How do isolated singularities look like ?

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Meaning of solution :

$$u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\}).$$

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$$u(x) \sim C_u \frac{1}{\|x\|^2}.$$

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Assume that $q \geq 2$. *Then,*

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Singularities in $\mathbb{R}^4$ (weak nonlinearities)

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Existence of singularities in $\mathbb{R}^4$

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Theorem (del Pino–Musso–Pacard '07)

For every $q \gtrapprox \frac{5}{3}$ *there exists a solution such that*

$$u(x) \to +\infty \quad \textit{nontangentially as } x \to 0.$$

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(iii) *If* $\frac{5}{3} < q < 5$, *then* (E') *admits a unique positive solution.*

Pohozaev identity

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Theorem

$$(q-5) \int\limits_{S^3_+} \|\nabla_{S^3} v(x)\|^2 x_4 \, d\sigma - 3(\ell(q-1)+3) \int\limits_{S^3_+} (v(x))^2 x_4 \, d\sigma$$
$$= -(q+1) \int\limits_{\partial S^3_+} \|\nabla_{S^3} v(x)\|^2 \, d\tau.$$

Uniqueness

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Theorem

For every $\ell \in \mathbb{R}$, problem has at most one positive solution

A question

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Question

Does problem have at most one positive solution for every $\ell \in \mathbb{R}$?

A priori estimate

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