Porous medium flow with nonlocal effects

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Nonlinear PDE and Boundary Value Problems
with Measure Data
honoring Prof. Marie-Françoise Bidaut-Véron
and Prof. Laurent Véron

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Outline

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PME with fractional diffusion operators

- We study two models for flow in porous media including nonlocal (long-range) diffusion effects.
- The first model is based on Darcy's law and the pressure is related to the density by an inverse fractional Laplacian operator. We prove existence of solutions that propagate with finite speed.
- The model has the very interesting property that mass preserving selfsimilar solutions can be found by solving an elliptic obstacle problem with fractional Laplacian for the pair pressure-density.
- We use entropy methods to show that the asymptotic behaviour is described after renormalization by these solutions which play the role of the Barenblatt profiles of the standard porous medium model.
- *This is a joint ongoing project with Luis Caffarelli, Univ. Texas.* Started in October 2007
  Last part also with Fernando Soria, UAM
- The second model is more in the spirit of fractional Laplacian flows, but nonlinear. It has infinite speed of propagation. *Joint work with local team: Arturo de Pablo, Fernando Quirós and Ana Rodriguez, Madrid.*

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Mr Laurent et son enfant

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Porous Medium equations

- The simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion.

- A more general model of nonlinear diffusion takes the divergence form

$$\partial_t H(u) = \nabla \cdot \vec{A}(x, u, Du) + \mathcal{B}(x, t, u, Du)$$

with monotonicity conditions on H and $\nabla_p \vec{A}(x, t, u, p)$ and structural conditions on $\vec{A}$ and $\mathcal{B}$. This generality includes Stefan Problems, p-Laplacian flows (including $p = \infty$ and total variation flow $p = 1$) and many others.

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The basics

- The equation is re-written for $m = 2$ as

$$\boxed{\frac{1}{2}u_t = u\Delta u + |\nabla u|^2}$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$\boxed{u_t = |\nabla u|^2}$$

This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$\boxed{v_t = (m-1)v\Delta v + |\nabla v|^2}$$

where $\boxed{v = cu^{m-1}}$ is the pressure; normalization $c = m/(m-1)$.
This separates $m > 1$ PME - from - $m < 1$ FDE

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These are the main topics of mathematical analysis (1958-2006):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit? C^k for some k?*
- Regularity and movement of interfaces: *C^k for some k?.*
- Asymptotic behaviour: *patterns and rates? universal?*
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- Generalizations: fast models, inhomogeneous media, anisotropic media, applications to physics, biology, geometry or image processing;

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  elliptic work by Brezis, Eidus, Kamin, Tesei; parabolic Rosenau and SK, work by Karlsen et al for general variable coefficients
- Dynamic boundary conditions.
  papers with E. Vitillaro (CPDE, M3AS, 2008, JMAA 2009);
- Nonlinear diffusion with fractional diffusion operators.
  Most works by Caffarelli, Silvestre, Cabré and others on stationary problems

Some of our lines of research

- Very fast diffusion flows, logarithmic diffusion, gradient diffusion (p-Laplacian) flows
  A Blanchet, M. Bonforte, J Dolbeault, G Grillo; with R. Iagar, ...
- Flows on manifolds:
  paper with Bonforte and Grillo in JEE, 2008;
  paper with Lu, Lei, Villani in arXiv, JMPA, 2009 ...
- Flows in inhomogeneous media.
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Outline

Nonlocal diffusion model

- The model arises from the consideration of a continuum, say, a fluid, represented by a density distribution $u(x,t) \geq 0$ that evolves with time following a velocity field $\mathbf{v}(\mathbf{x},\mathbf{t})$, according to the continuity equation

$$u_t + \nabla \cdot (u\,\mathbf{v}) = 0.$$

- We assume next that $\mathbf{v}$ derives from a potential, $\mathbf{v} = -\nabla p$, as happens in fluids in porous media according to Darcy's law, an in that case p is the pressure. But potential velocity fields are found in many other instances, like Hele-Shaw cells, and other recent examples.
- We still need a closure relation to relate u and p. In the case of gases in porous media, as modeled by Leibenzon and Muskat, the closure relation takes the form of a state law $p = f(u)$, where f is a nondecreasing scalar function, which is linear when the flow is isothermal, and a power of u if it is adiabatic.

  The linear relationship happens also in the simplified description of water infiltration in an almost horizontal soil layer according to Boussinesq. In both cases we get the standard porous medium equation, $u_t = c\Delta(u^2)$.

  See PME Book for these and other applications (more than a dozen!).

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Nonlocal diffusion model II. The problem

- The diffusion model with nonlocal effects we propose here uses a closure relation of the form $p = \mathcal{K}(u)$, where $\mathcal{K}$ is a linear integral operator, which we assume in practice to be the inverse of a fractional Laplacian. Hence, p es related to u through a fractional potential operator, $\mathcal{K} = (-\Delta)^{-s}$ with kernel

$$k(x, y) = c|x - y|^{-(n-2s)}$$

(i.e., a Riesz operator). We have $(-\Delta)^s p = u$.

- The diffusion model with nonlocal effects is thus given by the system

$$u_t = \nabla \cdot (u \nabla p), \quad p = \mathcal{K}(u). \tag{1}$$

where u is a function of the variables (x, t) to be thought of as a density or concentration, and therefore nonnegative, while p is the pressure, which is related to u via a linear operator $\mathcal{K}$.

- The problem is posed for $x \in \mathbb{R}^n$, $n \geq 1$, and $t > 0$, and we give initial conditions

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where u_0 is a nonnegative, bounded and integrable function in $\mathbb{R}^n$.

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Extreme cases

- If we take $s = 0$, $\mathcal{K} =$ the identity operator, we get the standard porous medium equation, whose behaviour is well-known, see references later.
- In the other end of the s interval, when $s = 1$ and we take $\mathcal{K} = -\Delta$ we get

$$(3) \qquad u_t = \nabla u \cdot \nabla p - u^2, \quad -\Delta p = u.$$

In one dimension this leads to $u_t = u_x p_x - u^2$, $p_{xx} = -u$. In terms of $v = -p_x = \int u\,dx$ we have

$$v_t = up_x + c(t) = -v_x v + c(t),$$

For $c = 0$ this is the Burgers equation $v_t + vv_x = 0$ which generates shocks in finite time but only if we allow for u to have two signs.

- A recent case is proposed by Ambrosio and Serfaty[1] in a model in superconductivity, and describing the evolution of the vortex-density. It is equivalent to our case $s = 1$ but posed in a bounded domain.
- The interpolation we study has better properties than $s = 1$ but is different in many properties from $s = 0$.

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Nonlocal diffusion model III

- The interest in using fractional Laplacians in modeling diffusive processes has a wide literature, especially when one wants to model long-range diffusive interaction, and this interest has been activated by the recent progress in the mathematical theory as represented in [AC][2], [CSS][3], or cf. the talks by J.M. Roquejoffre and L. Silvestre at FBP Congress in Stockholm 2008.
- A variant of the proposed model was studied by Lions and Mas-Gallic [LMG] [4] They study the regularization of the velocity field in the standard porous medium equation by means of a convolution kernel to get a system like ours, with a difference, namely that they assume the kernel to be smooth and integrable. Since the kernel of the fractional operator $(-\Delta)^s$ is $k(x,y) = |x-y|^{n-2s}$, we are away from that case, but it may serve as a regularization step below.

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Nonlocal diffusion model IV

- Modeling dislocation dynamics as a continuum.

  • A. K. Head. *Dislocation group dynamics II. Similarity solutions of the continuum approximation.* Phil. Mag. **26** (1972), 65–72.

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  This is a one-dimensional model. By integration in x they introduce viscosity solutions a la Crandall-Evans-LIons. Uniqueness holds.

- More generally, it could be assumed that $\mathcal{K}$ is an operator of integral type defined by convolution on all of $\mathbb{R}^n$, with the assumptions that is positive and symmetric. The fact the $\mathcal{K}$ is a homogeneous operator of degree $2s$, $0 < s < 1$, will be important in the proofs. An interesting variant would be $\mathcal{K} = (-\Delta + cI)^{-s}$. We are not exploring such extensions.

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Nonlocal diffusion model V

- Different versions to the evolution process are obtained when the pressure is related to u in other ways. Thus, we could relax the relation between the pressure and the density into the form

$$\partial_t p + (-\Delta)^s p = u.$$

- This reflection is motivated by a very important system, the Keller-Segel chemotaxis model,[5] in which the phenomenon which is modeled is not diffusion but concentration, and p is replaced by variable c proportional to the concentration of the chemical substance responsible for the aggregation of the population. A suitable general system is proposed in the form

$$u_t = \varepsilon \Delta u - \nabla \cdot (u \nabla c), \quad \delta c_t + \mathcal{K}^{-1} c = f(u, c). \tag{4}$$

The standard chemotaxis model uses $\mathcal{K} = -\Delta$ and $f(u, c) = u - bc$. In the limit case where δ and b are zero, and if we use as $\mathcal{K}$ an integral operator as described above, we get a model with a term like ours but different sign, $u_t = \varepsilon \Delta u - \nabla \cdot (u \nabla \mathcal{K}(u))$, as a consequence of the fact that c answers for aggregation and not diffusion.

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Alternative model

An alternative natural equation is

$$\partial_t u + (-\Delta)^s u^m = 0. \tag{5}$$

This model arises from stochastic differential equations when modelling for instance heat conduction with anomalous properties and one introduces jump processes into the modelling. The physical situation looks still a bit confusing to me.

Applied References: Stefano Olla and collab., his current work at Paris Dauphine: • G. Basile et C. Bernardin, S. Olla, *Momentum conserving model with anomalous thermal conductivity in low dimensional systems,* Phys. Rev. Lett. 96, 204303, 2006. • Jara, M., Komorowski, T., Olla, S., *Limit theorems for additive functionals of a Markov chain*. http://arxiv.org/abs/0809.0177.

Mathematics: • Athanasopoulos, I.; Caffarelli, L. A. *Continuity of the temperature in boundary heat control problems*, 2009, subm. to Adv. Math.
• Arturo de Pablo, Fernando Quirós, Ana Rodríguez, and Juan Luis Vázquez, *A fractional porous medium equation*, arXiv:1001.2383v1 [math.AP]

Outline

Main estimates[6]

The equation is $\partial_t u = \nabla \cdot (u \nabla K(u))$, posed in the whole space $\mathbb{R}^n$.

We consider $K = (-\Delta)^{-s}$ for some $0 < s < 1$ acting on Schwartz class functions defined in the whole space. It is a positive essentially self-adjoint operator.

We let $H = K^{1/2} = (-\Delta)^{-s/2}$.

We do formal calculations, assuming that $u \geq 0$ satisfies the required smoothness and integrability assumptions. This is to be justified later by approximation.

- Conservation of mass

$$\frac{d}{dt}\int u(x,t)\,dx = 0. \tag{6}$$

- First energy estimate:

$$\frac{d}{dt}\int u(x,t)\log u(x,t)\,dx = -\int |\nabla H u|^2\,dx. \tag{7}$$

- Second energy estimate

$$\frac{d}{dt}\int |Hu(x,t)|^2\,dx = -2\int u|\nabla K u|^2\,dx. \tag{8}$$

[6] Luis Caffarelli and J. L. Vázquez, *Nonlinear porous medium flow with fractional potential pressure*. arXiv:1001.0410v1 [math.AP].

Main estimates II

- Conservation of positivity: $u_0 \geq 0$ implies that $u(t) \geq 0$ for all times.
- L^∞ estimate. We prove that the L^∞ norm does not increase in time.
  Proof. At a point of maximum of u at time $t = t_0$, say $x = 0$, we have
  $$u_t = \nabla u \cdot \nabla P + u\,\Delta K(u).$$
  The first term is zero, and for the second we have $-\Delta K = L$ where $L = (-\Delta)_q$ with $q = 1 - s$ so that
  $$\Delta Ku(0) = -Lu(0) = -\int \frac{u(0) - u(y)}{|y|^{n+2(1-s)}}\,dy \leq 0.$$
  This concludes the proof.
- We did not find a clean comparison theorem, a form of the usual maximum principle is not proved.
- L^p bounds are conserved

Finite propagation in dimension $n = 1$. Solutions with compact support

- One of the most important features of the porous medium equation and other related degenerate parabolic equations is the property of finite propagation, whereby compactly supported initial data $u_0(x)$ give rise to solutions $u(x,t)$ that have the same property for all positive times, i.e., the support of $u(\cdot,t)$ is contained in a ball $B_{R(t)}(0)$ for all $t > 0$.
- One possible proof in the case of the PME is by constructing explicit weak solutions with that property and then using the comparison principle, that holds for that equation.
- Since we do not have such a general principle here, we have to devise a comparison method with a suitable family of "strict supersolutions", in fact excessive supersolutions. The technique has to be adapted to the peculiar form of the integral kernels involved in operator $\mathcal{K}$.

Solutions with compact support

- We begin with $n = 1$. We assume that our solution $u(x, t) \geq 0$ has bounded initial data $u_0(x) = u(x, t_0) \leq M$ with compact support and is such that

$$u_0 \text{ is below the parabola } a(x - b)^2,\ a, b > 0.$$

with graphs strictly separated. We may assume that u_0 is located under the left branch of the parabola. We take as comparison function

$$U(x, t) = a(Ct - (x - b))^2,$$

and argue at the first point and time where $u(x, t)$ touches U the left branch of the parabola from below.

Lemma

Assume that u is a bounded solution of equation (1) *with* $\mathcal{K} = (-\Delta)^{-s}$ *and* $0 < s < 1/2$. *Assume that u has compact support for all* $t > t_0$ *and is* C^2 *for* $u > 0$. *If C is large enough, then there is no contact from below between u and U at a point where* $u > 0$. *Actually, we can take* $C = C_0(n, s)M^{(1/2)+s}a^{(1/2)-s}$.

Conclusion is true for $n > 1$ with no extra restriction on s, $0 < s < 1$

Positivity. Instantaneous Boundedness

- Solutions that are positive at $t = 0$ remain positive for all times.
  Proof. Use an excessive subsolution.

- Solutions are bounded in terms of data in L^p, $1 \le p \le \infty$.
  Use (the de Giorgi or the Moser) iteration technique on the Caffarelli-Silvestre extension as in Caffarelli-Vasseur.
  Or use energy estimates based on the properties of the quadratic and bilinear forms associated to fractional , and then the iteration technique

- **Theorem** *Let u be a weak solution the IVP for the FPME with data $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, as constructed before. Then, there exists a positive constant C such that for every $t > 0$*

$$\sup_{x\in\mathbb{R}^n} |u(x,t)| \le C\, t^{-\alpha} \|u_0\|^{\gamma}_{L^1(\mathbb{R}^n)} \tag{9}$$

  with $\alpha = n/(n+2-2s)$, $\gamma = (2-2s)/((n+2-2s)$. The constant C depends only on n and s.

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Continuity

- Bounded weak solutions u are C^α if $s < 1/2$ and continuous with a uniform modulus for all $0 < s < 1$.
  Use variant of Luis local regularity theory for elliptic and parabolic PDE's. The crucial point is to get a local version of the energy inequalities that can be iterated. It involves a delicate manipulation of the bilinear forms associated to the fractional operator, with amounts to knowing well the H^s spaces and then doing nonlinear versions of the embeddings and bounds.

  Paper in preparation with L Caffarelli and Fernando Soria.

Outline

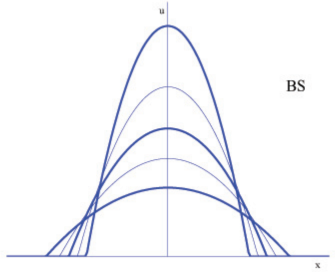
Barenblatt profiles and Asymptotics

- These profiles are the alternative to the Gaussian profiles.

  They are source solutions. Source *means that* $u(x,t) \to M\,\delta(x)$ *as* $t \to 0$.

  us
- Explicit formulas (1950, 52):

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)}$$



$\alpha = \frac{n}{2+n(m-1)}$

$\beta = \frac{1}{2+n(m-1)} < 1/2$

Height $u = Ct^{-\alpha}$

Free boundary at distance $|x| = ct^{\beta}$

Scaling law; anomalous diffusion versus Brownian motion

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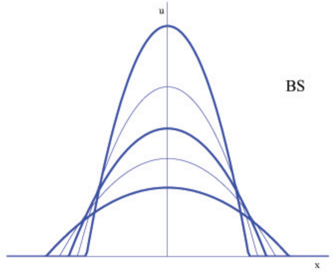

$\alpha = \frac{n}{2+n(m-1)}$

$\beta = \frac{1}{2+n(m-1)} < 1/2$

Height $u = Ct^{-\alpha}$

Free boundary at distance $|x| = ct^{\beta}$

Scaling law; anomalous diffusion versus Brownian motion

Asymptotic behaviour I
Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

$$\boxed{t^\alpha |u(x,t) - B(x,t)| \to 0}$$

For every $p \geq 1$ we have

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: α *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

- Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$.

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Asymptotic behaviour. II

- The rates. Carrillo-Toscani 2000. Using entropy functional with entropy dissipation control you can prove decay rates when $\int u_0(x)|x|^2\,dx < \infty$ (finite variance):

$$\|u(t) - B(t)\|_1 = O(t^{-\delta}),$$

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- Eventual geometry, concavity and convexity Result by Lee and Vazquez (2003): Here we assume compact support.There exists a time after which the pressure is concave, the domain convex, the level sets convex and

$$t\,\|(D^2 v(\cdot,t) - k\mathbf{I})\|_\infty \to 0$$

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Calculations of the entropy rates

- We rescale the function as $u(x,t) = r(t)^n \rho(y\, r(t), s)$ where $r(t)$ is the Barenblatt radius at $t+1$, and "new time" is $s = \log(1+t)$. Equation becomes
$$\rho_s = \mathrm{div}\,(\rho(\nabla \rho^{m-1} + \frac{c}{2}\nabla y^2)).$$
- Then define the entropy
$$E(u)(t) = \int (\frac{1}{m}\rho^m + \frac{c}{2}\rho y^2)\,dy$$
The minimum of entropy is identified as the Barenblatt profile.
- Calculate
$$\frac{dE}{ds} = -\int \rho|\nabla \rho^{m-1} + cy|^2\,dy = -D$$
Moreover,
$$\frac{dD}{ds} = -R, \quad R \sim \lambda D.$$
We conclude exponential decay of D and E in new time s, which is potential in real time t.

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Asymptotic behaviour

for the nonlocal PME [7]

[7] *Asymptotic behaviour of a porous medium equation with fractional diffusion,* Luis Caffarelli, Juan Luis Vazquez, Preprint

Rescaling for the NL-PME

We now begin the study of the large time behaviour.

- Inspired by the asymptotics of the standard porous medium equation, we define the renormalized flow through the transformation

(10) $$u(x,t) = t^{-\alpha} v(x/t^{\beta}, \tau)$$

with new time $\tau = \log(1+t)$. We also put $y = x/t^{\beta}$ as rescaled space variable. In order to cancel the factors including t explicitly, we get the condition on the exponents

(11) $$\alpha + (2-2s)\beta = 1$$

- We use the homogeneity of K in the form

(12) $$(Ku)(x,t) = t^{-\alpha+2s\beta}(Kv)(y,\tau).$$

- If we also want conservation of (finite) mass, then we must put $\alpha = n\beta$, and we arrive at the the precise value of the exponents:

$$\beta = 1/(n+2-2s), \quad \alpha = n/(n+2-2s).$$

Renormalized flow

- We thus arrive at the *nonlinear, nonlocal Fokker-Plank equation*

$$v_\tau = \nabla_y \cdot (v\,(\nabla_y K(v) + \beta y)) \tag{13}$$

- This formula implies a transformation for the pressure of the form

$$p(u)(x,t) = t^{-\sigma} p(v)(x/t^{\beta},\tau), \quad \text{with } \sigma = \alpha - 2s\beta = 1 - 2\beta = \frac{n-2s}{n+2-2s} < 1. \tag{14}$$

- **Stationary renormalized solutions**. They are the solutions $U(y)$ of

$$\nabla_y \cdot (U\,\nabla_y(P + a|y|^2)) = 0, \quad P = K(U). \tag{15}$$

where $a = \beta/2$, and β defined just above. Since we are looking for asymptotic profiles of the standard solutions of the NL-PME we also want $U \geq 0$ and integrable. The simplest possibility is integrating once and getting the radial version

$$U\,\nabla_y(P + a|y|^2)) = 0, \quad P = K(U), \quad U \geq 0. \tag{16}$$

The first equation gives an alternative choice that reminds of the complementary formulation of the obstacle problems.

Obstacle problem

- Indeed, if we solve the obstacle problem with fractional Laplacian we will obtain a unique solution $P(y)$ of the problem:

$$
\begin{aligned}
&P \geq \Phi, \quad U = (-\Delta)^s P \geq 0; \\
&\text{either } \ P = \Phi \text{ or } \ U = 0.
\end{aligned}
\tag{17}
$$

  with $0 < s < 1$. Here we have to choose as obstacle

$$\Phi = C - a|y|^2,$$

  where C is any positive constant and $a = \beta/2$. For uniqueness we also need the condition $P \to 0$ as $|y| \to \infty$.

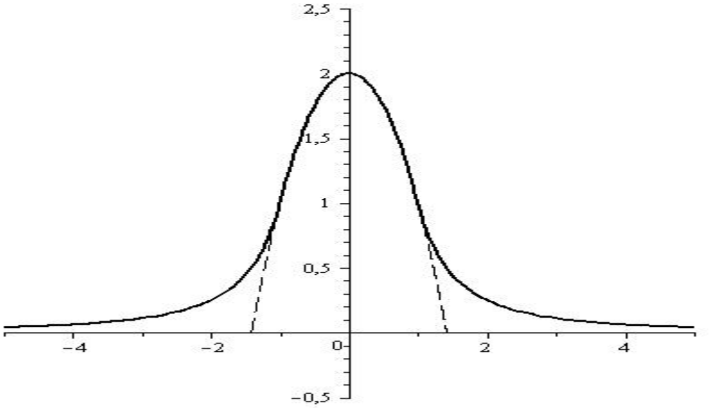
- The theory is developed in A-C-S, C-S-S, the solution is unique and belongs to the space H^{-s} with pressure in H^s. The solutions have $P \in C^{1,s}$ and $U \in C^{1-s}$.
- Note that for $C \leq 0$ the solution is trivial, $P = 0$, $U = 0$, hence we choose $C > 0$. We also note the the pressure is defined but for a constant, so that we may take without loss of generality $C = 0$ and take as pressure $\widehat{P} = P - C$ instead of P. But then $P \to 0$ implies that $\widehat{P} \to -C$ as $|y| \to \infty$, so we get a one parameter family of stationary profiles that we denote $U_C(y)$.

Barenblatt solutions of new type.

- The above solution of this obstacle problem produces a correct weak solution of the NL-PM equation with initial data a Dirac delta for the density u, i.e., it is the source-type or Barenblatt solution for this problem, which is a profile $U \geq 0$.
  It is positive in the contact set of the obstacle problem, $\mathcal{C} = \{|y| \leq R(C)\}$,
  and is zero outside, hence it has compact support.

- It is clear that R is smaller than the intersection of the parabola Φ with the axis $R_1 = (C/a)^{1/2}$.

- On the other hand, the pressure $P(|y|)$ is always positive and decays to zero as $|y| \to \infty$ according to fractional potential theory, cf Landau, Stein.
  The rate is $P = O(|y|^{2s-n})$.



The solution of the obstacle problem with parabolic obstacle

Estimates for the renormalized problem. Entropy dissipation.

- Our main problem is now to prove that these profiles are attractors for the renormalized flow.
- We review the estimates of Main Estimates Section above in order to adapt them to the renormalized problem. ç
- There is no problem is reproving mass conservation or positivity.
- First energy estimate becomes (recall that $H = K^{1/2}$)

$$
\begin{aligned}
&\frac{d}{d\tau}\int v(y,\tau)\log v(y,\tau)\,dy \\
&= -\textstyle\int |\nabla Hv|^2\,dy - \beta \int \nabla v\cdot y \\
&= -\int |\nabla Hv|^2\,dy + \alpha \int v.
\end{aligned}
\tag{18}
$$

Estimates for the renormalized problem. Entropy dissipation.

- However, the second energy estimate has an essential change. We need to define the entropy of the renormalized flow as

$$\mathcal{E}(v(\tau)) := \frac{1}{2}\int_{\mathbb{R}^n} (v\,K(v) + \beta y^2 v)\,dy \tag{19}$$

The entropy contains two terms. The first is

$$E_1(v(\tau)) := \int_{\mathbb{R}^n} v\,K(v)\,dy = \int_{\mathbb{R}^n} |Hv|^2\,dy, \quad H = K^{1/2}$$

hence positive. The second is the moment $E_2(v(\tau)) = M_2(v(\tau)) := \int y^2 v\,dy$, also positive. By differentiation we get

$$\frac{d}{d\tau}\mathcal{E}(v) = -\mathcal{I}(v), \qquad \mathcal{I}(v) := \int \left|\nabla(Kv + \frac{\beta}{2}y^2)\right|^2 v dy. \tag{20}$$

This means that whenever the initial entropy is finite, then $\mathcal{E}(v(\tau))$ is uniformly bounded for all $\tau > 0$, $\mathcal{I}(v)$ is integrable in $(0, \infty)$ and

$$\mathcal{E}(v(\tau)) + \iint \left|\nabla(Kv + \frac{\beta}{2}y^2)\right|^2 vdy\,dt \le \mathcal{E}(v_0).$$

Convergence.

- The standard idea is to let $t \to \infty$ in the renormalized flow. Since the entropy goes down there is a limit
$$E_* = \lim_{t\to\infty} \mathcal{E}(t) \geq 0.$$
Since u is bounded in L^1_x unif in t, and also ux^2 is bounded in L^1_x unif in t, and moreover $|\nabla H(u)| \in L^2_x$ unif in t, we have that $u(t)$ is a compact family that there is a subsequence $t_j \to \infty$ that converges in L^1_x and almost everywhere to a a limit $u_* \geq 0$. The mass of u_* is the same mass of u. One consequence is that the lim inf of the component $E_2(u(t_j))$ is equal or larger that $M_2(u_*)$.
- We also have $H(u) \in L^2_x$ uniformly in t. The boundedness of $\nabla H(u)$ in L^2_x implies the compactness of $H(u)$ in space, so that it converges along a subsequence to v_* . This allows to pass to the limit in $E_1(u(t_j))$ and obtain a correct limit. We have $v_* = H(u_*)$.
- Since the mass of u for large x is small On the other hand, also

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Convergence

- Now we get the consequence that for every $h > 0$ fixed

$$\int_{t_j}^{t_j+h} \int \left|\nabla(Ku + \frac{\beta}{2}x^2)\right|^2 udx\,dt \to 0.$$

  This implies that if $w(x,t) = Ku + \frac{\beta}{2}x^2$ and $w_h(x,t) = w(x,t+h)$, then $u_h|\nabla w_h|^2$ converges to zero as $h \to \infty$ in $L^1(\mathbb{R}^n \times (0,T))$. Then w_h converges to a constant in space wherever u is not zero, and that constant must be $Ku_* + \frac{\beta}{2}x^2$ along the said subsequence, hence constant also in time

- This means that the limit is a solution of the Barenblatt obstacle problem.

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Work to Do

- Study the regularity of the solutions
- Study the regularity of the free boundary
- Study fine asymptotic behaviour
- Study problems in bounded domains
- Decide conditions of uniqueness
- Decide conditions of comparison
- Write a performing numerical code
- Discuss the Stochastic Particle Models in the literature that involve long-range effects and anomalous diffusion parameters.

Thank you

Muchas gracias

♠

Mes meilleurs voeux pour

Un couple fortuné,

avec tant de théorèmes prouvés,

tants à prouver!

En plus, après 60 on est sage, ...

Thank you

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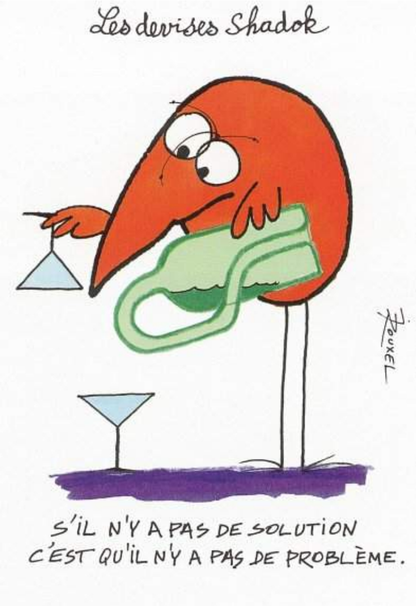
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