The case $q \leq k$ simpler.)
A different capacity: a compact set $E \subset \Omega$ is removable iff $\mathrm{cap}_{2k, \frac{q}{q-k}}(E) = 0$.
Viscosity solutions exist if and only if

$$\mu(E) \leq c\, \mathrm{cap}_{2k, \frac{q}{q-k}}(E).$$

More generally: $(-1)^k F_k[u] = \omega\, u^q + \mu$ (dyadic models, Wolff's potentials, two weight inequalities) (Nguyen-Verbitsky, 2006, 2008).

Hessian equations with natural growth terms

Consider the fundamental solution $\mathbf{u = u(x, x_0)}$:

$$\mathbf{F_k[-u] = \omega\, u^k + \delta_{x_0}, \quad u \geq 0.}$$

Theorem (Jaye-Verbitsky, 2010)

Let $\mathbf{1 \leq k < n/2}$. *Suppose* ω *is a positive measure satisfying* $\mathbf{|E|_\omega \leq C\, \mathrm{cap}_k(E)}$, *where* $\mathbf{C \leq C_0(k, n)}$. *Then there exists a minimal fundamental solution so that*

$$\mathbf{u(x, x_0) \leq c|x - x_0|^{2-\frac{n}{k}} \exp\left(c \int_0^{|x-x_0|} \left(\frac{\omega(B(x, r))}{r^{n-2k}}\right)^{1/k} \frac{dr}{r}\right) \cdot \exp\left(c \int_0^{|x-x_0|} \frac{\omega(B(x_0, r))}{r^{n-2k}} \frac{dr}{r}\right).}$$

The converse also holds with another $\mathbf{C(k, n)}$. *If* $\mathbf{k \geq n/2}$, *then* $\mathbf{\omega = 0}$.

Publications

1. “The fundamental solution of nonlinear equations with natural growth terms,” **preprint** (2010), joint with Benjamin Jaye, http://arxiv.org/pdf/1002.4664.pdf
2. “Global estimates for kernels of Neumann series, Green’s functions, and the conditional gauge,” **preprint** (2010), joint with Michael Frazier and Fedor Nazarov.
3. “Global Green’s function estimates,” **preprint** (2009), joint with Michael Frazier.
4. “Hessian Sobolev inequalities and their extensions,” **preprint** (2008).
5. “Singular quasilinear and Hessian equations and inequalities,” **J. Funct. Anal.** (2009), joint with Nguyen Cong Phuc.
6. “Quasilinear and Hessian equations of Lane-Emden type,” **Ann. Math.** (2008), joint with Nguyen Cong Phuc.
7. “Local integral estimates and removable singularities for quasilinear and Hessian equations with nonlinear source terms,” **Comm. PDE** (2006), joint with Nguyen Cong Phuc.