Singularities and phases of circle-valued Sobolev maps

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Nonlinear PDE and BVP with Measure Data
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Marie-François Bidaut-Véron and Laurent Véron

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Thus, there should be no possible comparison between the norm of sums of Dirac masses acting on $C_b^{1/2}$ and $W^{1,3}$

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Theorem

Let $m \in \mathbb{N}^*$, $P_1, \ldots, P_m, N_1, \ldots, N_m \in \mathbb{S}^2$ *and set*

$$T := \sum_{j=1}^{m} (\delta_{P_j} - \delta_{N_j})$$

Then (with C an absolute constant)

$$\|T\|_{(\dot{C}^{1/2})^*} \leq C \|T\|^{3/2}_{(\dot{W}^{1,3})^*}$$

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Let $2 < q < \infty$. Then (with $C = C(q)$ an absolute constant)

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Theorem from previous slide is the above theorem for $q = 3$

The origin of this inequality

Question: describe all maps $u : \mathbb{S}^n \to \mathbb{S}^1$ having a given Sobolev regularity

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Motivations

- Ginzburg-Landau equation
- Lifting problem in covering spaces

In order to simplify, we let $n = 2$
Thus we take maps $u : \mathbb{S}^2 \to \mathbb{S}^1$

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First guess...

...is that

$$\{u : \mathbb{S}^n \to \mathbb{S}^1 \ ; \ u \in W^{s,p}\} = \{e^{\imath\varphi} \ ; \ \varphi : \mathbb{S}^n \to \mathbb{R}, \varphi \in W^{s,p}\}$$

Theorem (Bourgain, Brezis, M., '00)

Equality

$$\{u : \mathbb{S}^2 \to \mathbb{S}^1 \; ; \; u \in W^{s,p}\} = \{e^{\imath\varphi} \; ; \; \varphi : \mathbb{S}^2 \to \mathbb{R}, \varphi \in W^{s,p}\}$$

holds precisely when $sp < 1$ *or* $sp \geq 2$

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  Regularizations of u are almost circle-valued
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Ideas

- $sp > 2$: maps are continuous
- $sp = 2$: maps are VMO
  Regularizations of u are almost circle-valued
  (Schoen-Uhlenbeck)
- $sp < 1$: do not start with smooth maps

It remains to describe $W^{s,p}(\mathbb{S}^2;\mathbb{S}^1)$ when the equality

$$\{u : \mathbb{S}^2 \to \mathbb{S}^1 \; ; \; u \in W^{s,p}\} = \{e^{\imath\varphi} \; ; \; \varphi : \mathbb{S}^2 \to \mathbb{R}, \varphi \in W^{s,p}\}$$

does not hold, i. e. when $1 \leq sp < 2$

Example of $u \in W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$ $(0 < s < 1, 1 < sp < 2)$

Example #1

$v = e^{\imath\varphi}$, $\varphi \in W^{s,p}$

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Example #2

$w = e^{\imath\psi}$, $\psi \in W^{1,sp}$

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Example #2 relies on the Gagliardo-Nirenberg embeddings

$$W^{\sigma,\rho} \cap L^\infty \subset W^{\theta\sigma,\rho/\theta},\ \theta \in (0,1)$$

except when $\sigma = \rho = 1$
Example: a map which is both in $W^{1,2}$ and bounded is in $W^{1/2,4}$ and in $W^{1/3,6}$

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Thus:

$$\psi \in W^{1,sp} \implies e^{\imath\psi} \in W^{1,sp} \cap L^\infty \implies e^{\imath\psi} \in W^{s,p}$$

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$$\xi(z) = \frac{z}{|z|}$$

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$\xi(z) = \dfrac{z}{|z|}$

Heuristics

ξ homogeneous of degree 0 $\Longrightarrow$ $D^s\xi$ homogeneous of degree $-s \Longrightarrow D^s\xi \in L^p$ (since $sp < 2$=space dimension)

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Assume that

- space dimension $n = 2$
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"Then the above examples are the only ones"

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We still assume dimension $n = 2$

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$$\langle T(u), \zeta \rangle := \frac{1}{2\pi} \int (u \wedge du) \wedge d\zeta$$

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(Brezis, Coron, Lieb)

Theorem (All maps have a singular set; Bourgain, Brezis, M.; Bousquet)

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Then

- *Maps not too ugly are dense in* $W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$
- *The map* $u \mapsto T(u)$ *extends by continuity from not too ugly maps to all maps in* $W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$
- *The result is of the form*

$$T(u) = \sum_{j=1}^{\infty} (\delta_{P_j} - \delta_{N_j})$$

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Assume that $u \in W^{q,1}$
Then $T(u)$ acts on $W^{1,q'} \cap C^{2-q}$

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Previous conditions are necessary and sufficient
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$$T = \sum_{j=1}^{\infty} (\delta_{P_j} - \delta_{N_j})$$

may be written as

- $T = T(u)$ *for some* $u \in W^{1,q}$ *iff it acts on* $W^{1,q'}$
- $T = T(u)$ *for some* $u \in W^{q,1}$ *iff it acts on* $W^{1,q'} \cap C^{2-q}$

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Theorem (M.)

For $T = \sum_{j=1}^{\infty}(\delta_{P_j} - \delta_{N_j})$*, acting on* $W^{1,q'}$ *is the same as acting on* $W^{1,q'} \cap C^{2-q}$

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More generally,

$$\{T(u)\ ;\ u \in W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)\}$$

depends only on the value of the product sp, not on s or p

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The above theorem implies the strange property of sums of Dirac masses

Range of $u \mapsto T(u)$?

The previous theorem relies on

Theorem (M.)

Assume that $1 \leq sp < 2$
Then each $u \in W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$ *may be written as*

$$u = ve^{\imath\varphi}, \ v \in W^{sp,1}, \ \varphi \in W^{s,p} + W^{1,sp}$$

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Special cases due to Bourgain-Brezis, Nguyen

Analysis of $W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$

Theorem (Splitting maps into examples $\#1$ to $\#3$; M.)

Assume that

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- *Compute $T(u)$*
- *Associate to $T(u)$ a "canonical map" ξ such that $T(\xi) = T(u)$*

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- *Compute* $T(u)$
- *Associate to* $T(u)$ *a "canonical map"* ξ *such that* $T(\xi) = T(u)$
- *Find explicitly* $\varphi \in W^{s,p}$, $\psi \in W^{1,sp}$ *such that*

$$u = v\,w\,\xi, \quad \text{with } v = e^{\imath\varphi},\ w = e^{\imath\psi}$$

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Thinks go also the other way around...

Generalizations

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Structure of $W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$ for the other values of s and p

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Points become $(n-2)$-manifolds

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Very little known

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Very little known
Expected result:
the set

$$\{T(u)\ ;\ u\in W^{s,p}(\mathbb{S}^{n+1};\mathbb{S}^n)\}$$

depends only on the value of the product sp, not on s or p

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Thank you for your attention!