CRITICAL NONLINEARITIES

IN PARTIAL DIFFERENTIAL

EQUATIONS

Introduction

Critical nonlinearities are divided into three classes:

(I) – Fujita type critical nonlinearities

(II) – Bernstein type critical nonlinearities

(III) – Sobolev type critical nonlinearities

All these three critical nonlinearities are related to "the Hamlet problem = to be or not to be global solutions to nonlinear PDEs".

In this lecture we consider the Fujita type critical nonlinearities.

EXAMPLES

1. Parabolic equation:

$$\begin{cases} u_t = \Delta u + u^p \text{ in } \mathbb{R}_+^{N+1} \quad (p > 1), \\ u|_{t=0} = u_0(x) \geq 0, \quad u_0(x) \not\equiv 0. \end{cases} \tag{1}$$

Local analysis:

$\forall p > 1\ \exists T = T(u_0):\ \ \exists u(t,x)$ for $0 < t < T$, $x \in \mathbb{R}^N$.

Global analysis:

? $T = \infty$! $\exists p_{\text{cr}} = 1 + \frac{2}{N}$ (H. Fujita '66)
$\forall p\colon 1 < p \leq p_{\text{cr}}\ \ \nexists u(t,x)\ \forall u_0 \geq 0:$
$\forall u_0 \geq 0\ \exists T_\infty = T_\infty(u_0) < \infty,\ \int\limits_{\mathbb{R}^N} u^p(t,x)\,dx \to +\infty,\ t \to T_\infty.$

2. Hyperbolic equation:

$$\begin{cases} u_{tt} = \Delta u + |u|^p \text{ in } \mathbb{R}_+^{N+1} \quad (p > 1), \\ u = u_0(x) \text{ for } t = 0, \\ u_t = u_1(x) \geq 0 \text{ for } t = 0. \end{cases} \tag{2}$$

Local analysis:

$\forall p > 1 \ \exists T = T(u_0, u_1)\colon \ \exists u(t, x)$ for $0 < t < T$, $x \in \mathbb{R}^N$

Global analysis:

? $T = \infty$! $\exists p_{\text{cr}} = \frac{N+1}{N-1}$ (T. Kato '80)

$\forall p\colon 1 < p \leq p_{\text{cr}}$ there is no global in t
(i.e., for all $t > 0$) solution to (2)

$\exists T_\infty = T_\infty(u_0) < \infty, \ \int\limits_{\mathbb{R}^N} u^p(t, x)\, dx \to +\infty, \ t \to T_\infty.$

METHODS

1. Comparison method (maximum principle, positivity of the fundamental solution of the corresponding linear operator).

2. Self-similar solutions.

This method essentially reduces the class of nonlinear equations to scalar second order ones, which possess the positivity property.

METHOD OF NONLINEAR CAPACITY (S.P., 1997)

It turns out that the blow-up of the solution is connected with the **nonlinear capacity** induced by the nonlinear operator.

"Each nonlinear operator has its own nonlinear capacity."

The general scheme: nonlinear operator $\longrightarrow$ nonlinear capacity $\longrightarrow$ capacity dimension $\longrightarrow$ blow-up conditions.

Advantages:

- generality of the method
- simpler method
- sharpness (non-improvability) of the obtained criteria
- stability of critical exponents with respect to nonlinear perturbations

Comment

$$\begin{cases} u_t = \Delta u + u^p \text{ in } \mathbb{R}^N \times (0, +\infty), \\ u = u_0 > 0 \text{ for } t = 0. \end{cases}$$

? $p_{\text{cr}} = 1 + 2/N$.

H. Fujita '66, $1 < p < 1 + 2/N$
J. Lions '69, $1 < p < 1 + 2/N$.

? $p = 1 + 2/N$

K. Hayakawa '73, $N = 2$
D. Aronson, H. Weinberger '78
$\forall N$: $p_{\text{cr}} = 1 + 2/N$.
$1 < p \leq 1 + 2/N$

$1978 - 1966 = 12$ years!

Example

$$\begin{cases} u_t = \Delta u + u^p \\ u_0(x) = 1 \end{cases} \Rightarrow u(t,x) = u(t):$$

$$\begin{cases} u_t = u^p \\ u(0) = 1 \end{cases} \Rightarrow u(t) = [1 - (p-1)t]^{-\frac{1}{p-1}}$$

$$\forall p > 1 \ (p > p_{\text{cr}}) \Rightarrow \text{ blow-up} \qquad T_\infty = \frac{1}{p-1}.$$

Problem

$$\begin{cases} u_t = \Delta u + |u|^p + f(t,x) \\ u|_{t=0} = u_0(x) \end{cases}$$

? $\quad p_{\text{cr}} = p_{\text{cr}}(N, f, u_0)$

Let $\displaystyle \int\limits_{|x|\le R} \int\limits_0^T f(t,x)\,dt\,dx\Big|_{T=R^2} + \int\limits_{|x|\le R} u_0(x)\,dx \ge c_d(1+R^\gamma)$

with $\gamma \ge 0$, $c_d > 0$ (without assumptions $f \ge 0$, $u_0 \ge 0$!)

$$\text{Then } p_{\text{cr}} = \begin{cases} +\infty, & \text{if } \gamma \ge N, \\ 1 + \frac{2}{N-\gamma}, & \text{if } 0 \le \gamma < N. \end{cases}$$

Estimate of the existence domain

$T_\infty < T_* = (c_*/c_d)^\theta$, $\quad \theta = \dfrac{2(p-1)}{(\gamma - N)p + N + 2 - \gamma}$ for $1 < p < p_{\text{cr}}$

$R_\infty < R_* = T_*^{1/2}$, $\qquad \theta$ is the sharp exponent!

How does the nonlinear capacity "appear" and "work"?

Pilot example: $A(u) = -\Delta u - u^q$

$$\begin{cases} -\Delta u \geq u^q \text{ in } \mathbb{R}^N, \\ u \geq 0, \quad q > 1. \end{cases}$$

Definition: $u \in L^q_{\text{loc}}(\mathbb{R}^N), u \geq 0$: $\int u^q \psi \leq -\int u \Delta \psi \ \forall \psi \in C_0^2(\mathbb{R}^N), \psi \geq 0$.
$X_{\text{loc}}(\mathbb{R}^N) = \{u \in L^q_{\text{loc}}(\mathbb{R}^N), \ u \geq 0\}$

Let $e = e_R = \{x \in \mathbb{R}^N \mid |x| \leq R\}$, $\psi \in C_0^2(\mathbb{R}^N)$, $\psi \geq 0$, $\psi(x) = \begin{cases} 1, & |x| \leq R, \\ 0, & |x| \geq \kappa R \ (\kappa > 1). \end{cases}$

$$\int u^q \psi \leq -\int \Delta u \cdot \psi = -\int u \Delta \psi \leq \left(\int u^q \psi \right)^{1/q} \left(\int \frac{|\Delta \psi|^{q'}}{\psi^{q'-1}} \right)^{1/q'}$$

$$\int\limits_{|x| \leq R} u^q \leq \int u^q \psi \leq \int \frac{|\Delta \psi|^{q'}}{\psi^{q'-1}} \to \inf_{\psi} \int \frac{|\Delta \psi|^{q'}}{\psi^{q'-1}} = \text{Cap}(A, e_R)$$

Thus the capacity = the optimal a priori estimate in a certain class of test functions.

How does the capacity “work”?

Evidently, if $\text{Cap}(A, e_R) \to 0$ for $R \to \infty$, there is no nontrivial global (i.e., $\forall R > 0$) solution.

We have $\text{Cap}(A, e_R) \leq \int \frac{|\Delta\psi|^{q'}}{\psi^{q'-1}}$, $\psi(x) = \psi_0\left(\frac{r}{R}\right)$, $\quad \psi_0 \geq 0, \quad \psi_0 \in C^2$,

$$\psi_0(\rho) = \begin{cases} 1, & \rho \leq 1, \\ 0, & \rho \geq \kappa > 1, \end{cases} \quad \int \frac{|\Delta\psi|^{q'}}{\psi^{q'-1}} = R^{N-2q'} \int \frac{|\Delta\psi_0|^{q'}}{\psi_0^{q'-1}}$$

Hence if $1 < q < q_{\text{cr}} = \begin{cases} \frac{N}{N-2} & \text{for } N > 2, \\ +\infty & \text{for } N = 1, 2, \end{cases}$ then

$$\text{Cap}(A, e_R) \to 0 \text{ for } R \to \infty \Rightarrow \int\limits_{\mathbb{R}^N} u^q = 0,\ u \geq 0 \Rightarrow u = 0 \text{ a.e. in } \mathbb{R}^N.$$

The limit case: $N = 2q'$, $\text{Cap}(A, e_R) \leq c_* < \infty\, \forall R > 1 \Rightarrow \int\limits_{\mathbb{R}^N} u^q < \infty$

$$\int\limits_{|x|\leq R} u^q \leq \int u^q\psi \leq -\int u\Delta\psi = -\int\limits_{\text{supp}\Delta\psi} u\Delta\psi \leq$$

$$\Big(\int\limits_{\text{supp}\Delta\psi} u^q\psi\Big)^{1/q}\Big(\int \frac{|\Delta\psi|^{q'}}{\psi^{q'-1}}\Big)^{1/q'} \leq c_*^{1/q'} \cdot \Big(\int\limits_{R\leq|x|\leq\kappa R} u^q\psi\Big)^{1/q} \to 0 \text{ for } R \to \infty.$$

A counterexample

$\forall q > q_{\text{cr}} \quad (N > 2)$

$$\exists u(x) = A(1 + |x|^2)^\lambda > 0$$

with $\dfrac{2-N}{2} < \lambda < -\dfrac{1}{q-1}$

and $A = 2|\lambda| \cdot (N - 2 + 2\lambda) > 0$:

$$-\Delta u \geq u^q \text{ in } \mathbb{R}^N, \quad N > 2.$$

Stability of the critical exponent

$$A(u) := -\Delta u - u^q, \quad u \geq 0, \quad q > 1.$$

$$q_{\text{cr}} = \begin{cases} \frac{N}{N-2} & \text{for } N > 2, \\ +\infty & \text{for } N \leq 2. \end{cases}$$

Let $\tilde{A}(u) = -\text{div}(a(x,u,Du) \cdot Du) - b(x,u,Du)$, $u \geq 0$

$$\text{with } \begin{cases} 0 < a_0 \leq a(x,u,Du) \leq a_1, \\ b(x,u,Du) \geq b_0|u|^q, \; b_0 > 0. \end{cases} \qquad (H)$$

Then $q_{\text{cr}}(\tilde{A}) = q_{\text{cr}}(A)$.

Moreover, for the operator $\tilde{A}_1$:

$$\tilde{A}_1(u) = -\text{div}\left(a(x,u,Du)\frac{Du}{\sqrt{1+|Du|^2}}\right) - b(x,u,Du)$$

under assumptions (H) we have

$$q_{\text{cr}}(\tilde{A}_1) = q_{\text{cr}}(\tilde{A}) = q_{\text{cr}}(A).$$

In particular, for

$$A_1(u) = -\text{div}\frac{Du}{\sqrt{1+|Du|^2}} - u^q, \;\; u \geq 0$$

we have

$$q_{\text{cr}}(A_1) = q_{\text{cr}}(A).$$

The concept of nonlinear capacity

(S.P., Doklady RAS, 1997)

$\Omega \subset \mathbb{R}^N, \quad X(\Omega) \subset L_{1,\mathrm{loc}}(\Omega)$

$A\colon X(\Omega) \to X'(\Omega).$

$e \subset \Omega$ is compact

$C_0^\infty(e,\Omega) = \{\psi \in C_0^\infty(\Omega) \,|\, 0 \le \psi \le 1 \text{ in } \Omega \text{ and } \psi = 1 \text{ in } e\}.$

Definition 1. We call the quantity

$$\text{Cap}_A(e,\Omega) = \inf_{\psi \in C_0^\infty(e,\Omega)} \sup_{u \in X(\Omega)} \int_\Omega A(u)\psi \, dx \qquad (3)$$

nonlinear capacity.

Remark. For coercive operators A, we introduce an extra restriction

$$\|u\|_X \leq \|\psi\|_X.$$

For anti-coercive operators A, we remove this restriction.

Example

1. $A(u) = -\Delta u$

$$X(\Omega) = W^{1,2}_{\text{loc}}(\Omega)$$

$$\text{Cap}_A(e,\Omega) = \inf_{\psi \in C_0^\infty(e,\Omega)} \sup_{u \in W^{1,2}_{\text{loc}}(\Omega)} \left\{ -\int_\Omega \Delta u \cdot \psi \, dx \,\middle|\, \|u\|_{1,2} \le \|\psi\|_{1,2} \right\}$$

$$= \inf_{\psi \in C_0^\infty(e,\Omega)} \int_\Omega |\nabla \psi|^2 \, dx$$

$\text{Cap}_\Delta(e,\Omega)$ is the classical (harmonic) capacity

2. $A(u) = -\Delta_p u := -\mathrm{div}(|Du|^{p-2} Du) \quad (p > 1)$

$X(\Omega) = W^{1,p}_{\mathrm{loc}}(\Omega)$

$$\begin{aligned}\mathrm{Cap}_A(e,\Omega) &= \inf_{\psi \in C_0^\infty(e,\Omega)} \sup_{u \in W^{1,p}_{\mathrm{loc}}(\Omega)} \left\{ -\int_\Omega \Delta_p u \cdot \psi \, dx \,\middle|\, \|u\|_{1,p} \le \|\psi\|_{1,p} \right\} \\ &= \inf_{\psi \in C_0^\infty(e,\Omega)} \int_\Omega |\nabla \psi|^p \, dx\end{aligned}$$

$\mathrm{Cap}_{\Delta_p}(e,\Omega)$ is the p–harmonic capacity

Definition 2. Let M be a multiplier corresponding to the operator A. Define A_M as

$$A_M = MA,$$

so that $MA\colon X_{\mathrm{loc}}(\Omega) \to D'(\Omega)$. Then

$$\begin{aligned} &\mathrm{Cap}_{A_M}(e,\Omega)\colon = \mathrm{Cap}_{MA}(e,\Omega) \\ &= \inf_{\psi\in C_0^\infty(e,\Omega)} \sup_{u\in X(\Omega)} \int_\Omega MA(u)\psi\, dx. \end{aligned} \tag{4}$$

Example 1. Nonlinear elliptic capacity

$$A(u) := -\Delta_p u - u^q, \ p > 1, \ q > p - 1$$

$$X_{\text{loc}}(\Omega) = \{u \in W^{1,p}_{\text{loc}}(\Omega) | u \geq 0, \int_{\text{loc}} |Du|^p u^\alpha + \int_{\text{loc}} u^{q+\alpha} < \infty.\}$$

Let $M(u) = u^\alpha$ with $1 - p < \alpha < 0$. Then

$$\text{Cap}_{A_M}(e, \Omega) = \inf_{\psi \in C_0^\infty(e,\Omega)} \left\{ \frac{1}{c} \int_\Omega \frac{|D\psi|^\gamma}{\psi^{\gamma-1}} dx \right\}$$

with $c > 0$ and $\gamma = \dfrac{p(q+\alpha)}{q-p+1}$.

Remark. If one replaces $\psi \to \zeta$: $\psi = \zeta^\gamma$, then

$$\text{Cap}_{A_M}(e,\Omega) = \inf_{\psi \in C_0^\infty(e,\Omega)} \left\{ \frac{\gamma^\gamma}{c} \int_\Omega |D\zeta|^\gamma dx \right\}.$$

This implies

$$q_{\text{cr}} = \begin{cases} \dfrac{N(p-1)}{N-p} & \text{for } N > p, \\ +\infty & \text{for } p \geq N. \end{cases}$$

For $p = 2$, one has $q_{\text{cr}} = \dfrac{N}{N-2}$ $(N > 2)$ –

the J. Serrin exponent.

Example 2. Nonlinear parabolic capacity

$$A(u) := \frac{\partial u}{\partial t} - D(u^\sigma |Du|^{p-2} Du) - u^q, \;\; u \geq 0 \;\; \text{in } \mathbb{R}_+^{N+1},$$
$$p > 1, \;\; \frac{p}{N} + \sigma + p - 1 > 1, \;\; q > \max\{1, \sigma + p - 1\}.$$

$$\mathrm{Cap}_{A_M}(e,\Omega) = \inf_{\psi \in C_0^\infty(e,\Omega)} \left\{ \iint_\Omega \frac{|D\psi|^{\gamma_1}}{\psi^{\gamma_1 - 1}} + \frac{|\psi_t|^{\gamma_2}}{\psi^{\gamma_2 - 1}} : \psi = \psi(t,x), \, \Omega \subset \mathbb{R}_+^{N+1} \right\}$$

$$\text{with } \gamma_1 = \frac{p(q+\alpha)}{q - (\sigma + p - 1)}, \;\; \gamma_2 = \frac{q+\alpha}{q-1} \;\; (-1 < \alpha \leq 0).$$

Hence,

$$q_{\mathrm{cr}} = \frac{p}{N} + \sigma + p - 1.$$

It is the Galaktionov exponent – '82, '94.

Stability of critical exponents in $X^+_{\mathrm{loc}}(\mathbb{R}^{N+1}_+)$

Example.

$$\tilde{A}(u) := \frac{\partial u}{\partial t} - \mathrm{div}(a(\dots)u^\sigma |Du|^{p-2}Du) - b(\dots),$$

$u \geq 0$ in $\mathbb{R}^{N+1}_+$

with $\begin{array}{l} a(\dots) = a(t,x,u,Du,\dots), \\ b(\dots) = b(t,x,u,Du,\dots) \end{array}$ under assumptions:

$$\begin{cases} 0 < a_0 \leq a(\dots) \leq a_1, \\ b(x,u,Du) \geq b_0|u|^q, \ b_0 > 0. \end{cases}$$

We have

$$q_{\mathrm{cr}}(\tilde{A}) = q_{\mathrm{cr}}(A) = \frac{p}{N} + \sigma + p - 1$$

under the previous assumptions on p, q and σ.

The generalized Zeldovich–Kompaneets–Barenblatt equation

$$A_0(u) := u_t + (-1)^k \Delta^k |u|^m - |u|^p \text{ in } \mathbb{R}_+^{N+1}$$

with $p > m \geq 1$.

We have $$\mathrm{Cap}_{A_0}(e, \Omega) = \inf_{\psi \in C_0^\infty(e,\Omega)} \left\{ \iint_\Omega \frac{|D^k \psi|^{q'}}{\psi^{q'-1}} + \frac{|\psi_t|^{p'}}{\psi^{p'-1}} \right\}$$

with $q = \dfrac{p}{m}$, $q' = \dfrac{q}{q-1}$, $p' = \dfrac{p}{p-1}$.

$$p_{\mathrm{cr}}(A_0) = m + \frac{2k}{N}$$

If $\begin{cases} A_0(u) \geq f(t,x) & \text{in } \mathbb{R}^{N+1}_+, \\ u = u_0(x) & \text{for } t = 0,\ x \in \mathbb{R}^N \end{cases}$

with $\displaystyle\int_0^T \int_{B_R} f(t,x)\,dt\,dx \Big|_{T=R^\theta} + \int_{B_R} u_0(x)\,dx \geq c_d(1 + R^\gamma),$

$\theta = 2k\dfrac{p-1}{p-m}$ and $c_d > 0$, then

$$p_{\text{cr}}(A_0, f, u_0) = \begin{cases} m + \dfrac{2k}{N-\gamma} & \text{for } 0 \leq \gamma < N, \\ +\infty & \text{for } \gamma \geq N. \end{cases}$$

Stability of the critical exponent

Consider $\tilde{A}(u) = u_t + \Delta^k a(t,x,u) - b(t,x,u),$

$$|a(t,x,u)| \leq c|u|^m, \qquad b(t,x,u) \geq b_0|u|^p$$

with $p > m > 1$, $b_0 > 0$. We have

$$p_{\text{cr}}(\tilde{A}) = p_{\text{cr}}(A_0).$$

Example 3. Nonlinear hyperbolic capacity

$$A(u):\ = \frac{\partial^2 u}{\partial t^2} - \sum_{l \le |\alpha| \le L} D^\alpha A_\alpha(t,x,u) - b(t,x,u)|u|^q \text{ in } \mathbb{R}_+^{N+1}$$

with $|A_\alpha(t,x,u)| \le c|u|^p,\ p > 0,\ b(t,x,u) \ge b_0 > 0.$

$$\mathrm{Cap}_A(e,\Omega) \le C \inf_{\psi \in C_0^\infty(e,\Omega)} \left\{ \iint_\Omega \frac{|D\psi|^{\gamma_1}}{\psi^{\gamma_1 - 1}} + \frac{|\psi_{tt}|^{\gamma_2}}{\psi^{\gamma_2 - 1}} : \psi = \psi(t,x),\ \Omega \subset \mathbb{R}_+^{N+1} \right\}$$

$$\text{with } \gamma_1 = \frac{q}{q-p},\ \gamma_2 = \frac{q}{q-1}.$$

This implies

$$q_{\mathrm{cr}} = \begin{cases} \dfrac{2N + l}{2N - l} & \text{for } 2N - l > 0, \\ +\infty & \text{for } 2N - l \le 0. \end{cases}$$

Corollary

$$A(u) \colon = \frac{\partial^2 u}{\partial t^2} - \Delta u - |u|^q \text{ in } \mathbb{R}^{N+1}_+$$

$$p = 1, l = L = 2, q > 1$$

$$\mathrm{Cap}_A(e, \Omega) \leq C \inf_{\psi \in C_0^\infty(e,\Omega)} \left\{ \iint_\Omega \frac{|\Delta\psi|^\gamma + |\psi_{tt}|^\gamma}{\psi^{\gamma-1}} \colon \psi = \psi(t,x),\ \Omega \subset \mathbb{R}^{N+1}_+ \right\}$$

with $\quad \gamma = \dfrac{q}{q-1}$

Hence,

$$q_{\mathrm{cr}}|_{p=1,l=2} = \frac{2N \cdot 1 + 2}{2N \cdot 1 - 2} = \frac{N+1}{N-1} \ (N > 1),$$

i.e., the Kato exponent.

Nonlocal nonlinearities

Nonlinear integral equations and inequalities.

Liouville theorems

Example 1. $u(x) = \int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\beta}} |u(y)|^q dy$

For $0 < \beta < N$ we have $q_{\text{cr}} = \frac{N}{N-\beta}$.

Let $1 < q \le q_{\text{cr}}$.

Then there are no nontrivial solutions.

Example 2. Consider a more general case

$$L(u) \geq \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\beta}} |u(y)|^q dy$$

with $\beta < N$, where L is a linear differential operator.

Then there exists $q_{\text{cr}} > 1$ such that for

$1 < q \leq q_{\text{cr}}$ there is no nontrivial solution.

Example 2.1. $u_t - \Delta u \geq \int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\beta}} |u(y)|^q dy$

$$q_{\rm cr} = \frac{N+2}{N-\beta} \quad (0 < \beta < N).$$

Example 2.2. $u_{tt} - \Delta u \geq \int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\beta}} |u(y)|^q dy$

$$q_{\rm cr} = \begin{cases} \dfrac{N+1}{N-1-\beta} & \text{for } 0 < \beta < N-1, \\ +\infty & \text{for } N-1 \leq \beta < N. \end{cases}$$

Consider the following Cauchy problem:

$$\begin{cases} \dfrac{\partial^k u}{\partial t^k} - \sum\limits_{l\le|\alpha|\le m} D^\alpha a_\alpha(t,x,u) \ge \int\limits_{\mathbb{R}^N} |x-y|^{\beta-N}|u(t,y)|^q\,dy + f(t,x), \\ \dfrac{\partial^i u}{\partial t^i}(0,x) = u_i(x), \quad i = 0,\dots,k-1. \end{cases} \tag{5}$$

Theorem. *Let $k \ge 1$, $l \ge 1$, $N > \beta > 0$ and functions $u_i \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^N)$ be such that*

$$\liminf_{R\to\infty} \int\limits_{B_R} u_{k-1}(x)\,dx \ge 0, \qquad \liminf_{R,T\to\infty} \int\limits_0^T \int\limits_{B_R} f(t,x)\,dx\,dt \ge 0.$$

Let a_0 satisfy the estimate $|a_0(t,x,u)| \le c_0|u|^p$ with some $c_0 > 0$,

$p > 0$ and $q > \max\{1,p\}$. Then problem (5) has no nontrivial solutions for $q \le q_{\mathrm{cr}} = \begin{cases} \dfrac{(kN-\beta)p+l+\beta}{(N-\beta-l)k+l} & \text{if } (N-\beta-l)k+l>0, \\ +\infty & \text{if } (N-\beta-l)k+l \le 0. \end{cases}$

Corollary

Example (the generalized John-Kato problem)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u \geq \displaystyle\int_{\mathbb{R}^N} |x-y|^{\beta-N} |u(t,y)|^q \, dy + f(t,x) \text{ in } \mathbb{R}^{N+1}_+, \\ u(0,x) = u_0(x) \text{ in } \mathbb{R}^N, \\ \dfrac{\partial u}{\partial t}(0,x) = u_1(x) \text{ in } \mathbb{R}^N, \end{cases} \tag{6}$$

where $q > 1$ *and* $N > \beta > 0$. *Let functions* $u_0, u_1 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^N)$,

$$\liminf_{R\to\infty} \int_{B_R} u_1(x)\, dx \geq 0, \quad \liminf_{R,T\to\infty} \int_0^T \int_{B_R} f(t,x)\, dx\, dt \geq 0.$$

Then problem (6) *has no nontrivial solutions for*

$$q \leq q_{\mathrm{cr}} = \begin{cases} \dfrac{N+1}{N-1-\beta} & \text{if } 0 < \beta < N-1, \\ +\infty & \text{if } N-1 \leq \beta < N. \end{cases}$$

The Kuramoto-Sivashinsky equation

Let Ω_0 be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega_0$ and $0 \in \Omega_0$.

Let $\Omega = \mathbb{R}^N \setminus \overline{\Omega}_0$. Consider the problem in Q_T:

$$u_t + \Delta^2 u + \Delta u = |Du|^2 \text{ in } Q_T, \tag{7}$$

$$u = \frac{\partial}{\partial n}(u + \Delta u) = 0 \text{ on } \Gamma_T, \tag{8}$$

$$u = u_0(x) \text{ for } t = 0,\ x \in \Omega. \tag{9}$$

Here $Q_T = (0, T) \times \Omega$ and $\Gamma_T = (0, T) \times \partial\Omega_0$.

The problem is considered in the space $W^{1,4}_{2,\text{loc}}(Q_T)$.

Theorem. *Let*

$$\overline{\lim_{R\to\infty}} \frac{I_R(u_0)}{R^{\frac{3N-2}{2}}} = +\infty \quad \text{for } N > 2,$$

$$\overline{\lim_{R\to\infty}} \frac{I_R(u_0)}{R^2 \ln R} = +\infty \quad \text{for } N = 2,$$

$$\overline{\lim_{R\to\infty}} \frac{I_R(u_0)}{R} = +\infty \quad \text{for } N = 1,$$

where $I_R(u_0) := \int\limits_{\Omega_R} u_0(x)\,dx$. *Then problem* (7) – (9) *has no global solution for all* $t > 0$. *Moreover, there is no solution* $u(t,x)$ *for* $x \in \Omega_R$ *with* $R > R_*$ *and* $t > T_{R_*}$, *where* R_* *and* T_{R_*} *are determined by the initial data of the problem.*

Remark. Note that the asymptotic exponent of the behavior of the initial data u_0, namely $\dfrac{3N-2}{2}$ for $N>2$, is sharp (non-improvable). This follows from a corresponding counterexample demonstrating existence of a global solution to the problem for

$$|u_0(x)| \le C|x|^{\frac{N-2}{2}} \quad (N>2,\ R>1),$$

which implies

$$\left|\int\limits_{\Omega_R} u_0(x)\,dx\right| \le CR^{\frac{3N-2}{2}} \text{ for } N>2,\ R>1.$$

Schemes of application of nonlinear capacity

I. The explicit scheme

Let $A(u) \geq f(u) \geq 0$ in $X_{\text{loc}}(\mathbb{R}^N)$,

$f(u) = 0 \Rightarrow u = 0$ a.e. in $\mathbb{R}^N$.

Fix $B_R \subset \mathbb{R}^N$ and $\psi \geq 0$,

$\psi \in C_0^\infty(\mathbb{R}^N),\ \psi(x) = 1$ for $x \in B_R$.

Then $\int\limits_{B_R} f(u) \leq \int f(u)\psi \leq \int A(u)\psi \leq \text{Cap}_A(B_R, \mathbb{R}^N) \to 0 \Rightarrow$

$\Rightarrow u = 0$ a.e. in $\mathbb{R}^N$.

Estimate of the "size" of the possible existence domain. A scheme

$A(u) \geq h \geq 0$ in $X_{\text{loc}}(\mathbb{R}^N)$

$$\int h\psi \leq \int A(u)\psi \leq \text{Cap}_A(B_R, \mathbb{R}^N) \text{ for } R \geq 1$$

If $\displaystyle\int_{B_R} h \geq c_1 \cdot R^\kappa$ for $R \geq 1$ and $\text{Cap}_A(B_R, \mathbb{R}^N) \leq c_2 \cdot R^\theta$ for $R \geq 1$, then

$$c_1 R^\kappa \leq c_2 R^\theta, \qquad \kappa > \theta$$

$$R_* = \max\left\{1, (c_2/c_1)^{1/(\kappa-\theta)}\right\}$$

II. The implicit scheme

Let there exist a nonnegative functional of the form

$$\int_{\mathbb{R}^N} E(u)\,dx, \qquad u \in X^+_{\text{loc}}(\mathbb{R}^N):$$

$$E(u) \geq 0 \text{ and } \int_{\Omega} E(u)\,dx = 0 \Rightarrow u = 0 \text{ a.e. in } \Omega.$$

Let for any solution $u \in X^+_{\text{loc}}(\mathbb{R}^N)$ of the inequality $A(u) \geq 0$ there hold

$$\int_{\Omega} E(u)\psi \leq c_0 \text{Cap}_A(e_R, \mathbb{R}^N).$$

Then, if $\text{Cap}_A(e_R, \mathbb{R}^N) \to 0$ for $R \to \infty$, one has $u = 0$ a.e. in $\mathbb{R}^N$.

Some generalizations

The k-th order entropy and its aplications

Let for a certain non-stationary problem in $\mathbb{R}_+^{N+1}$ there hold an inequality of the form

$$\frac{\partial H_0}{\partial t} - \sum_{i=1}^{N} \frac{\partial H_i}{\partial x_i} \geq W \text{ in } \mathbb{R}_+^{N+1} \qquad (10)$$

on the solutions of the problem under consideration from the corresponding class. Here $H_0, H_1, \ldots, H_N$ and W are functions depending on t, x, the solution u and its derivatives of order $k \geq 0$ that belong to the class C^1 in the range of the variables.

Let these functions satisfy the inequality

$$W \geq c_0(|H_0|^q + \sum_{i=1}^{N} |H_i|^{q_i}) \tag{11}$$

with some constants $c_0 > 0$, $q, q_1, \ldots, q_N > 1$ in the whole range of the arguments under consideration.

Definition 1. A pair of functions $(H_0, \overline{H})$, where H_0 is a C^1 scalar function and $\overline{H} = (H_1, \ldots, H_N)$ is a C^1 vector function satisfying inequalities (10)–(11) on the solutions of the problem, is called the k-th order entropy for the original non-stationary problem.

An essential characteristic of this concept is the quantity

$$\theta = \sum_{i=1}^{N} \frac{q'}{q'_i} + 1 - q' \text{ with } q' = \frac{q}{q-1} \text{ and } q'_i = \frac{q_i}{q_i - 1} \ (i = 1, \ldots, N). \tag{12}$$

Application of the k-th order entropy to the problem of existence of global solutions to non-stationary problems

Note that the considered class of solutions is determined by conditions

$$\begin{cases} H_0,\ H_i,\ W \in L^1_{\text{loc}}(\mathbb{R}^{N+1}_+), \\ H_0|_{t=0} \in L^1_{\text{loc}}(\mathbb{R}^N). \end{cases} \tag{13}$$

Definition 2. A generalized solution of the non-stationary problem satisfying conditions (13) is called a k-entropy one, if for this solution there exists the k-th order entropy satisfying inequality (10) in the sense of distributions $D'_+(\mathbb{R}^{N+1}_+)$.

Theorem. *Let the non-stationary problem under consideration have the k-th order entropy with $\theta \le 0$.*

Then this non-stationary problem does not admit a nontrivial global k-entropy solution in the whole $\mathbb{R}^{N+1}_+$ if the initial conditions are such that

$$H_0|_{u(0,x)} \ge 0. \tag{14}$$

The Hamilton–Jacobi equation with a nonnegative Hamiltonian

$$\begin{cases} \dfrac{\partial u}{\partial t} = H(u, Du), & (x,t) \in \mathbb{R}^2_+, \\ u(0,x) = u_0(x) \geq 1, \ \ x \in \mathbb{R}. \end{cases} \tag{15}$$

Define for this problem the 1st order entropy, i.e. vector function (H_0, H_1) with $H_0 = H_0(u,p)$, $H_1 = H_1(u,p)$ $(p \Leftrightarrow Du)$ satisfying relations (5)–(6) on smooth solutions of equation (15).

Validity of inequality (10) for all values of the corresponding arguments by (11) implies

$$\frac{\partial H_1}{\partial p} = \frac{\partial H}{\partial p} \cdot \frac{\partial H_0}{\partial p}. \tag{16}$$

Choosing

$$H_0(u,p) = u^a |p|^b, \qquad u \geq 1,\ p \in \mathbb{R},$$

we get

$$\frac{\partial H_1}{\partial p} = b u^a |p|^{b-2} p \frac{\partial H}{\partial p}.$$

Hence for

$$W(u,p) := \frac{\partial H_0}{\partial t} - \frac{\partial H_1}{\partial x} \qquad \text{with} \qquad \frac{\partial u}{\partial t} = H(u,p),\ p = \frac{\partial u}{\partial x}$$

we find

$$W(u,p) = a(1-b)u^{a-1}|p|^b H(u,p) + b(b-1)u^{a-1} p \int (aH(u,s) + uH_u(u,s))|s|^{b-2} ds.$$

Application of the previous theorem yields the following result.

Theorem. *Let a nonnegative C^1 Hamiltonian be such that there exist parameters a and $b \in \mathbb{R}$, for which and for functions H_0, H_1 and W defined above there holds*

$$W \geq c_0(|H_0|^q + |H_1|^{q_1})$$

for all $u \geq 1$ and $p \in \mathbb{R}$ with some $c_0 > 0$, $q > 1$ and $q_1 > 1$ such that

$$\theta = \frac{q'}{q_1'} + 1 - q' \leq 0.$$

Then there exists no nontrivial global solution of the Cauchy problem (15) with a C^1 initial function $u_0(x) \geq 1$.

Example. Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = u^{\mu}|Du|^{\lambda}, & (x,t) \in \mathbb{R}^2_+, \\ \\ u(0,x) = u_0(x) \geq 1, & x \in \mathbb{R} \end{cases} \tag{17}$$

with $u_0 \in C^1(\mathbb{R})$. We restrict ourselves to the case $\lambda > 1$, $\lambda + \mu > 1$. Then for parameters a and b such that $b > 1$ and $a + b + \lambda + \mu - 1 \leq 0$ there hold all assumptions of the theorem with

$$q = \frac{b+\lambda}{b}, \; q_1 = \frac{b+\lambda}{b+\lambda-1} \text{ and } \theta = \frac{1-b}{\lambda} < 0.$$

Hence there exists no nontrivial entropy solution of the Cauchy problem (17) for all $t > 0$. In particular, there is no nontrivial (i.e., $u \not\equiv$const) C^2_x solution of this problem for all $t > 0$.

A mathematical model of a roof crash

$$\frac{\partial^2 u}{\partial t^2} - \mathrm{div}\frac{Du}{\sqrt{1+|Du|^2}} = -k|u|^{q-1}u + h(x) \quad \text{in } \mathbb{R}^N, \quad k > 0$$

The principal steady state: $u \leq 0$ (the u–axis is directed downwards)

$$-\mathrm{div}\frac{Du}{\sqrt{1+|Du|^2}} = k|u|^q + h(x),\ \ h(x) \equiv H_0 \text{ in } \mathbb{R}^N.$$

Then for $R > R_\infty$ there exists no solution in the ball B_R.

For R_∞, we have the estimate

$$R_\infty \leq R_* = \left(\frac{c_*}{H_0}\right)^{\frac{1}{2q'}},$$

where $q' = \dfrac{q}{q-1}$, $c_* = c_*(N,k)$.

NONLINEAR CAPACITY AND EXISTENCE

Example

$$\begin{cases} \dfrac{du}{dt} = u^2 \\ \\ u\big|_{t=0} = u_0 \end{cases} \Rightarrow u(t) = \frac{u_0}{1 - u_0 t} \Rightarrow \tag{18}$$

$$\Rightarrow \quad u_0 < 0 \quad \exists \text{ global solution}$$

$$\Rightarrow \quad u_0 > 0 \quad \exists \text{ blow-up solution with } T_\infty = \frac{1}{u_0}$$

NONLINEAR CAPACITY APPROACH

$$\int_0^T u^2\varphi = u\varphi\Big|_0^T - \int_0^T u\varphi'$$

$$\varphi(t) \geq 0, \quad \varphi \in C_0^1(\mathbb{R})$$

$$\varphi(T) = 0, \ \ \varphi(0) = 1$$

$$\int_0^T u^2\varphi \leq -u_0 + \left(\int_0^T u^2\varphi\right)^{1/2} \left(\int_0^T \frac{\varphi'^2}{\varphi}\right)^{1/2} \tag{19}$$

Scaling:

$$\varphi(t) = \varphi_0(\tau), \quad \tau = t/T,$$

$$\left.\begin{array}{l} \varphi_0 \geq 0, \qquad \varphi_0(0) = 1, \ \ \varphi_0(1) = 0 \\ \varphi_0 \in C_0^1(\mathbb{R}) \end{array}\right. \qquad \text{(S)}$$

Then $\int\limits_0^T \frac{\varphi'^2}{\varphi}\, dt = \frac{1}{T}\int\limits_0^1 \frac{{\varphi_0'}^2}{\varphi_0}\, d\tau$ and (19) $\Rightarrow$

$$\int\limits_0^T u^2\varphi \leq -u_0 + \frac{C_{\varphi_0}^{1/2}}{\sqrt{T}} \left(\int\limits_0^T u^2\varphi\right)^{1/2} \qquad (20)$$

with $C_{\varphi_0} = \int\limits_0^1 \frac{{\varphi_0'}^2}{\varphi_0}\, d\tau.$

Consider

$$\min\left\{\int_0^1 \frac{{\varphi_0'}^2}{\varphi_0}\,d\tau : \varphi_0 \in S\right\}.$$

After replacement $\varphi_0(\tau) \to \psi_0(\tau)$: $\quad \varphi_0 = \psi_0^2$

we get $\displaystyle\int_0^1 \frac{{\varphi_0'}^2}{\varphi_0}\,d\tau = 4\int_0^1 {\psi_0'}^2\,d\tau.$

Then

$$\min\left\{\int_0^1 {\psi_0'}^2 d\tau : \psi_0 \in S\right\} = 1.$$

Thus $C_{\varphi_0} = 4$ and (20) $\Rightarrow$

$$\begin{aligned} &\int_0^T u^2\varphi \le -u_0 + \frac{2}{\sqrt{T}}\left(\int_0^T u^2\varphi\right)^{1/2} \Rightarrow \\ &\left[\left(\int_0^T u^2\varphi\right)^{1/2} - \frac{1}{\sqrt{T}}\right]^2 \le -u_0 + \frac{1}{T}. \end{aligned} \tag{21}$$

Consequently:

If $u_0 < 0 \Rightarrow -u_0 + 1/T > 0 \Rightarrow \exists$ solution for any $T > 0$.

If $u_0 > 0$ and $T < 1/u_0 \Rightarrow -u_0 + 1/T > 0$, and there is solution for $T < 1/u_0$.

If $u_0 > 0$ and $T \geq 1/u_0$, we have $-u_0 + 1/T \leq 0$ and there is no solution for $T \geq 1/u_0$.

Thus: $$T_\infty = 1/u_0$$

is the same as before from the explicit form of the solution!

In this talk results obtained jointly with Professor Enzo Mitidieri were used.

Further development:

Critical exponents $\rightarrow$

$\rightarrow$ regularity of solutions $\rightarrow$

$\rightarrow$ the 19th Hilbert problem

David Hilbert – Mathematical Problems, 1900

"...the essence of mathematical science is such that each real progress in it comes together with finding stronger techniques and simpler methods that facilitate the understanding of earlier theories and replace complicated old arguments..."