Singularities versus boundedness of solutions of superlinear elliptic boundary value problems

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M.-F. Bidaut-Véron (1998):

"...Up to now, our proof is limitated to the case

$$q < N(N+2)/(N-1)^2,$$

but we hope to extend it up to the case

$$q < (N+2)/(N-2)\ldots.$$"

We are mainly interested in the

existence / nonexistence of singular solutions

for a given class of elliptic equations or systems.

In particular, we study the impact of boundary conditions on the existence of **boundary singularities**.

In the nonexistence case our methods also guarantee uniform **a priori estimates** of solutions.

$\Omega \subset \mathbb{R}^N$ bounded smooth, $N \geq 3$, $|f(x,u)| \leq C(1+|u|^p)$

$$-\Delta u = f(x,u) \quad \text{in } \mathcal{D}'(\Omega)$$

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$$u, \ f = f(\cdot, u) \in L^1_{\text{loc}}(\Omega), \qquad \int_\Omega (u\Delta\varphi + f\varphi)\, dx = 0 \quad \forall \varphi \in \mathcal{D}(\Omega)$$

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Locally $u = G * (f\psi) + w$, where

$$\left.\begin{array}{l} G(x) = \dfrac{c_N}{|x|^{N-2}} \in L^q_{\mathrm{loc}} \quad \forall q < \dfrac{N}{N-2} \\ \psi \in L^\infty \text{ cutoff} \Rightarrow f\psi \in L^1 \\ w \in L^\infty_{\mathrm{loc}} \text{ harmonic} \end{array}\right\} \Rightarrow u \in L^q_{\mathrm{loc}}$$

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If $p < \dfrac{N}{N-2}$: $f \in L^{q/p}_{\text{loc}}$ $(q/p > 1) \Rightarrow u \in L^{\tilde{q}}_{\text{loc}}$ $(\tilde{q} > q)$

$$\Rightarrow f \in L^{\tilde{q}/p}_{\text{loc}} \Rightarrow \ldots (\text{bootstrap}) \cdots \Rightarrow u \in L^\infty_{\text{loc}}$$

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Typical singular solution for $p > p^*$: $u(x) = \frac{c_{p,N}}{|x|^{2/(p-1)}}$

Isolated singularities studied by Brezis, Lions, Gidas, Spruck, Aviles, Pacard, Caffarelli, Bidaut-Véron, Véron, ...

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Question: Impact of boundary conditions on p^*

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \qquad |f(x,u)| \le C(1+|u|^p)$$

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very weak solution: $u \in L^1(\Omega)$, $f(\cdot,u) \in L^1(\Omega)$,

$$\int_\Omega (u\Delta\varphi + f(x,u)\varphi)\,dx = 0 \qquad \forall \varphi \in C^2(\overline{\Omega}),\ \frac{\partial \varphi}{\partial \nu}\Big|_{\partial\Omega} = 0$$

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Equiv.: $\quad u(x) = \int_\Omega G_{\mathcal{N}}(x,y)\big(f(y,u(y)) + u(y)\big)\,dy,$

$$\text{where} \quad \begin{cases} -\Delta G_{\mathcal{N}}(x,\cdot) + G_{\mathcal{N}}(x,\cdot) = \delta_x & \text{in } \Omega, \\ \dfrac{\partial G_{\mathcal{N}}(x,\cdot)}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

$$G_{\mathcal{N}}(x,y) \sim |x-y|^{2-N}$$

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Critical exponent: $p^* = \frac{N}{N-2}$

- $p < p^* \quad \Rightarrow \quad u \in L^\infty(\Omega) \quad (L^p - L^q\text{-estimates})$
- $p \geq p^* \quad \Rightarrow \quad \exists\ f,\ u \notin L^\infty(\Omega)$

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where $\begin{cases} -\Delta G_{\mathcal{D}}(x,\cdot) = \delta_x & \text{in } \Omega, \\ G_{\mathcal{D}}(x,\cdot) = 0 & \text{on } \partial\Omega \end{cases}$

$$G_{\mathcal{D}}(x,y) \sim \min\left(1, \frac{\delta(x)\delta(y)}{|x-y|^2}\right)|x-y|^{2-N}$$

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Critical exponent: $p^*_{\mathcal{D}} = \frac{N+1}{N-1}$

- $p < p^*_{\mathcal{D}} \quad \Rightarrow \quad u \in L^\infty(\Omega) \quad (L^p_\delta - L^q_\delta\text{-estimates})$
- $p \ge p^*_{\mathcal{D}} \quad \Rightarrow \quad \exists\ f,\ u \notin L^\infty(\Omega)$

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$$p^*_{\mathcal{D}} \le p < p^* = \frac{N}{N-2} \quad \Rightarrow \quad \text{boundary singularity}$$

Boundedness and a priori bounds for $p < p^*_{\mathcal{D}}$:

- Bidaut-Véron, Vivier 2000; Bidaut-Véron, Yarur 2002; Q., Souplet 2004

Singular solutions for $p \geq p^*_{\mathcal{D}}$:

- Souplet 2005 $f(x,u) = a(x)u^p$, $a \in L^\infty(\Omega)$, $p > p^*_{\mathcal{D}}$
- Del Pino, Musso, Pacard 2007 $f(u) = u^p$, $p^*_{\mathcal{D}} \leq p < p^*_{\mathcal{D}} + \varepsilon$
- Bidaut-Véron, Ponce, Véron 2007-2009+ behavior of singularities

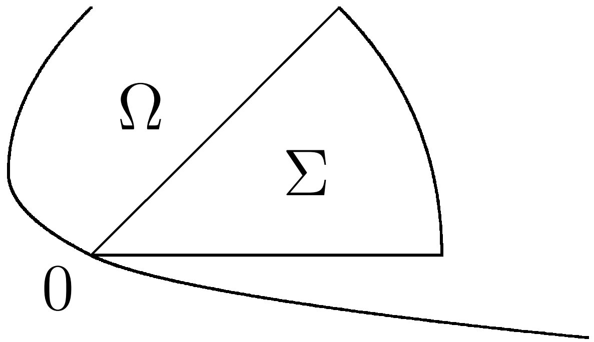
Ω not smooth: $p^*_{\mathcal{D}} = p^*_{\mathcal{D}}(\Omega)$ ($= \frac{N}{N-1}$ for hypercubes)

- McKenna, Reichel 2007; Horák, McKenna, Reichel 2009

Singular solution of $-\Delta u = a(x)u^p$ for $p > p^*_{\mathcal{D}}$:

$$\phi(x) := \frac{\mathbf{1}_\Sigma(x)}{|x|^{\alpha p}}, \qquad \alpha = \frac{2}{p-1},$$

$\Sigma = \Sigma_1 \cap B_R \subset \Omega, \quad \Sigma_1$ – cone with vertex at $0 \in \partial\Omega$



$$p > p^*_{\mathcal{D}} \;\Rightarrow\; \phi \in L^1_\delta(\Omega)$$

$$-\Delta u = \phi \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$\Rightarrow \quad u(x)^p \geq c\phi(x)$$

$$\Rightarrow \quad -\Delta u = a(x)u^p \quad \text{for some } a \in L^\infty$$

Singular solution of $-\Delta u = u^p$ for $p \in [p^*_{\mathcal{D}}, p^*_{\mathcal{D}} + \varepsilon)$:

Set $\alpha = \frac{2}{p-1}$, $z = \frac{x}{|x|}$.

1. Singular solution of the form

$$u(x) = |x|^{-\alpha}\phi(z_N)(1 + o(1)), \quad x_N > 0 \qquad (p \in (p^*_{\mathcal{D}}, p^*_{\mathcal{D}} + \varepsilon))$$

$$u(x) = c|x|^{-\alpha}\log(1/|x|)^{-\alpha/2} z_N(1 + o(1)), \quad x_N > 0, \ |x| < 1 \qquad (p = p^*_{\mathcal{D}})$$

2. Fixed point argument in a general domain

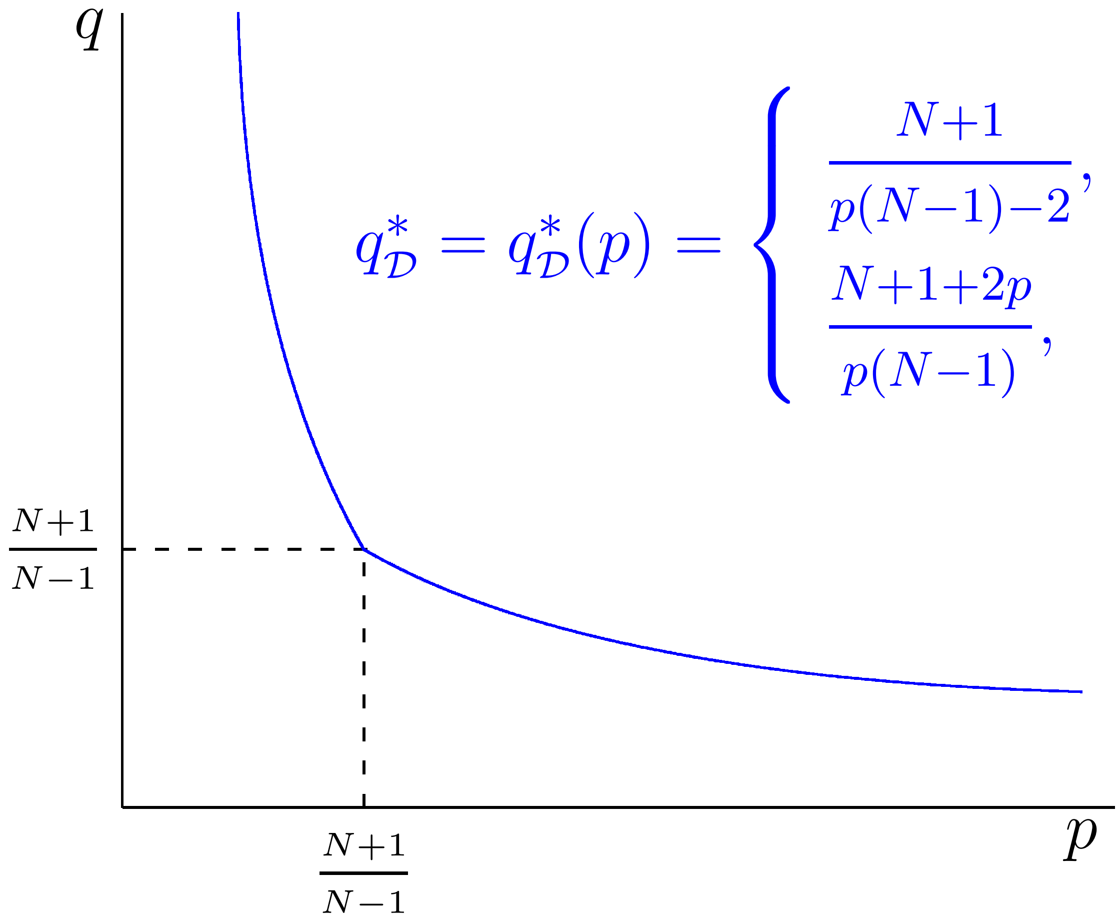
(using invertibility of Δ with singular RHS)

$$-\Delta u = f(x,v) \quad \text{in } \Omega, \qquad |f(x,v)| \le C(1+|v|^p)$$

$$-\Delta v = g(x,u) \quad \text{in } \Omega, \qquad |g(x,u)| \le C(1+|u|^q)$$

$$(\mathcal{D}) \quad \begin{cases} -\Delta u = f(x,v) & \text{in } \Omega, \\ -\Delta v = g(x,u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{aligned} |f(x,v)| &\le C(1+|v|^p) \\ |g(x,u)| &\le C(1+|u|^q) \end{aligned}$$

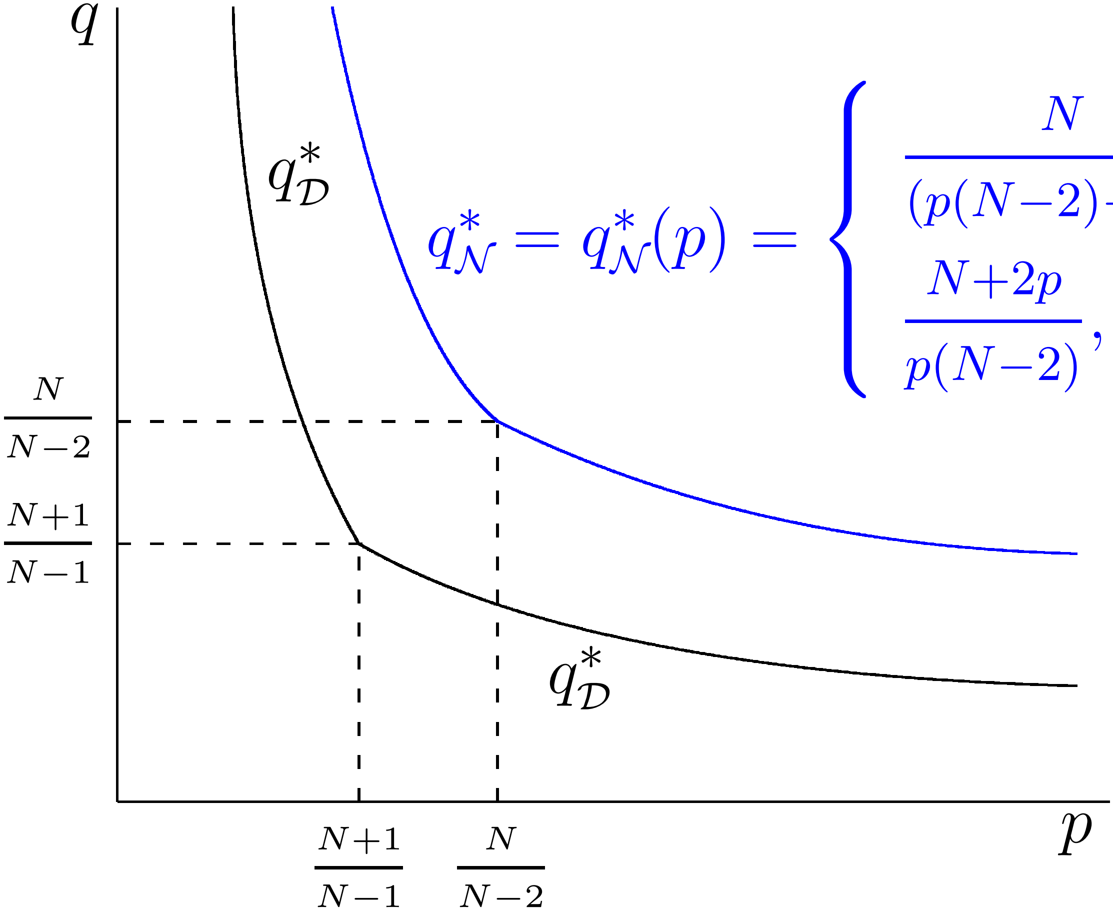
$$q^*_{\mathcal{D}} = q^*_{\mathcal{D}}(p) = \begin{cases} \dfrac{N+1}{p(N-1)-2}, & p < \dfrac{N+1}{N-1} \\[2ex] \dfrac{N+1+2p}{p(N-1)}, & p \ge \dfrac{N+1}{N-1} \end{cases}$$



$$q < q^*_{\mathcal{D}} \;\Rightarrow\; u, v \in L^\infty, \qquad q > q^*_{\mathcal{D}} \;\Rightarrow\; \exists\, f, g,\; u, v \notin L^\infty$$

$$(\mathcal{N})\quad \begin{cases} -\Delta u = f(x,v) & \text{in } \Omega, \qquad |f(x,v)| \le C(1+|v|^p) \\ -\Delta v = g(x,u) & \text{in } \Omega, \qquad |g(x,u)| \le C(1+|u|^q) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

$$q^*_{\mathcal{N}} = q^*_{\mathcal{N}}(p) = \begin{cases} \dfrac{N}{(p(N-2)-2)_+}, & p < \dfrac{N}{N-2} \\ \dfrac{N+2p}{p(N-2)}, & p \ge \dfrac{N}{N-2} \end{cases}$$



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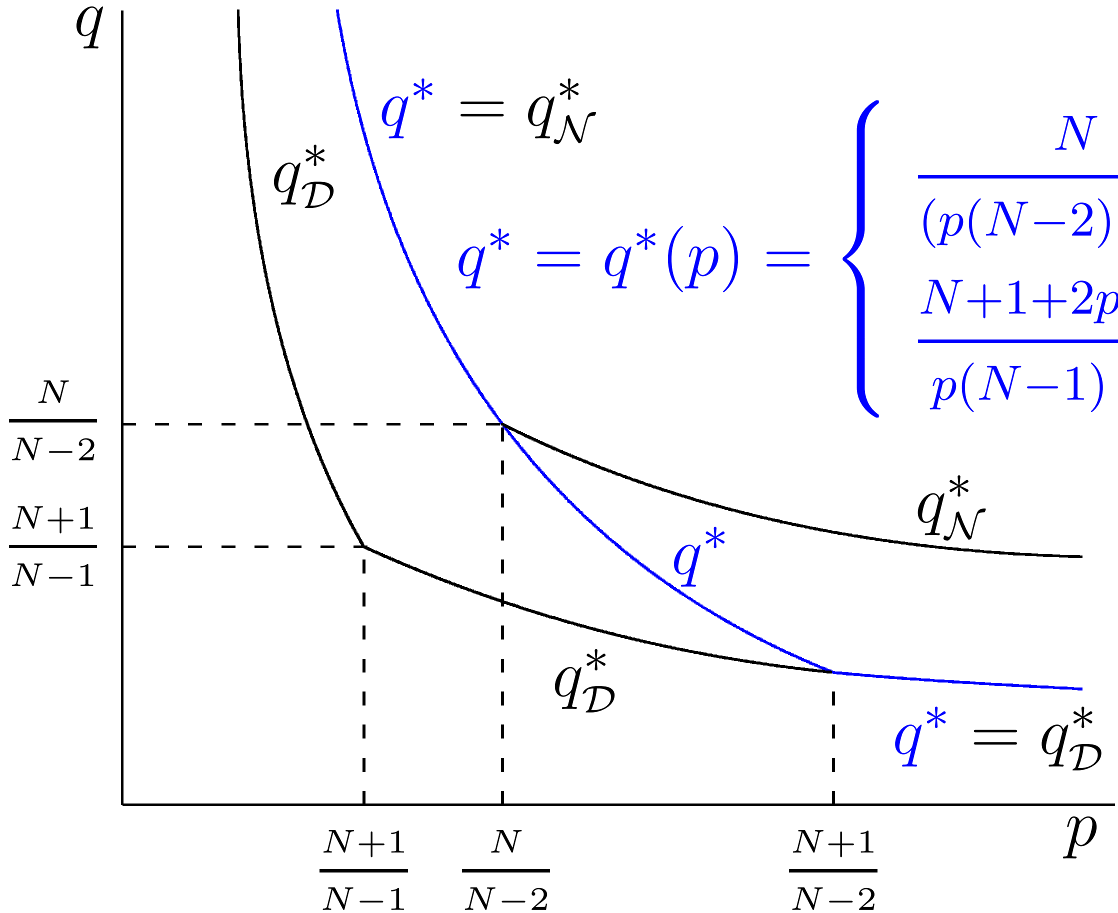

$$q^* = q^*(p) = \begin{cases} \dfrac{N}{(p(N-2)-2)_+}, & p < \dfrac{N+1}{N-2} \\ \dfrac{N+1+2p}{p(N-1)}, & p \ge \dfrac{N+1}{N-2} \end{cases}$$

$$q < q^* \;\Rightarrow\; u, v \in L^\infty, \qquad q > q^* \;\Rightarrow\; \exists\, f, g,\; u, v \notin L^\infty$$

Scaling exponents:

$$\alpha := \frac{2(p+1)}{(pq-1)_+}, \qquad \beta := \frac{2(q+1)}{(pq-1)_+}$$

u, v solutions of $-\Delta u = v^p, \ -\Delta v = u^q, \quad \lambda > 0$

$\Rightarrow u_\lambda(x) := \lambda^\alpha u(\lambda x), \ v_\lambda(x) := \lambda^\beta v(\lambda x)$ are also solutions

$$q < q^*_{\mathcal{D}}(p) \quad \Leftrightarrow \quad \max(\alpha, \beta) > N - 1$$

$$q < q^*_{\mathcal{N}}(p) \quad \Leftrightarrow \quad \max(\alpha, \beta) > N - 2$$

$$q < q^*(p) \quad \Leftrightarrow \quad \max(\beta - (N-2), \alpha - (N-1)) > 0$$

Idea of the proof of boundedness
(and a priori estimates in the superlinear case):
Alternate bootstrap and

- $(\mathcal{N})$ $\dots L^p - L^q$ estimates
- $(\mathcal{D})$ $\dots L^p_\delta - L^q_\delta$ estimates
- $(\mathcal{DN})$ $\dots L^p \cap L^{\tilde p}_{\delta^\xi} - L^q_\delta$ and $L^p_\delta - L^q \cap L^{\tilde q}_{\delta^\xi}$ estimates

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- $(\mathcal{N})$ ... $L^p - L^q$ estimates
- $(\mathcal{D})$ ... $L^p_\delta - L^q_\delta$ estimates
- $(\mathcal{DN})$ ... $L^p \cap L^{\tilde{p}}_{\delta^\xi} - L^q_\delta$ and $L^p_\delta - L^q \cap L^{\tilde{q}}_{\delta^\xi}$ estimates

References:

$(\mathcal{D})$,$(\mathcal{N})$ Q., Souplet 2004; Souplet 2005 $(f(x,u,v),\ g(x,u,v))$

Improvements: Li 2010, Kosírová 2010

Systems with more than 2 components: Li 2009

$(\mathcal{DN})$ Kelemen, Q. 2010

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g(x,u) & \text{on } \partial\Omega, \end{cases} \qquad |g(x,u)| \le C(1+|u|^r)$$

Q., Reichel 2008: The critical exponent is $r^* = \frac{N-1}{N-2}$

(also for more general nonlinear equations)

Systems with nonlinear boundary conditions:

Kosírová (work in progress)

Singular solutions of $-\Delta u = 0$ in Ω, $\frac{\partial u}{\partial \nu} = g(x,u)$ on $\partial\Omega$:

1. If $\Omega = \{x : x_N < 0\}$, $g(x,u) = u^r$, $r > r^* = \frac{N-1}{N-2}$,

   then there exists solution of the form

$$u(x) = |x|^{-1/(p-1)} h(\theta), \quad \text{where} \quad \cos\theta = \frac{x_N}{|x|}.$$

2. For bounded Ω with a flat boundary piece, $N \in \{3,4\}$

$$g(x,u) = u^r - u, \qquad r \in (r^*, r^* + \varepsilon),$$

   perturbation arguments (and variational methods)

   guarantee existence of multiple singular solutions.