The Fujita Exponent
for
Semilinear Heat Equations
with
Quadratically Decaying Potentials
or
in an Exterior Domain

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Consider the classical semilinear heat equation

$$\begin{aligned} &u_t = \Delta u + u^p \text{ in } R^n \times (0, T); \\ &u(x,0) = \phi(x) \gneq 0 \text{ in } R^n, \end{aligned} \tag{1}$$

where $p > 1$, $n \geq 1$ and $T \in (0, \infty]$.

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- The above result goes back to **Fujita ('66)** in the case $p \neq p^*$. Various proofs of blow-up in the borderline case $p = p^*$ can be found in *Aronson and Weinberg ('76), Kobayashi, Siaro and Tanaka ('77) and RP ('97).*

More recently, **Zhang ('01)** considered the consequences of adding a potential term:

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  i. If $0 \leq V(x) \leq \frac{\omega}{1+|x|^b}$, *with* $b > 2$ *and* $\omega > 0$, *then* $p^* = 1 + \frac{2}{n}$, *and consequently the potential does not affect the critical exponent;*

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- **He also noted that it is unclear whether or not p^* is finite in the case that $V(x) \sim \frac{\omega}{|x|^2}$, with $\omega < 0$.**

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- Let $\alpha = \alpha(\omega, n)$ denote the larger root of the equation
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- Since we are assuming here that $\omega > 0$, one has $\alpha(\omega, n) > 0$ and thus $p^*(\omega) < 1 + \frac{2}{n}$.

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- Ishige's proof involved comparison with a solution to the radially symmetric linear equation $v_t=\Delta v-\hat{V}(|x|)v$, where $\hat{V}(r)\sim\frac{\omega}{r^2}$ as $r\to\infty$. The **asymptotic space-time behavior** of this equation appears in *Ishige and Kabeya (08)*.

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- **What happens when $\omega<0$?**

Our method will require us to consider more generally

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- But now we want to consider (2) with $V(x) \sim \frac{\omega}{|x|^2}$ for large $|x|$ with $\omega < 0$ and also with $\omega > 0$.

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  Let $-\frac{1}{4}(n-2)^2 \le \omega < 0, \ n \ge 3.$
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  - **Remark.** We also have a similar result for $\omega > 0$, with $n \ge 2$.

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- **Theorem 1** *Let* $-\frac{1}{4}(n-2)^2 \le \omega < 0$*,* $n \ge 3$*.*
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  ii. If $V(x) \le \frac{\omega}{|x|^2}$**, for sufficiently large** $|x|$, then there are no global solutions for $1 < p \le p^*(\omega, m)$;
- **Theorem (**Zhang) *iii. If* $\frac{\omega}{1+|x|^b} \le V(x) \le 0$, *for* $b > 2$ *and* $\omega < 0$ **with** $|\omega|$ **sufficiently small**, then $p^* = 1 + \frac{2}{n}$;

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- **What happens when $\omega < -\frac{1}{4}(n-2)^2$, $n \ge 2$?**

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- **First let's see what happens at $\omega = -\frac{1}{4}(n-2)^2$.**
  For $\omega \geq -\frac{(n-2)^2}{4}$:
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- **It turns out that if $\omega < -\frac{(n-2)^2}{4}$, then $p^* = \infty$ and there are no global solutions for any $p > 1$. So there is a discontinuity in $p^*(\omega,m)$ at $\omega = -\frac{(n-2)^2}{4}$!**

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  If there exists a bounded domain $D \subset R^n$ *for which the principal eigenvalue* $\lambda_{0;D}(-\Delta + V) < 0$*, then there are no global solutions to* (6) *for any* $p > 1$*; that is,* $p^* = \infty$*.*

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If inf *spec*$(-\Delta + V; R^n) < 0$, *or equivalently, if there exists a bounded domain* $D \subset R^n$ *for which the principal eigenvalue* $\lambda_{0;D}(-\Delta + V) < 0$, *then there are no global solutions for any* $p > 1$*; that is,* $p^* = \infty$.

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- Corollary

  Let $n \geq 2$*. Let* $V(x) \leq \frac{\omega}{|x|^2}$ *with* $\omega < -\frac{(n-2)^2}{4}$*, for* $|x| > \epsilon$ *and* $\epsilon > 0$ *sufficiently small. Then there are no global solutions to* (7) *for any* $p > 1$*; that is,* $p^* = \infty$*.*

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  - **Theorem (**Zhang) *iii. If* $\frac{\omega}{1+|x|^b} \le V(x) \le 0$, *for* $b > 2$ *and* $\omega < 0$ **with** $|\omega|$ **sufficiently small,** then $p^* = 1 + \frac{2}{n}$;

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  Let $-\frac{(n-2)^2}{4} \leq \omega < 0$*. If* $V(x) \geq \frac{\omega}{|x|^2}$*, then there exist global solutions for* $p > p^*(\omega, m)$*.*

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- (9) **is solvable for** γ **iff** $p > p^*(\omega, m)$.

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- There are no global solution to (10) if and only if there are no global solutions to (11). **So it suffices to study blow up for** (11).

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- So the **solution to** (12) **is dominated by the solution to**

$$\begin{aligned}&w_t = w_{rr} + \frac{N-1}{r}w_r + \hat{a}(r)w^p \text{ in } (r_0,\infty)\times(0,T);\\ &w(r_0,t)=0,\ t>0,\end{aligned}$$

where

$$N\equiv n+2\alpha(\omega,n)\ge 2,$$
$$\hat{a}(r)=r^{\alpha(p-1)}a(r),$$
$$c_1r^M\le\hat{a}(r)\le c_2r^M, \text{ for large } r,$$
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  one has $p^* = 1 + \frac{(2+m)^+}{n}$.
- **So it is reasonable to suspect that no global solution exists for** (13) **if** $p \leq 1 + \frac{(2+M)^+}{N}$.

- $$\begin{aligned} & w_t = w_{rr} + \frac{N-1}{r} w_r + \hat{a}(r) w^p \text{ in } (r_0, \infty) \times (0, T); \\ & w(r_0, t) = 0, \ \ t > 0, \end{aligned} \tag{13}$$

  $$N \equiv n + 2\alpha(\omega, n) \geq 2$$

  $$c_1 r^M \leq \hat{a}(r) \leq c_2 r^M, \ \ M \equiv \alpha(\omega, n)(p-1) + m.$$

- Recall that for

  $$\begin{aligned} & u_t = \Delta u + a u^p \text{ in } R^n \times (0, T); \\ & c_1 |x|^m \leq a(x) \leq c_2 |x|^m, \text{ for large } |x|, \end{aligned}$$

  one has $p^* = 1 + \frac{(2+m)^+}{n}$.
- **So it is reasonable to suspect that no global solution exists for** (13) **if** $p \leq 1 + \frac{(2+M)^+}{N}$.
- Of course, N **is in general fractional** and also, **we have placed the Dirichlet b.c. at** $r = r_0$ **which serves to make solutions smaller.**

- $$\begin{aligned} & w_t = w_{rr} + \frac{N-1}{r} w_r + \hat{a}(r) w^p \text{ in } (r_0, \infty) \times (0, T); \\ & w(r_0, t) = 0, \ \ t > 0, \end{aligned} \tag{14}$$

  $$N \equiv n + 2\alpha(\omega, n) \geq 2$$

  $$\hat{a}(r) = r^{\alpha(p-1)} a(r), \text{ so for large } \ r,$$

  $$c_1 r^M \leq \hat{a}(r) \leq c_2 r^M, \ M \equiv \alpha(\omega, n)(p-1) + m.$$

  It is reasonable to suspect that no global solution exists for (14) **if**

  $$p \leq 1 + \frac{(2+M)^+}{N}. \tag{15}$$

- $$\begin{aligned} & w_t = w_{rr} + \frac{N-1}{r} w_r + \hat{a}(r) w^p \text{ in } (r_0, \infty) \times (0, T); \\ & w(r_0, t) = 0, \ \ t > 0, \end{aligned} \tag{14}$$

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  $$p \leq 1 + \frac{(2+M)^+}{N}. \tag{15}$$

- But (15) is equivalent to

  $$p \leq 1 + \frac{(2+m)^+}{n + \alpha(\omega, n)}.$$

- $$\begin{aligned} &w_t = w_{rr} + \frac{N-1}{r} w_r + \hat{a}(r) w^p \text{ in } (r_0, \infty) \times (0, T); \\ &w(r_0, t) = 0, \;\; t > 0, \end{aligned} \tag{14}$$

  $$N \equiv n + 2\alpha(\omega, n) \geq 2$$

  $$\hat{a}(r) = r^{\alpha(p-1)} a(r), \text{ so for large } \; r,$$

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  It is reasonable to suspect that no global solution exists for (14) **if**

  $$p \leq 1 + \frac{(2+M)^+}{N}. \tag{15}$$

- But (15) is equivalent to

  $$p \leq 1 + \frac{(2+m)^+}{n + \alpha(\omega, n)}.$$

- Or equivalently, $p \leq p^*(\omega, m)$.

- $$\begin{aligned} &u_t = \Delta u + au^p \text{ in } R^n \times (0,T);\\ &c_1|x|^m \le a(x) \le c_2|x|^m, \text{ for large } |x|;\\ &\text{There are no global solutions for } p \le 1 + \frac{(2+m)^+}{n}. \end{aligned} \tag{16}$$

  $$\begin{aligned} &w_t = w_{rr} + \frac{N-1}{r} w_r + \hat{a}(r) w^p \text{ in } (r_0,\infty) \times (0,T);\\ &w(r_0,t) = 0,\ t > 0;\\ &c_1 r^M \le \hat{a}(r) \le c_2 r^M, \text{ for large } r\\ &\textbf{Need to show that there are no global solutions for}\\ &p \le 1 + \frac{(2+M)^+}{N}. \end{aligned} \tag{17}$$

- $$\begin{aligned} & u_t = \Delta u + a u^p \text{ in } R^n \times (0, T); \\ & c_1 |x|^m \le a(x) \le c_2 |x|^m, \text{ for large } |x|; \\ & \text{There are no global solutions for } p \le 1 + \frac{(2+m)^+}{n}. \end{aligned} \tag{16}$$

  $$\begin{aligned} & w_t = w_{rr} + \frac{N-1}{r} w_r + \hat{a}(r) w^p \text{ in } (r_0, \infty) \times (0, T); \\ & w(r_0, t) = 0, \ t > 0; \\ & c_1 r^M \le \hat{a}(r) \le c_2 r^M, \text{ for large } r \\ & \textbf{Need to show that there are no global solutions for} \\ & p \le 1 + \frac{(2+M)^+}{N}. \end{aligned} \tag{17}$$

- The proof of (16) uses heavily the explicit form of the **Gaussian kernel** for the linear part of (16). To use that type of proof for (17) requires use of heat kernel $q_{N,r_0}(t, r, \rho)$ for linear part of (17).

- $$u_t = \Delta u + au^p \text{ in } R^n \times (0,T); \qquad \text{No global solutions for } p \le 1 + \frac{(2+m)^+}{n}. \tag{18}$$

  $$w_t = w_{rr} + \frac{N-1}{r} w_r + \hat{a}(r) w^p \text{ in } (r_0, \infty) \times (0,T);$$
  $$w(r_0, t) = 0, \ t > 0;$$

  Need to show no global solutions for $p \le 1 + \frac{(2+M)^+}{N}.$ (19)

  Proof of (18) uses the explicit form of the **Gaussian kernel** for the linear part of (18).

- $$\begin{gathered} u_t = \Delta u + au^p \text{ in } R^n \times (0,T); \\ \text{No global solutions for } p \le 1 + \frac{(2+m)^+}{n}. \end{gathered} \tag{18}$$

  $$\begin{aligned} & w_t = w_{rr} + \frac{N-1}{r} w_r + \hat{a}(r) w^p \text{ in } (r_0,\infty) \times (0,T); \\ & w(r_0,t) = 0,\ \ t > 0; \\ & \textbf{Need to show no global solutions for } \ p \le 1 + \frac{(2+M)^+}{N}. \end{aligned} \tag{19}$$

  Proof of (18) uses the explicit form of the **Gaussian kernel** for the linear part of (18).
- Let $q_{N,r_0}(t,r,\rho)$ be kernel for linear part of (19).

-
$$u_t = \Delta u + a u^p \text{ in } R^n \times (0, T);$$
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  Proof of (18) uses the explicit form of the **Gaussian kernel** for the linear part of (18).
- Let $q_{N,r_0}(t, r, \rho)$ be kernel for linear part of (19).
- The corresponding heat kernel on **whole space** is

$$q_N(t, r, \rho) \equiv \exp(-\frac{r^2+\rho^2}{4t}) \frac{\rho^{N-1}}{2t(r\rho)^{\frac{N}{2}-1}} I_{\frac{N}{2}-1}(\frac{r\rho}{2t}),$$

where I_ν is the modified Bessel func. of order ν.

- $$u_t = \Delta u + au^p \text{ in } R^n \times (0, T);$$
  $$\text{No global solutions for } p \leq 1 + \frac{(2+m)^+}{n}. \tag{20}$$

$$w_t = w_{rr} + \frac{N-1}{r} w_r + \hat{a}(r) w^p \text{ in } (r_0, \infty) \times (0, T);$$
$$w(r_0, t) = 0, \ \ t > 0;$$

$$\textbf{Need to show no global solutions for } \ p \leq 1 + \frac{(2+M)^+}{N}. \tag{21}$$

Let $q_{N,r_0}(t, r, \rho)$ be kernel for linear part of (21).
The corresponding heat kernel on **whole space** is
$q_N(t, r, \rho) \equiv \exp(-\frac{r^2+\rho^2}{4t}) \frac{\rho^{N-1}}{2t(r\rho)^{\frac{N}{2}-1}} I_{\frac{N}{2}-1}(\frac{r\rho}{2t})$.
If we can use $q_N(t, r, \rho)$ **instead of** $q_{N,r_0}(t, r, \rho)$**, then we can prove** (21) **via method of** (20)**.**

- $$u_t = \Delta u + au^p \text{ in } R^n \times (0,T);$$
  $$\text{No global solutions for } p \le 1 + \frac{(2+m)^+}{n}. \tag{20}$$
  $$w_t = w_{rr} + \frac{N-1}{r} w_r + \hat{a}(r) w^p \text{ in } (r_0,\infty) \times (0,T);$$
  $$w(r_0,t) = 0, \ t > 0;$$
  $$\textbf{Need to show no global solutions for } \ p \le 1 + \frac{(2+M)^+}{N}. \tag{21}$$
  Let $q_{N,r_0}(t,r,\rho)$ be kernel for linear part of (21).
  The corresponding heat kernel on **whole space** is
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  If we can use $q_N(t,r,\rho)$ **instead of** $q_{N,r_0}(t,r,\rho)$**, then we can prove** (21) **via method of** (20)**.**
- **Need:**
  $q_{N,r_0}(t,r,\rho) \ge c q_N(Kt,r,\rho), \ r,\rho \ge r_0 + 1,$ **for some** $K > 0$.

- $$\begin{aligned} & w_t = w_{rr} + \frac{N-1}{r} w_r \text{ in } (r_0, \infty) \times (0, T); \\ & w(r_0, t) = 0, \ \ t > 0, \\ & \qquad\qquad N \equiv n + 2\alpha(\omega, n) \end{aligned} \tag{22}$$

  $q_{N,r_0}(t, r, \rho)$ is heat kernel for (22).

- $$\begin{aligned} &w_t = w_{rr} + \frac{N-1}{r} w_r \text{ in } (r_0, \infty) \times (0, T); \\ &w(r_0, t) = 0, \ \ t > 0, \\ &\qquad\qquad N \equiv n + 2\alpha(\omega, n) \end{aligned} \tag{22}$$

  $q_{N,r_0}(t, r, \rho)$ is heat kernel for (22).
- The corresponding heat kernel on whole space is

  $$q_N(t, r, \rho) \equiv \exp(-\frac{r^2 + \rho^2}{4t}) \frac{\rho^{N-1}}{2t(r\rho)^{\frac{N}{2}-1}} I_{\frac{N}{2}-1}(\frac{r\rho}{2t}),$$

- $$\begin{aligned} & w_t = w_{rr} + \frac{N-1}{r} w_r \text{ in } (r_0,\infty)\times(0,T); \\ & w(r_0,t) = 0, \ \ t>0, \\ & \qquad\qquad N \equiv n + 2\alpha(\omega,n) \end{aligned} \tag{22}$$

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- Recall: $\alpha(\omega,n) = \frac{2-n+\sqrt{(n-2)^2+4\omega}}{2}$, $\omega \ge -\frac{(n-2)^2}{4}$.

  So $N > 2$ unless $\omega = -\frac{(n-2)^2}{4}$.

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  $$N \equiv n + 2\alpha(\omega, n)$$
  $q_{N,r_0}(t, r, \rho)$ is heat kernel for (22).
- The corresponding heat kernel on whole space is
  $$q_N(t, r, \rho) \equiv \exp\left(-\frac{r^2 + \rho^2}{4t}\right) \frac{\rho^{N-1}}{2t(r\rho)^{\frac{N}{2}-1}} I_{\frac{N}{2}-1}\left(\frac{r\rho}{2t}\right),$$
- Recall: $\alpha(\omega, n) = \frac{2-n+\sqrt{(n-2)^2+4\omega}}{2}$, $\omega \geq -\frac{(n-2)^2}{4}$.
  So $N > 2$ unless $\omega = -\frac{(n-2)^2}{4}$.
- For $N > 2$, the Markov diffusion process corresponding to the generator $\frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr}$ is **transient.** Using this and the **parabolic Harnack inequality, Grigor'yan and Saloff-Coste ('02)** proved that
  $$q_{N,r_0}(t, r, \rho) \geq c q_N(Kt, r, \rho), \text{ for } r, \rho > r_0+1, \text{ for some } K > 0.$$

- **Fujita Exponent in an Exterior Domain**

  Let $B_1 = \{x \in R^n : |x| < 1\}$,

$$\begin{aligned}
&u_t = \Delta u + a u^p \text{ in } (R^n - \bar{B}_1) \times (0, T);\\
&u(x,t) = 0,\ |x| = 1,\ t \geq 0;\\
&u(x,0) = \phi(x) \gneq 0,\ \text{in } R^n - \bar{B}_1,\\
&\text{where } \ c_1|x|^m \leq a(x) \leq c_2|x|^m,\ \text{for large } |x|.
\end{aligned} \tag{23}$$

$$\text{Let } p^* = 1 + \frac{(2+m)^+}{n}$$

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$$\text{Let } p^* = 1 + \frac{(2+m)^+}{n}$$

- Theorem
  Let $n \geq 2$.
  i. If $1 \leq p \leq p^*$, then there exist global solutions to (23);
  ii. If $p > p^*$, then there are no global solutions to (23).

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- Theorem

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  - In the case $a \equiv 1$, $n \geq 3$ and $p \neq p^*$, this was proved by **Bandle and Levine ('89)**.

- **Fujita Exponent in an Exterior Domain**
  Let $B_1 = \{x \in R^n : |x| < 1\}$,

$$\begin{aligned} & u_t = \Delta u + a u^p \text{ in } (R^n - \bar{B}_1) \times (0, T); \\ & u(x,t) = 0, \ |x| = 1, \ t \ge 0; \\ & u(x,0) = \phi(x) \gneq 0, \text{ in } R^n - \bar{B}_1, \\ & \text{where } \ c_1|x|^m \le a(x) \le c_2|x|^m, \text{ for large } |x|. \end{aligned} \tag{23}$$

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- Theorem
  Let $n \ge 2$.
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  - In the case $a \equiv 1$, $n \ge 3$ and $p \ne p^*$, this was proved by **Bandle and Levine ('89)**.
  - For proof of the theorem , we use estimates on heat kernel for $u_t = \Delta u$ in an exterior domain, obtained by **Grigor'yan and Saloff-Coste ('02).**