Very singular solutions of semi-linear parabolic equations with degenerate absorption potential

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1. "Fundamental" and "very singular" solutions of semilinear parabolic equations

1.1 "Fundamental" solutions to:

$$(P_1)\begin{cases} u_t - \Delta u + h_0|u|^{p-1}u = 0 \text{ in } Q,\ p > 1 \\ u(0,x) = k\,\delta(x)\,,\ \ \forall\, k > 0, \end{cases}$$

where $Q = \mathbb{R}^N \times\ (0,T)$, $h_0 = const > 0$, $N \geqslant 1$. (H.Brezis, A.Friedman -1983). If p is supercritical

$$p \geqslant p_0 := 1 + \frac{2}{N}\,,$$

then (P_1) does not have solution; if p is subcritical: $p < p_0$, then (P_1) has unique solution $u_k(t,x) \geqslant 0$ and:

$$u_k(t,0) \sim t^{-\frac{N}{2}} \text{ as } t \to 0$$

1.2 Very singular solutions (v.s. solution): (H.Brezis, L.Peletier, D.Terman – 1985):
If $p < p_0$ (subcritical case), then there exists function $u_\infty(t,x)$:

$$u_\infty := \lim_{k\to\infty} u_k\ \ \forall (t,x) \in Q : u_\infty(0,x) = 0\ \forall\, x \neq 0,$$

and u_∞ is solution of equation (P_1) with infinite initial mass:

$$\int_{R^N} u_\infty(t,x)dx \to \infty \text{ as } t \to o;$$

moreover,

$$u_\infty(t,x) = t^{-\frac{1}{p-1}} f(\frac{x}{\sqrt{t}}),$$

where $f(\eta)$ is positive (and radial) solution of the problem

$$(E_1) \begin{cases} -\Delta f - (\frac{\eta}{2}, \nabla f) - \frac{1}{p-1} f + h_0 |f|^{p-1} f = 0 \text{ in } \mathbb{R}^N \\ \lim\limits_{|\eta|\to\infty} |\eta|^{\frac{2}{p-1}} f(\eta) = 0. \end{cases}$$

2. Degenerate absorption potential.

$$(P_2) \begin{cases} u_t - \Delta u + h(t,x)|u|^{p-1} u = 0, \\ u(0,x) = k\,\delta(x)\,, \;\; k \to \infty, \end{cases}$$

where $h(t,x) \geqslant 0$, $h(0,0) = 0$

Problem: existence, nonexistence and structure of solution $u_\infty(t,x)$ in dependence of h.

2.1 The case $h = h(x) = |x|^\beta$, $\beta > -2$.

Th.1. If $p \geqslant p_{cr.} := 1 + \frac{\beta+2}{N}$ (or $\beta \leqslant N(p-1)-2$), then problem (P_2) does not have any solution and, more general, arbitrary isolated initial singularity is removable.

If $p < p_{cr} = 1 + \frac{\beta+2}{N}$ (or $\beta > N(p-1) - 2$), then for any $k < \infty$ there exists solution $u_k(t,x)$ of (P_2) and there exists v.s. solution $u_\infty(t,x) := \lim\limits_{k\to\infty} u_k(t,x)$:

$$u_\infty(t,x) := t^{-\frac{(2+\beta)}{2(p-1)}} f^*(\frac{x}{\sqrt{t}}),$$

where $f^*(\eta) \geqslant 0$ is solution of:

$$(E_2) \begin{cases} -\Delta f - (\frac{1}{2}\eta, \nabla f) - \frac{2+\beta}{2(p-1)} f + |\eta|^\beta f^p = 0 \\ \lim\limits_{|\eta|\to\infty} |\eta|^{\frac{2+\beta}{p-1}} f(\eta) = 0. \end{cases}$$

Remark 1. If $\beta \to \infty$ then $f^*(0) := f^*_\beta(0) \to \infty$
Remark 2. Existence by generalization of M.Escobedo-O.Kavian variational method:

$$\inf\{I(v) : v(\eta) \in H^1_K(\mathbb{R}^N)\}, K = K(\eta) = \exp(\frac{|\eta|^2}{4})$$

$$I(v) = \int\limits_{R^N} (|\nabla_\eta v|^2 - \tfrac{2+\beta}{2(p-1)} v^2 + \tfrac{2}{p+1} |\eta|^\beta |v|^{p+1}) K \, d\eta$$

2.2 **The case of flat degenerate potential**:

$$h(x)|x|^{-\alpha} \to 0 \quad \text{as } |x| \to 0 \quad \forall\, \alpha > 0.$$

Th.2. (sufficient condition of existence of v.s.s.) Assume that $h(x)$ is continuous, positive in $\mathbb{R}^N \setminus \{0\}$ and satisfies flatness condition:

$$|x|^2 \ln\left(\frac{1}{h(x)}\right) \leqslant \omega(|x|) \Leftrightarrow h(x) \geqslant \exp\left(-\frac{\omega(|x|)}{|x|^2}\right),$$

where nondecreasing $\omega(s) > 0$ satisfies Dini-like condition

$$\int_0^1 s^{-1}\omega(s)ds < \infty\,.$$

Then for arbitrary $p > 1$, $k > 0$ there exists solution u_k of (P_2) and v.s. solution u_∞ with infinite initial mass:

$$u_\infty(t,x) = \lim_{k\to\infty} u_k(t,x),$$

and the following estimate holds:

$$\int_{\mathbb{R}^N} u_\infty(t,x)^2 dx \leqslant c_1 t \exp[c_2(\Phi^{-1}(c_3 t))^{-2}]\ \forall\, t > 0,$$

where Φ^{-1} is reverse function of

$$\Phi(\tau) := \int_0^\tau \frac{\omega(s)}{s} ds,$$

constants c_1, c_2, c_3 depend on N, p only.

Remark 3. Conditions of Th. 2 are satisfied if:

$h(x) \geqslant c\,\exp(-|x|^{\theta-2}) \quad \forall\, x \in \mathbb{R}^N$, $c =$ const > 0,

with some $\theta > 0$.

Th.3. (sufficient condition for nonexistence of v.s.solution-existence of "raisor blade" (R.B.)-solution).

Assume that continuous $h(x) \geqslant 0$ satisfies:

$$\liminf_{x\to 0} |x|^2 \ln(\frac{1}{h(x)}) > 0 \Leftrightarrow$$

$$\exists \omega_0 = const > 0 : h(x) \leqslant c\,\exp(-\frac{\omega_0}{|x|^2}).$$

Then there exists $u_\infty(t,x) := \lim_{k\to\infty} u_k(t,x)$ and $u_\infty(t,x)$ is solution of equation (P_2) in domain $Q \setminus \{(t,x) : |x| = 0\}$. Moreover,

$$u_\infty(t,0) = \infty \quad \forall\, t > 0\,(\text{R.B. - solution}).$$

Let $U(x) := \lim_{t\to\infty} u_\infty(t,x)$. Then U is minimal large solution of

$$-\Delta u + h(x)u^p = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

i.e. the smallest solution, which satisfies:

$$\int_{B(\varepsilon)} u(x)dx = \infty \quad \forall\, \varepsilon > 0, \quad B(\varepsilon) = \{x : |x| < \varepsilon\}.$$

Open problem: to characterize $u_\infty(t,x)$ if potential h satisfies:

$$h(x) \sim \exp(-\omega(|x|)|x|^{-2}) \text{ as } |x| \to 0\,,$$

where $\omega(s) \to 0$ as $s \to 0$ and

$$\int_0^1 \omega(s)s^{-1}\,ds = \infty\,.$$

3. Time dependent potential $h(t) \geq 0$.

3.1 Th.4. If $h(t) = t^\alpha$, $\alpha > 0$, then the problem

$$(P_3)\begin{cases} u_t - \Delta u + t^\alpha u^p = 0 \\ u(0,x) = k\,\delta(x) \end{cases}$$

has solution $u_k(t,x)$, if

$$p < p_{\text{cr}} := 1 + \tfrac{2(1+\alpha)}{N}.$$

Moreover, there exists v.s. solution

$$u_\infty(t,x) = \lim_{k\to\infty} u_k(t,x) = t^{-\frac{1+\alpha}{p-1}} f_\alpha\Big(\frac{|x|}{\sqrt{t}}\Big)$$

where $f_\alpha(r)$ is solution of the problem:

$$(E_3)\begin{cases} f''(r) + (\frac{N-1}{r} + \frac{r}{2})f'(r) + \frac{1+\alpha}{p-1}f - f^p = 0 \\ f'(0) = 0,\ \lim\limits_{r\to\infty} r^{\frac{2(1+\alpha)}{p-1}} f(r) = 0. \end{cases}$$

Solvability of (E_3) by standard (E.-K.)- method.

Th.5. (M.Marcus, L.Veron-2002.). If

$$h(t) = \exp(-\omega_0 t^{-1}),\ \omega_0 > 0,$$

then

$$u_\infty(t,x) = U(t) := ((q-1)\int_0^t h(s)ds)^{-\frac{1}{q-1}}$$

- complete initial blow-up.

Th.6. If $h(t) \geqslant c\exp(-\frac{\omega(t)}{t}),\ c > 0,$ where continuous, nondecreasing function $\omega(s) \geqslant 0$ satisfies Dini-like condition:

$$\int_0^1 \omega(s)^{1/2} s^{-1} ds < \infty\,,$$

then $u_\infty(t,x) := \lim\limits_{k\to\infty} u_k(t,x)$ is v.s. solution with single-point initial blow-up at (0,0).
The proof demonstrates essentially new application of local energy method. Its main ideas and steps are the following:

Consider solutions $u_k(t,x)$ of the approximating problem:

$$(P_k) \quad \begin{cases} u_t - \Delta u + h(t)u^p = 0, \\ u(0,x) = u_{0,k} = M_k^{1/2} h_k^{\frac{N}{2}} \delta_k(x) \in L_2(\mathbb{R}^{\mathbb{N}}), \end{cases}$$

where $\delta_k(x) = c_0 h_k^{-N} \chi_{B_0(h_k)}$, $h_k = \exp(-\beta k)$, $\beta > 0$, $h(t) = \exp(-\omega(t)t^{-1})$.
$\operatorname{supp} \delta_k = B_0(h_k) = \{x : |x| < h_k\}$, $\|\delta_k\|_{L_1} = 1$, $\delta_k \to \delta(x)$ weakly as $k \to \infty$, $\|\delta_k\|_{L_2}^2 = c_0 h_k^{-N}$.
Therefore, $\|u_{0,k}\|_{L_2}^2 = c_0 M_k$;
assumptions for the sequence $\{M_k\}$:

$$\|u_{0,k}\|_{L_1} = M_k^{1/2} h_k^{\frac{N}{2}} \to \infty \text{ as } k \to \infty.$$

Introduce families of subdomains:

$$\begin{aligned} &\Omega(s) = \mathbb{R}^N \cap \{|x| > s\} \ \forall\, s > 0, \\ &Q^\tau(s) = \Omega(s) \times (0,\tau) \ \forall\, \tau \in (0,T), \\ &Q_\tau(s) = \Omega(s) \times (\tau, T), \quad Q_\tau = \mathbb{R}^N \times (\tau, T), \end{aligned}$$

and energy functions:

$$I_k(\tau) := \int_{Q_\tau} (|\nabla_x u_k|^2 + |u_k|^2 + h(t)|u_k|^{p+1}) dxdt = I(\tau) \quad \forall\, \tau < T.$$

$$E_k(\tau, s) := \int_{Q^\tau(s)} |\nabla_x u_k|^2 dxdt + \int_{\Omega(s)} |u_k(\tau,x)|^2 dx = E(\tau, s) \quad s > 0.$$

Step 1. To prove Th. 1 it is sufficient to find optimal sequence $\{M_k\}$, such that the following uniform a'priory estimate holds:

$$I_k(\tau) + E_k(\tau, s) \leq C = C(\tau, s) < \infty \qquad \forall\, k \in \mathbb{N},\ \forall\, \tau > 0,\ \forall\, s > 0,$$

where constant C does not depend on k.

Step 2. Energy functions $I_k(\tau)$ satisfies ordinary differential inequality (ODI) of the following structure

$$I_k(\tau) \leq c h(\tau)^{-\frac{2}{p+1}} (-I_k'(\tau))^{\frac{2}{p+1}} + (\text{"lower order terms"})$$

and, as consequence, the following uniform "absorption" a'priory estimate holds:

$$I_k(\tau) \leq c_1 \left(\int_0^\tau h(r)dr \right)^{-\frac{2}{p-1}} \approx c_1 \exp\left(\frac{2\omega(t)}{(p-1)\tau} \right) \qquad \forall\, \tau > 0,\ \forall\, k \in \mathbb{N}.$$

Step 3. Using method of introducing of parameter of O.A.Oleinik we deduce the energy relationship of the structure:

$$E_k(\tau, s_2) \leq c_2 E_k(\tau, s_1) \exp\left(-\frac{s_2^2 - s_1^2}{c_3 \tau} \right) \qquad \forall\, s_2 \geq s_1 \geq \max(\tau^{1/2}, \exp(-\beta k)),$$

which is main "diffusion" a'priory estimate.

Step 4. By standard way the following global energy estimate of solution u_k holds:

$$I_k(0) + E_k(\tau, 0) \leq \|u_{0,k}\|^2_{L_2} = c_0 M_k, \qquad \forall\, \tau > 0,\ \forall\, k \in \mathbb{N}.$$

Step 5. Fix now arbitrary $\bar{k} \in \mathbb{N}$ and, keeping in the mind estimate from Step 2, define $\tau_{\bar{k}}$ by:

$$I_{\bar{k}}(\tau_{\bar{k}}) \leq c_1 \exp\left(\frac{2\omega\tau_{\bar{k}}}{(p-1)\tau_{\bar{k}}}\right) := 2^{-1} M_{\bar{k}-1} \Rightarrow$$
$$\tau_{\bar{k}} = \frac{2\omega(\tau_{\bar{k}})}{(p-1)[\ln(2c_1)^{-1} + \ln M_{\bar{k}-1}]}.$$

Step 6. As optimal sequence $\{M_k\}$ we choose:

$$M_k := \exp\exp k,$$

therefore $\tau_{\bar{k}} \approx \frac{2\omega(\tau_{\bar{k}})}{(p-1)\exp(\bar{k}-1)} = \frac{2e\omega(\tau_{\bar{k}})}{(p-1)\exp\bar{k}}$.

Step 7. Define now shift $s_{\bar{k}}$ by:

$$c_4 M_{\bar{k}} \exp\left(-\frac{s^2_{\bar{k}}}{c_3\tau_{\bar{k}}}\right) = 2^{-1} M_{\bar{k}-1},$$

where $c_4 = c_2 \exp(c_3^{-1})$; c_2 and c_3 are from Step 3. Then from Step 3 in virtue of global estimate from Step 4 it follows

$$E_{\bar{k}}(\tau_{\bar{k}}, s_{\bar{k}}) \leq 2^{-1} M_{\bar{k}-1}.$$

Step 8. From definitions of $\tau_{\bar{k}},\ s_{\bar{k}}$ it follows due to optimal choice of $\{M_k\}$:

$$s_{\bar{k}} = c_5 \omega(\tau_{\bar{k}})^{1/2}.$$

Step 9. Combining estimate from Step 5 and Step 7 we get:

$$E_{\bar{k}}(\tau, s_{\bar{k}}) \le I_{\bar{k}}(\tau_{\bar{k}}) + E_{\bar{k}}(\tau_{\bar{k}}, s_{\bar{k}}) \le M_{\bar{k}-1} \quad \forall\, \tau > 0.$$

Step 10. This estimate we use as global estimate instead of estimate from Step 4 for next round of computations. Namely, keeping in the mind estimate from Step 2, define $\tau^{\bar{k}-1}$ by:

$$I_{\bar{k}}(\tau_{\bar{k}-1}) \le c_1 \exp\left(\frac{2\omega\tau_{\bar{k}-1}}{(p-1)\tau_{\bar{k}-1}}\right) := 2^{-1} M_{\bar{k}-2} \Rightarrow$$
$$\tau_{\bar{k}-1} \approx \frac{2e\omega(\tau_{\bar{k}-1})}{(p-1)\exp(\bar{k}-1)}.$$

Step 11. Define $s_{\bar{k}-1}$ similar to $s_{\bar{k}}$ from Step 7:

$$c_2 M_{\bar{k}-1} \exp\left(-\frac{s_{\bar{k}-1}^2}{c_3 \tau_{\bar{k}-1}}\right) = 2^{-1} M_{\bar{k}-2}.$$

Then it follows from main “diffusion” energy estimate (Step 3):

$$E_{\bar{k}}(\tau_{\bar{k}-1}, s_{\bar{k}} + s_{\bar{k}-1})$$
$$\le c_2 E_{\bar{k}}(\tau_{\bar{k}-1}, s_{\bar{k}}) \exp\left(-\frac{s_{\bar{k}-1}^2}{c_3 \tau_{\bar{k}-1}}\right) \le \text{(due to Step 9)}$$
$$\le c_2 M_{\bar{k}-1} \exp\left(-\frac{s_{\bar{k}-1}^2}{c_3 \tau_{\bar{k}-1}}\right) \le 2^{-1} M_{\bar{k}-2}.$$

Step 12. Summing last estimate with estimate from Step 10 we obtain:

$$E_{\bar{k}}(\tau, s_{\bar{k}} + s_{\bar{k}-1}) \le I_{\bar{k}}(\tau_{\bar{k}-1}) + E_{\bar{k}}(\tau_{\bar{k}-1}, s_{\bar{k}} + s_{\bar{k}-1}) \le M_{\bar{k}-2} \quad \forall\, \tau > 0$$

where, as it follows from definition of $s_{\bar{k}-1}$ in Step 11:

$$s_{\bar{k}-1} = c_5 \omega(\tau_{\bar{k}-1})^{1/2}.$$

It completes the second round of computation. After $(\bar{k} - l)$ such a rounds we get:

$$E_{\bar{k}}\left(\tau, \sum_{i=l}^{\bar{k}} s_i\right) \le M_l \quad \forall\, \tau > 0.$$

It is easy to check that

$$\sum_{i=l}^{\bar{k}} s_i = c_5 \sum_{i=l}^{\bar{k}} \omega(\tau_i)^{1/2} \le c \sum_{i=l}^{\infty} \left[\omega\left(\frac{\omega_0}{d_1 \exp i}\right)\right]^{1/2} \le c \int_0^{\frac{\omega_0}{d_1 \exp l}} \frac{\omega(s)^{1/2}}{s} ds \to 0 \text{ as } l \to \infty.$$

Due to Step 1 last inequalities complete the proof.

3.2 Exponential semilinearity:

$$(P_3)\begin{cases} u_t - \Delta u + h(t)(\exp u - 1) = 0 \\ u(0,x) = k\,\delta(x)\,, \;\; k \to \infty\,. \end{cases}$$

Th.7. If $h(t) \sim \exp(-\exp(\frac{\omega_0}{t}))$, $\omega_0 = const > 0$, then for arbitrary $k > 0$ there exists solution $u_k(t,x)$ of (P_3) and

$$u_\infty(t,x) := \lim_{k\to\infty} u_k(t,x) = U(t)\,,$$

thus complete initial blow-up occurs.
If $h(t) \sim \exp(-\exp(\frac{\omega(t)}{t}))$, for some positive, non-decreasing $\omega(s)$ with

$$\int_0^1 \omega(s)^{1/2} s^{-1} ds < \infty$$

then $u_\infty(t,x)$ is v.s.solution with single-point initial blow-up at $(0,0)$

3.3 More general equations:

$$(P_4)\begin{cases} u_t - \Delta u^m + h(t)u^q = 0\,, \\ u(0,x) = k\,\delta(x)\,,\ \ k \to \infty\,. \end{cases}$$

Th.8. Let $q > m > 1$. If $h(t) \sim t^{\frac{q-m}{m-1}}\omega(t)^{-1}$, $\omega(t) \to 0$ as $t \to 0$, and

$$\int_0^1 \omega(s)^\theta s^{-1} ds < \infty\,,\ \theta = \frac{m^2-1}{[N(m-1)+2(m+1)](q-1)}$$

then u_∞ has single-point initial blow-up at $(0,0)$.

4. General degenerate potential $h(t,x) \geqslant 0$.
Let $\Gamma := \{(t,x) : h(t,x) = 0\}$, $(0,0) \in \Gamma$.
Problem: to find conditions on Γ and on the asymptotic of $h(t,x)$ near to Γ, which guarantee propagation or nonpropagation of initial singularity along Γ.

4.1 The case $\Gamma = \Gamma_N := \{(t,x) = (t,x',x_N) : t = 0, x' = 0\}$.

Let $h(t,x) = h(\rho(t,x)),\ \rho(t,x) = \max\{t, |x'|^2\}$.

$$(P_N) \qquad \begin{cases} u_t - \Delta u + h(\rho(t,x))u^p = 0, & p > 1 \\ u(0,x) = k\delta(x). \end{cases}$$

Th.9. Let $h(s) \sim \exp(-\omega_0 s^{-1})$ as $s \to 0,\ \omega_0 > 0$. Then $u_k(t,x) \to U(t,x')$ as $k \to \infty$, where

$$\begin{cases} U_t - \Delta U + h(\rho(t,x'))U^p = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^{\mathbb{N}-\mathbb{1}} \\ U(0,x') = \infty\delta(x') \end{cases}$$

Th.10. Let $h(x) \sim \exp(-\frac{\omega(s)}{s})$:

$$\int_0^1 \frac{\omega(s)^{1/2}}{s} ds < \infty.$$

Then $u_k(t,x) \to u_\infty : u_\infty(0,x) = \infty\delta(x)$.