

# Inverse spectral results for Schrödinger operators on the unit interval

T. Raoux

Université de Reims

Technion, Haifa - March 4, 2010

*Work in collaboration with L. Amour (Reims) and J. Faupin (Bordeaux)*

# Plan

- 1 Introduction
- 2 Short history
- 3 A little trip into geophysics
- 4 Recent advances
- 5 Our results
- 6 Elements of proof

# Plan

- 1 Introduction
- 2 Short history
- 3 A little trip into geophysics
- 4 Recent advances
- 5 Our results
- 6 Elements of proof

## Schrödinger operator on $[0, 1]$

We consider the Schrödinger operator on  $[0, 1]$

$$A_{q,h,H} = -\frac{d^2}{dx^2} + q \quad (1)$$

with boundary conditions

$$u'(0) + hu(0) = 0, \quad u'(1) + Hu(1) = 0. \quad (2)$$

where  $q \in L^1([0, 1], \mathbb{R})$  and  $(h, H) \in \mathbb{R}^2$ .

## Schrödinger operator on $[0, 1]$

The operator  $A_{q,h,H}$  is selfadjoint on  $L^2(0, 1)$ .

Its spectrum consists of an increasing sequence of simple eigenvalues :

$$\sigma(A_{q,h,H}) = \{\lambda_j(q, h, H) , j \in \mathbb{N}\}$$

Asymptotic expansion as  $j \rightarrow +\infty$  :

$$\lambda_j(q, h, H) = j^2\pi^2 + 2(H - h) + \int_0^1 q(x)dx + o(1) . \quad (3)$$

# The considered inverse problem

To what extent is it possible to recover the potential  $q$  from spectral data?

Which spectral data are needed to determine the potential  $q$ ?

$$\sigma(A_{q_1, h, H}) = \sigma(A_{q_2, h, H}) \implies q_1 = q_2 ?$$

# Plan

- 1 Introduction
- 2 Short history**
- 3 A little trip into geophysics
- 4 Recent advances
- 5 Our results
- 6 Elements of proof

## Does the spectrum for a single $h$ determine the potential ?

Theorem (V.A. Ambarzumian - 1929)

Suppose that the set of eigenvalues of the problem

$$-y'' + q(x)y = \lambda y, \quad y'(0) = y'(1) = 0$$

is exactly the sequence  $(\pi n)^2$ ,  $n = 0, 1, 2, \dots$

Then  $q \equiv 0$ .

In other words :  $\sigma(A_{q,0,0}) = \sigma(A_{0,0,0}) \implies q = 0$ .

But if  $\tilde{q}(x) = q(1-x)$ , then  $\sigma(A_{\tilde{q},h,-h}) = \sigma(A_{q,h,-h})$

$$\sigma(A_{q_1,h,H}) = \sigma(A_{q_2,h,H}) \not\Rightarrow q_1 = q_2$$

# Does the spectrum for a single $h$ determine the potential ?

## Theorem (V.A. Ambarzumian - 1929)

Suppose that the set of eigenvalues of the problem

$$-y'' + q(x)y = \lambda y, \quad y'(0) = y'(1) = 0$$

is exactly the sequence  $(\pi n)^2$ ,  $n = 0, 1, 2, \dots$

Then  $q \equiv 0$ .

In other words :  $\sigma(A_{q,0,0}) = \sigma(A_{0,0,0}) \implies q = 0$ .

But if  $\tilde{q}(x) = q(1-x)$ , then  $\sigma(A_{\tilde{q},h,-h}) = \sigma(A_{q,h,-h})$

$$\sigma(A_{q_1,h,H}) = \sigma(A_{q_2,h,H}) \not\Rightarrow q_1 = q_2$$

# Two spectra determine the potential

## Theorem (G. Borg - 1946)

Let  $q_1, q_2 \in L^1(0, 1)$ ,  $h, h', H \in \mathbb{R}$  ( $h \neq h'$ ).

$$\left. \begin{array}{l} \sigma(A_{q_1, h, H}) = \sigma(A_{q_2, h, H}) \\ \sigma(A_{q_1, h', H}) = \sigma(A_{q_2, h', H}) \end{array} \right\} \implies q_1 = q_2 .$$

# A new point of view

Theorem (H. Hochstadt & B. Lieberman - 1978)

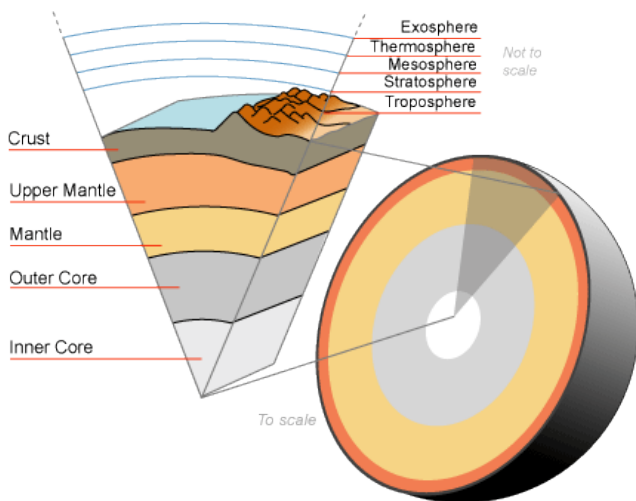
Let  $q_1, q_2 \in L^1(0, 1)$  and  $h, H \in \mathbb{R}$ .

$$\left. \begin{array}{l} q_1(x) = q_2(x) \text{ on } [\frac{1}{2}, 1] \\ \sigma(A_{q_1, h, H}) = \sigma(A_{q_2, h, H}) \end{array} \right\} \implies q_1 = q_2 \text{ sur } (0, 1) .$$

# Plan

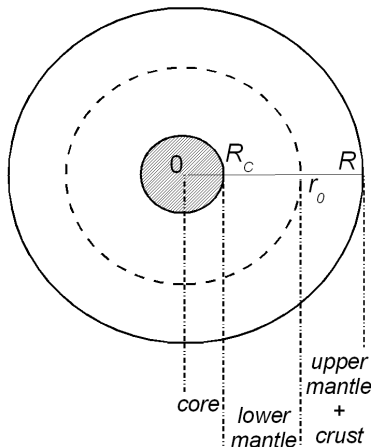
- 1 Introduction
- 2 Short history
- 3 A little trip into geophysics**
- 4 Recent advances
- 5 Our results
- 6 Elements of proof

# The internal structure of the Earth



# The internal structure of the Earth

O.H. Hald, *Geophys. J. R. astr. Soc.* - 1980



Physical quantities:

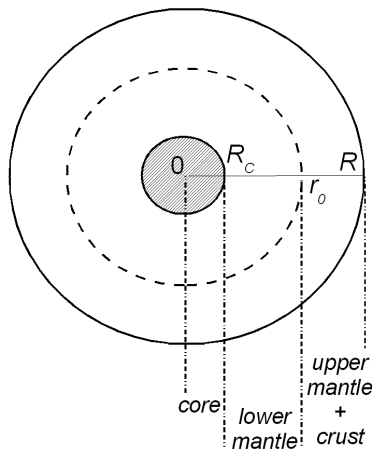
- Density  $\rho$ ,
- Incompressibility  $k$ ,
- Rigidity  $\mu$ ,  $\lambda = k - (2/3)\mu$ ,

*Lamé parameters*

- Velocity  $\alpha$  of the P-waves,
- Velocity  $\beta$  of the S-waves,

ASSUMPTION I - All these functions are supposed to be radially symmetric (and defined on  $[R_c, R]$ ):

## The internal structure of the Earth



ASSUMPTION II - The Earth consists of perfect elastic and isotropic material

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \beta = \sqrt{\frac{\mu}{\rho}}.$$

Seismic data give  $\alpha$  and  $\beta$ .

Goal: obtain  $\rho$  on  $[R_c, R]$ .

## Determination of $\rho$

ASSUMPTION III - The lower mantle is chemically homogeneous and devoid of phase changes.

So  $\rho$  may be determined on  $[R_c, r_0]$  by using the *Adams-Williamson's equation* (where  $\Phi = \alpha^2 - (4/3)\beta^2$ ):

$$\left\{ \begin{array}{l} \rho' = -\frac{Gm\rho}{r^2\Phi} \\ m' = 4\pi r^2\rho \\ \rho(R_c) = \rho_c, \quad m(R_c) = m_c. \end{array} \right. \quad \text{for } R_c \leq r \leq r_0$$

## Determination of $\rho$

The equations for the torsional modes of a spherically symmetric non-rotating earth lead to this eigenvalue problem:

$$-(r^4 \mu u')' + \frac{(n+2)(n-1)}{r^2} r^4 \mu u = \omega^2 r^4 \rho u \text{ for } R_c \leq r \leq R$$

$$u'(R_c) = u'(R) = 0 .$$

## Determination of $\rho$

After using a *Liouville transformation*,

$$r = R - \varphi(x), \quad r \in [r_0, R] \iff x \in [0, \frac{\pi}{2}], \quad r \in [R_c, r_0] \iff x \in [\frac{\pi}{2}, \pi]$$

$$f(x) = r^2 \sqrt[4]{\rho(r)\mu(r)} = r^2 \sqrt{\rho(r)\beta(r)}, \quad y(x) = f(x)u(r)$$

the problem becomes

$$-y'' + \underbrace{\left[ \frac{f''(x)}{f(x)} + K^2 \frac{(n+2)(n-1)\mu}{r^2} \frac{\mu}{\rho} \right]}_{=q} y = \omega^2 K^2 y \text{ pour } 0 \leq x \leq \pi$$

$$y'(0) + \underbrace{(-f'(0)/f(0))}_{=h} y(0) = y'(\pi) + \underbrace{(-f'(\pi)/f(\pi))}_{=H} y(\pi) = 0.$$

## Determination of $\rho$

After using a *Liouville transformation*,

$$r = R - \varphi(x), \quad r \in [r_0, R] \iff x \in [0, \frac{\pi}{2}], \quad r \in [R_c, r_0] \iff x \in [\frac{\pi}{2}, \pi]$$

$$f(x) = r^2 \sqrt[4]{\rho(r)\mu(r)} = r^2 \sqrt{\rho(r)\beta(r)}, \quad y(x) = f(x)u(r)$$

the problem becomes

$$-y'' + \underbrace{\left[ \frac{f''(x)}{f(x)} + K^2 \frac{(n+2)(n-1)\mu}{r^2} \frac{\mu}{\rho} \right]}_{=q} y = \omega^2 K^2 y \text{ pour } 0 \leq x \leq \pi$$

$$y'(0) + \underbrace{(-f'(0)/f(0))}_{=h} y(0) = y'(\pi) + \underbrace{(-f'(\pi)/f(\pi))}_{=H} y(\pi) = 0.$$

## An improvement of Hochstadt & Lieberman's result

### Theorem (O. Hald - 1980)

Let  $q_1, q_2 \in L^1(0, 1)$  and  $h_1, h_2, H \in \mathbb{R}$ .

$$\left. \begin{array}{l} q_1(x) = q_2(x) \text{ on } [\frac{1}{2}, 1] \\ \sigma(A_{q_1, h_1, H}) = \sigma(A_{q_2, h_2, H}) \end{array} \right\} \implies q_1 = q_2 \text{ on } (0, 1), h_1 = h_2 .$$

**Remark** - If  $q_1, q_2$  are *continuous* the conclusion holds under the weaker assumption that the eigenvalues are the same except possibly one.

## An improvement of Hochstadt & Lieberman's result

### Theorem (O. Hald - 1980)

Let  $q_1, q_2 \in L^1(0, 1)$  and  $h_1, h_2, H \in \mathbb{R}$ .

$$\left. \begin{array}{l} q_1(x) = q_2(x) \text{ on } [\frac{1}{2}, 1] \\ \sigma(A_{q_1, h_1, H}) = \sigma(A_{q_2, h_2, H}) \end{array} \right\} \implies q_1 = q_2 \text{ on } (0, 1), h_1 = h_2 .$$

**Remark** - If  $q_1, q_2$  are *continuous* the conclusion holds under the weaker assumption that the eigenvalues are the same except possibly one.

# Plan

- 1 Introduction
- 2 Short history
- 3 A little trip into geophysics
- 4 Recent advances**
- 5 Our results
- 6 Elements of proof

## Some notations

In the sequel, we will denote by

- $A$  or  $A_i$  one of the operators  $A_{q_i, h_i, H}$ ,  $i = 1, 2$ ;
- $\lambda_j$  or  $\lambda_j(A_i)$  the eigenvalue  $\lambda_j(q_i, h_i, H)$ ;
- $S$  some subset of  $\sigma(A_{q_1, h_1, H}) \cap \sigma(A_{q_2, h_2, H})$ ;

For  $E = \sigma(A)$  or  $S$  and  $t \in \mathbb{R}_+$ , we will use the notation:

$$n_E(t) = \#\{\lambda \in E \mid \lambda \leq t\}.$$

## F. Gesztesy and B. Simon

### Theorem (F. Gesztesy & B. Simon - 1999)

Let  $h_1, h_2, H \in \mathbb{R}$  and  $q_1, q_2 \in L^1([0, 1])$  such that

- $\exists a \in (0, \frac{1}{2})$   $q_1 = q_2$  on  $[a, 1]$
- $\exists S \subset \sigma(A_{q_1, h_1, H}) \cap \sigma(A_{q_2, h_2, H})$  such that for large enough  $t \in \mathbb{R}$ ,

$$n_S(t) \geq 2an_{\sigma(A)}(t) + \frac{1}{2} - a \quad (4)$$

Then  $q_1 = q_2$  on  $(0, 1)$  and  $h_1 = h_2$ .

Remark - If  $q_1, q_2 \in C^{2k}([a - \varepsilon, a + \varepsilon])$ , (4) becomes  
 $n_S(t) \geq 2an_{\sigma(A)}(t) + \frac{1}{2} - a - (k + 1)$ .

## F. Gesztesy and B. Simon

### Theorem (F. Gesztesy & B. Simon - 1999)

Let  $h_1, h_2, H \in \mathbb{R}$  and  $q_1, q_2 \in L^1([0, 1])$  such that

- $\exists a \in (0, \frac{1}{2})$   $q_1 = q_2$  on  $[a, 1]$
- $\exists S \subset \sigma(A_{q_1, h_1, H}) \cap \sigma(A_{q_2, h_2, H})$  such that for large enough  $t \in \mathbb{R}$ ,

$$n_S(t) \geq 2an_{\sigma(A)}(t) + \frac{1}{2} - a \quad (4)$$

Then  $q_1 = q_2$  on  $(0, 1)$  and  $h_1 = h_2$ .

**Remark** - If  $q_1, q_2 \in C^{2k}([a - \varepsilon, a + \varepsilon])$ , (4) becomes

$$n_S(t) \geq 2an_{\sigma(A)}(t) + \frac{1}{2} - a - (k + 1).$$

# Plan

- 1 Introduction
- 2 Short history
- 3 A little trip into geophysics
- 4 Recent advances
- 5 Our results**
- 6 Elements of proof

## The $L^p$ -case

### Theorem (L. Amour & T.R. - 2007)

Let  $h_1, h_2, H \in \mathbb{R}$  and  $q_1, q_2 \in L^1([0, 1])$  such that

- $\exists a \in (0, \frac{1}{2}] \quad \exists p \in [1, +\infty)$  s.t.  
 $q_1 = q_2$  on  $[a, 1]$  and  $q_1 - q_2 \in L^p(0, a)$ .
- $\exists S \subset \sigma(A_{q_1, h_1, H}) \cap \sigma(A_{q_2, h_2, H})$ ,  $S$  infinite, and  $\exists C > 0$  s.t. for large enough  $t \in S$ ,

$$2an_{\sigma(A)}(t) + C \geq n_S(t) \geq 2an_{\sigma(A)}(t) + \frac{1}{2p} - 2a.$$

Then  $q_1 = q_2$  on  $(0, 1)$  and  $h_1 = h_2$ .

## The $L^p$ -case

### Corollary (1)

Let  $h_1, h_2, H \in \mathbb{R}$  and  $q_1, q_2 \in L^1([0, 1])$  be such that

$q_1 = q_2$  on  $[\frac{1}{4}, 1]$  and  $\{\lambda_{2j}(q_1, h_1, H)\}_j = \{\lambda_{2j}(q_2, h_2, H)\}_j$ .

Then  $q_1 = q_2$  on  $(0, 1)$  and  $h_1 = h_2$ .

### Corollary (2)

Let  $h_1, h_2, H \in \mathbb{R}$  and  $q_1, q_2 \in L^1([0, 1])$  be such that

$q_1 = q_2$  on  $[\frac{1}{4}, 1]$  and  $\{\lambda_{2j+1}(q_1, h_1, H)\}_j = \{\lambda_{2j+1}(q_2, h_2, H)\}_j$ .

Then  $q_1 = q_2$  on  $(0, 1)$  and  $h_1 = h_2$ .

## The $L^p$ -case

### Corollary (1)

Let  $h_1, h_2, H \in \mathbb{R}$  and  $q_1, q_2 \in L^1([0, 1])$  be such that

$q_1 = q_2$  on  $[\frac{1}{4}, 1]$  and  $\{\lambda_{2j}(q_1, h_1, H)\}_j = \{\lambda_{2j}(q_2, h_2, H)\}_j$ .

Then  $q_1 = q_2$  on  $(0, 1)$  and  $h_1 = h_2$ .

### Corollary (2)

Let  $h_1, h_2, H \in \mathbb{R}$  and  $q_1, q_2 \in L^1([0, 1])$  be such that

$q_1 = q_2$  on  $[\frac{1}{4}, 1]$  and  $\{\lambda_{2j+1}(q_1, h_1, H)\}_j = \{\lambda_{2j+1}(q_2, h_2, H)\}_j$ .

Then  $q_1 = q_2$  on  $(0, 1)$  and  $h_1 = h_2$ .

## The $L^p$ -case

Set  $S = \{\lambda_{2j}\}_j$  and  $S' = \{\lambda_{2j+1}\}_j$ .

$$n_S(\lambda_{2j}) = n_S(\lambda_{2j+1}) = n_{S'}(\lambda_{2j+1}) = j + 1 ,$$

$$n_{\sigma(A)}(\lambda_{2j}) = 2j + 1, \quad n_{\sigma(A)}(\lambda_{2j+1}) = 2j + 2 .$$

$$\frac{1}{2}n_{\sigma(A)}(\lambda_{2j}) + \frac{1}{2} = n_S(\lambda_{2j}) > \frac{1}{2}n_{\sigma(A)}(\lambda_{2j}), \quad n_{S'}(\lambda_{2j+1}) = \frac{1}{2}n_{\sigma(A)}(\lambda_{2j+1})$$

while

$$n_S(\lambda_{2j+1}) \not\geq \frac{1}{2}n_{\sigma(A)}(\lambda_{2j+1}) + \frac{1}{4}, \quad n_{S'}(\lambda_{2j+1}) \not\geq \frac{1}{2}n_{\sigma(A)}(\lambda_{2j+1}) + \frac{1}{4} .$$

## The $L^p$ -case

Set  $S = \{\lambda_{2j}\}_j$  and  $S' = \{\lambda_{2j+1}\}_j$ .

$$n_S(\lambda_{2j}) = n_S(\lambda_{2j+1}) = n_{S'}(\lambda_{2j+1}) = j + 1 ,$$

$$n_{\sigma(A)}(\lambda_{2j}) = 2j + 1, \quad n_{\sigma(A)}(\lambda_{2j+1}) = 2j + 2 .$$

$$\frac{1}{2}n_{\sigma(A)}(\lambda_{2j}) + \frac{1}{2} = n_S(\lambda_{2j}) > \frac{1}{2}n_{\sigma(A)}(\lambda_{2j}), \quad n_{S'}(\lambda_{2j+1}) = \frac{1}{2}n_{\sigma(A)}(\lambda_{2j+1})$$

while

$$n_S(\lambda_{2j+1}) \not\geq \frac{1}{2}n_{\sigma(A)}(\lambda_{2j+1}) + \frac{1}{4}, \quad n_{S'}(\lambda_{2j+1}) \not\geq \frac{1}{2}n_{\sigma(A)}(\lambda_{2j+1}) + \frac{1}{4} .$$

## The $L^p$ -case

Set  $S = \{\lambda_{2j}\}_j$  and  $S' = \{\lambda_{2j+1}\}_j$ .

$$n_S(\lambda_{2j}) = n_S(\lambda_{2j+1}) = n_{S'}(\lambda_{2j+1}) = j + 1 ,$$

$$n_{\sigma(A)}(\lambda_{2j}) = 2j + 1, \quad n_{\sigma(A)}(\lambda_{2j+1}) = 2j + 2 .$$

$$\frac{1}{2}n_{\sigma(A)}(\lambda_{2j}) + \frac{1}{2} = n_S(\lambda_{2j}) > \frac{1}{2}n_{\sigma(A)}(\lambda_{2j}) , \quad n_{S'}(\lambda_{2j+1}) = \frac{1}{2}n_{\sigma(A)}(\lambda_{2j+1})$$

while

$$n_S(\lambda_{2j+1}) \not\geq \frac{1}{2}n_{\sigma(A)}(\lambda_{2j+1}) + \frac{1}{4} , \quad n_{S'}(\lambda_{2j+1}) \not\geq \frac{1}{2}n_{\sigma(A)}(\lambda_{2j+1}) + \frac{1}{4} .$$

## The $L^p$ -case

Set  $S = \{\lambda_{2j}\}_j$  and  $S' = \{\lambda_{2j+1}\}_j$ .

$$n_S(\lambda_{2j}) = n_S(\lambda_{2j+1}) = n_{S'}(\lambda_{2j+1}) = j + 1 ,$$

$$n_{\sigma(A)}(\lambda_{2j}) = 2j + 1, \quad n_{\sigma(A)}(\lambda_{2j+1}) = 2j + 2 .$$

$$\frac{1}{2}n_{\sigma(A)}(\lambda_{2j}) + \frac{1}{2} = n_S(\lambda_{2j}) > \frac{1}{2}n_{\sigma(A)}(\lambda_{2j}) , \quad n_{S'}(\lambda_{2j+1}) = \frac{1}{2}n_{\sigma(A)}(\lambda_{2j+1})$$

while

$$n_S(\lambda_{2j+1}) \not\geq \frac{1}{2}n_{\sigma(A)}(\lambda_{2j+1}) + \frac{1}{4} , \quad n_{S'}(\lambda_{2j+1}) \not\geq \frac{1}{2}n_{\sigma(A)}(\lambda_{2j+1}) + \frac{1}{4} .$$

## The $W^{k,p}$ -case

Theorem (L. Amour, J. Faupin & T.R. - 2009)

Fix  $k \in \{0, 1, 2\}$ . Let  $h_1, h_2, H \in \mathbb{R}$  and  $q_1, q_2 \in W^{k,1}([0, 1])$  such that

- $\exists a \in (0, \frac{1}{2}] \quad \exists p \in [1, +\infty]$   
 $q_1 = q_2$  on  $[a, 1]$  and  $q_1 - q_2 \in W^{k,p}(0, a)$ .
- $\exists S \subset \sigma(A_{q_1, h_1, H}) \cap \sigma(A_{q_2, h_2, H})$  s.t. for large enough  $t \in \sigma(A)$ ,

$$n_S(t) \geq 2a n_{\sigma(A)}(t) - \frac{k}{2} + \frac{1}{2p} - \frac{1}{2} - a.$$

Then  $q_1 = q_2$  on  $(0, 1)$  and  $h_1 = h_2$ .

## Corollary

Let  $h_1, h_2, H \in \mathbb{R}$  and  $q_1, q_2 \in L^1([0, 1])$  such that

- $q_1 = q_2$  on  $[\frac{1}{2}, 1]$  and  $q_1 - q_2 \in L^\infty(0, \frac{1}{2})$ .
- $\exists S \subset \sigma(A_{q_1, h_1, H}) \cap \sigma(A_{q_2, h_2, H})$  s.t. for large enough  $t \in \sigma(A)$ ,

$$n_S(t) \geq n_{\sigma(A)}(t) - 1.$$

Then  $q_1 = q_2$  on  $(0, 1)$  and  $h_1 = h_2$ .

The assumption on the spectra expresses that the eigenvalues are the same, except possibly one. This is similar to the result of O. Hald, but without assuming that  $q_1, q_2$  are themselves continuous.

# Plan

- 1 Introduction
- 2 Short history
- 3 A little trip into geophysics
- 4 Recent advances
- 5 Our results
- 6 Elements of proof**

## The Cauchy problem

For  $z \in \mathbb{C}$ ,  $q \in L^1(0, 1)$ ,  $h \in \mathbb{R}$ , we denote by  $\psi(\cdot, z, q, h)$  the solution to the Cauchy problem

$$-\psi'' + q\psi = z^2\psi, \quad \psi(0) = 1, \quad \psi'(0) = -h.$$

- $\psi(x, \cdot, q, h)$  is an entire function.
- for  $z \in \mathbb{C}$ , we have

$$z^2 \in \sigma(A_{q,h,H}) \iff \psi'(1, z, q, h) - H\psi(1, z, q, h) = 0.$$

Fix  $q_1, q_2, h_1, h_2$  satisfying the assumptions of Theorem [AFR] and set  $\psi_i(x, z) = \psi(x, z, q_i, h_i)$ ,  $A_i = A_{q_i, h_i, H}$  for  $i = 1, 2$ .

## Introduction of an entire function $f$

We define for all  $z \in \mathbb{C}$

$$f(z) = \int_0^a \left( \psi_1(x, z) \psi_2(x, z) - \frac{1}{2} \right) (q_1(x) - q_2(x)) dx .$$

$$z^2 \in \sigma(A_1) \cap \sigma(A_2) \implies f(z) = 0 .$$

*Proof* : 1) asymptotic expansion of the  $\lambda_j(A_i)$ 's  $\implies$

$$2(h_1 - h_2) + \int_0^1 (q_2(x) - q_1(x)) dx = 0$$

2)

$$0 = \int_0^1 (-\psi_1'' + (q_1 - z^2)\psi_1)\psi_2 - (-\psi_2'' + (q_2 - z^2)\psi_2)\psi_1 dx$$

$$\underbrace{=}_{\text{int. by part}} f(z) + 2(h_1 - h_2) + \int_0^1 (q_2(x) - q_1(x)) dx$$

## Introduction of an entire function $f$

We define for all  $z \in \mathbb{C}$

$$f(z) = \int_0^a \left( \psi_1(x, z) \psi_2(x, z) - \frac{1}{2} \right) (q_1(x) - q_2(x)) dx .$$

$$z^2 \in \sigma(A_1) \cap \sigma(A_2) \implies f(z) = 0 .$$

*Proof :* 1) asymptotic expansion of the  $\lambda_j(A_i)$ 's  $\implies$

$$2(h_1 - h_2) + \int_0^1 (q_2(x) - q_1(x)) dx = 0$$

2)

$$0 = \int_0^1 (-\psi_1'' + (q_1 - z^2)\psi_1)\psi_2 - (-\psi_2'' + (q_2 - z^2)\psi_2)\psi_1 dx$$

$$\underbrace{=}_{\text{int. by part}} f(z) + 2(h_1 - h_2) + \int_0^1 (q_2(x) - q_1(x)) dx$$

## Introduction of an entire function $f$

We define for all  $z \in \mathbb{C}$

$$f(z) = \int_0^a \left( \psi_1(x, z) \psi_2(x, z) - \frac{1}{2} \right) (q_1(x) - q_2(x)) dx .$$

$$z^2 \in \sigma(A_1) \cap \sigma(A_2) \implies f(z) = 0 .$$

*Proof* : 1) asymptotic expansion of the  $\lambda_j(A_i)$ 's  $\implies$

$$2(h_1 - h_2) + \int_0^1 (q_2(x) - q_1(x)) dx = 0$$

2)

$$0 = \int_0^1 (-\psi_1'' + (q_1 - z^2)\psi_1)\psi_2 - (-\psi_2'' + (q_2 - z^2)\psi_2)\psi_1 dx$$

$$\underbrace{=}_{\text{int. by part}} f(z) + 2(h_1 - h_2) + \int_0^1 (q_2(x) - q_1(x)) dx$$

## Estimate of the function $f$

There exists  $C = C(p, \|q_1 - q_2\|_{W^{k,p}([0,a])}) > 0$  s.t.

$$\forall \varepsilon > 0, \exists \delta_\varepsilon = \delta(\varepsilon, p, a, \|q_1 - q_2\|_{W^{k,p}([0,a])}) > 0$$

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon = 0 \quad \text{and} \quad \forall z \in \mathbb{C}, \quad |f(z)| \leq C \frac{e^{2|\Im z|a}}{|\Im z|^{k+1-\frac{1}{p}}} (e^{-\varepsilon|\Im z|} + \delta_\varepsilon)$$

for all  $z \in \mathbb{C}$ .

Suppose that  $f \neq 0$ .

Set  $n_f(t) = \#\{z \in \mathbb{C} \mid f(z) = 0 \text{ and } |z| \leq t\}$ .

Thanks to *Jensen's formula* and the previous estimate we have

$\forall R > 0$

$$\begin{aligned} \int_0^R \frac{n_f(t)}{t} dt &= \frac{1}{2\pi} \int_0^{2\pi} \ln |f(Re^{i\theta})| d\theta - \ln |f(0)| \\ &\leq \frac{4a}{\pi} R - \left(k + 1 - \frac{1}{p}\right) \ln R \\ &\quad + \underbrace{\frac{1}{2\pi} \int_0^{2\pi} \ln(e^{-\varepsilon R |\sin \theta|} + \delta_\varepsilon) d\theta}_{\rightarrow -\infty \text{ quand } R \rightarrow +\infty} + O(1) \end{aligned}$$

Since  $n_{S^{1/2}}(t) \leq n_f(t)$  for all  $t$ ,

$$\lim_{R \rightarrow +\infty} \int_0^R \frac{n_{S^{1/2}}(t)}{t} dt - \frac{4a}{\pi} R + \left(k + 1 - \frac{1}{p}\right) \ln R = -\infty$$

while the sequence

$$\left( \int_0^{\sqrt{\lambda_j}} \frac{n_{S^{1/2}}(t)}{t} dt - \frac{4a}{\pi} \sqrt{\lambda_j} + \left(k + 1 - \frac{1}{p}\right) \ln \sqrt{\lambda_j} \right)_{j \in \mathbb{N}}$$

is bounded from below (estimate of  $n_S(t)$  in [AFR] + asymp. exp. of  $\lambda'_j$ 's).

Finally we have  $f \equiv 0$ .

It follows that  $q_1 = q_2$  (as in Levin's book *Distribution of zeros of entire functions*, or see also a shorter proof in [AR]).

Therefore  $h_1 = h_2$  since  $2(h_1 - h_2) = \int_0^1 (q_1 - q_2)$ .