Higher order elliptic boundary value problems in non-smooth domains

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Statement of the problem

Ω is an arbitrary bounded domain in $\mathbb{R}^n$, $f \in C_0^\infty(\Omega)$.

We say that u is a variational solution of the Dirichlet problem if

$$(-\Delta)^m u = f, \qquad u \in \mathring{W}^{m,2}(\Omega)$$

where $\mathring{W}^{m,2}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W^{m,2}(\Omega)} = \Big(\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)} \Big)^{1/2}.$$

How smooth is the solution?

Sobolev: $W^{m,2} \hookrightarrow C^k$ for $k < m - \frac{n}{2}$, $2m > n$

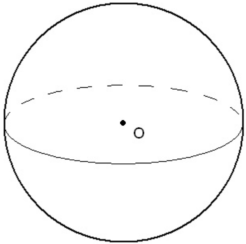
S.M., V.M., 2006-2009

- Polyharmonic equation ($\Delta^m u = f$), $\Omega \subset \mathbb{R}^3$ arbitrary:
  $\nabla^{m-1} u \in L^\infty(\Omega)$ /sharp/
  Biharmonic equation: $\nabla u \in L^\infty(\Omega)$ /sharp/
- Pointwise estimates on the Green function /sharp/
- Polyharmonic equation ($\Delta^m u = f$), $\Omega \subset \mathbb{R}^n$ arbitrary:
  all k such that $\nabla^k u \in L^\infty(\Omega)$ /sharp/
- Necessary and sufficient conditions for $u \in C^{m-1}(\overline{\Omega})$, $\Omega \in \mathbb{R}^3$, polyharmonic capacity
- Biharmonic equation, $\Omega \subset \mathbb{R}^n$ convex
  $\nabla^2 u \in L^\infty(\Omega)$

Arbitrary domain

Question 1: Let Ω be an arbitrary domain in $\mathbb{R}^n$.
Is $|\nabla^{m-1}u|$ bounded in Ω?

- $n \geq 4$ NO



Case I: $m \geq 3$, $\Omega = B(O,1) \setminus \{O\} \subset \mathbb{R}^4$

$$u(x) := \eta(x) \sum_{|\alpha|=m-3} c_\alpha D^\alpha \left(|x|^{2m-4} \log |x|\right)$$

where $\eta \in C_0^\infty(B(O,1/2))$
and $\eta = 1$ in $B(O,1/4)$.

Then u satisfies
$(-\Delta)^m u = f \in C_0^\infty(\Omega)$, $u \in \mathring{W}^{m,2}(\Omega)$.

But $|\nabla^{m-1}u| \sim \log |x| \notin L^\infty(\Omega)$

Arbitrary domain

Question 1: Let Ω be an arbitrary domain in $\mathbb{R}^n$.
Is $|\nabla^{m-1} u|$ bounded in Ω?

- $n \geq 4$ NO
  Case II: $m = 2$
  $K_1 := \{x \in \mathbb{R}^4 : \ x_4 > |x| \cos \alpha/2\}$, $\alpha \in [\pi, 2\pi)$
  (circular cone)
  Then $|x| \, (c_1 \cos\theta + c_2 \, \theta / \sin\theta) \, , \quad \cos\theta = x_4/|x|,$
  is a solution of $\Delta^2 u = 0$ in K_1. It satisfies zero Dirichlet data iff $\tan\alpha = \alpha$ ($\alpha \approx 1.4303\pi$). That is, the 1st eigenvalue of the corresponding operator pencil $\lambda_1 = 1$.
  Then for $K \supset K_1$ we have $\lambda_1 < 1$, $u \sim |x|^{\lambda_1} \Longrightarrow \nabla u \notin L^\infty(K)$
  $\Longrightarrow U := \eta u$, η as before, is a solution of $\Delta^2 U = f \in C_0^\infty(\Omega)$ in $\Omega = K \cap B(O,1)$, $U \in \mathring{W}^{2,2}(\Omega)$, while $\nabla U \notin L^\infty(\Omega)$

Arbitrary domain

Question 1: Let Ω be an arbitrary domain in $\mathbb{R}^n$.
Is $|\nabla^{m-1}u|$ bounded in Ω?

- $n \geq 4$ NO
- $n = 2, 3$ YES

Theorem

Let Ω be an arbitrary domain in $\mathbb{R}^3$ or $\mathbb{R}^2$ and u be a solution of the Dirichlet problem for the polyharmonic equation with $f \in C_0^\infty(\Omega)$. Then

$$|\nabla^{m-1}u| \in L^\infty(\Omega).$$

We gain one extra order of smoothness in comparison with the result of Sobolev embedding theorem.

Green function on an arbitrary 3-dim domain

Theorem

Let Ω be an arbitrary domain in $\mathbb{R}^3$. Then for every $x, y \in \Omega$

$$\left|\nabla_x^{m-1}\nabla_y^{m-1}(G(x,y) - \Gamma(x-y))\right| \leq \frac{C}{\max\{|x-y|,\, d(x),\, d(y)\}},$$

where $\Gamma(x-y) = C_m|x-y|^{2m-3}$ is the fundamental solution for $(-\Delta)^m$, $d(x)$ is the distance from x to the boundary. In particular,

$$|\nabla_x^{m-1}\nabla_y^{m-1}G(x,y)| \leq C|x-y|^{-1} \qquad \forall\, x, y \in \Omega,$$

where G is the Green function for biharmonic equation and C is an absolute constant.

Dirichlet problem

Theorem

Let Ω be an arbitrary domain in $\mathbb{R}^3$ and

$$\Delta^m u = \sum_{|\alpha|=m-1} D^\alpha f_\alpha, \qquad u \in \mathring{W}^{m,2}(\Omega).$$

Then

$$|\nabla^{m-1} u(x)| \le C \sum_{|\alpha|=m-1} \int_\Omega \frac{|f_\alpha(y)|}{|x-y|}\, dy, \quad x \in \Omega$$

Corollary

$$\|\nabla^{m-1} u\|_{L^\infty(\Omega)} \le C \sum_{|\alpha|=m-1} \|f_\alpha\|_{L^p(\Omega)}, \qquad p > 3/2$$

Regularity in higher dimensions

Question 2: What happens in higher dimensions?

Theorem

Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 3$, and u be a solution of the Dirichlet problem for the polyharmonic equation with $f \in C_0^\infty(\Omega)$. Then

$$|\nabla^k u| \in L^\infty(\Omega),$$

for $k = m - \frac{n}{2} + \frac{1}{2}$ when n is odd and $k = m - \frac{n}{2}$ when n is even.

Green function estimates, n odd

The main estimate:

$$|\nabla_x^{m-\frac{n}{2}+\frac{1}{2}} \nabla_y^{m-\frac{n}{2}+\frac{1}{2}} G(x,y)| \leq C\,|x-y|^{-1}$$

Lower order estimates: if $0 \leq i, j \leq m - \frac{n}{2} + \frac{1}{2}$, $i + j \leq 2m - n$,

$$|\nabla_x^i \nabla_y^j G(x,y)| \leq \quad C \min\left\{1, \left(\frac{d(x)}{|x-y|}\right)^{m-\frac{n}{2}+\frac{1}{2}-i}, \left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}+\frac{1}{2}-j}\right\} \frac{1}{|x-y|^{n-2m+i+j}}$$

$$\times \max\left\{\frac{d(x)}{|x-y|}, \frac{d(y)}{|x-y|}, 1\right\}^{2m-n-i-j}$$

Green function estimates, n odd

In particular,

$$|G(x,y)| \le C d(x)^{m-\frac{n}{2}+\frac{1}{2}} |x-y|^{m-\frac{n}{2}-\frac{1}{2}},$$

when $d(x) \le d(y) \le |x-y|$;

$$|G(x,y)| \le C\, d(x)^{m-\frac{n}{2}+\frac{1}{2}} \frac{d(y)^{2m-n}}{|x-y|^{m-\frac{n}{2}+\frac{1}{2}}},$$

when $d(x) \le |x-y| \le d(y)$;

$$|G(x,y)| \le C\, d(y)^{2m-n},$$

when $|x-y| \le d(x) \le d(y)$;

the remaining cases can be recovered from symmetry.

Green function estimates, n even

The main estimate:

$$|\nabla_x^{m-\frac{n}{2}}\nabla_y^{m-\frac{n}{2}}G(x,y)| \leq C\,\log\left(1+\frac{\min\{d(x),d(y)\}}{|x-y|}\right)$$

Lower order estimates: if $0 \leq i,j \leq m-\frac{n}{2}$

$$|\nabla_x^i\nabla_y^j G(x,y)| \leq \quad C\min\left\{1,\left(\frac{d(x)}{|x-y|}\right)^{m-\frac{n}{2}-i},\left(\frac{d(y)}{|x-y|}\right)^{m-\frac{n}{2}-j}\right\}\times$$
$$\times\frac{1}{|x-y|^{n-2m+i+j}}\max\left\{\frac{d(x)}{|x-y|},\frac{d(y)}{|x-y|},1\right\}^{2m-n-i-j}\times$$
$$\times\log\left(1+\frac{\min\{d(x),d(y)\}}{|x-y|}\right),$$

Green function estimates, n even

In particular,

$$|G(x,y)| \le C d(x)^{m-\frac{n}{2}} |x-y|^{m-\frac{n}{2}} \log\left(1 + \frac{d(x)}{|x-y|}\right),$$

when $d(x) \le d(y) \le |x-y|$;

$$|G(x,y)| \le C\, d(x)^{m-\frac{n}{2}} \frac{d(y)^{2m-n}}{|x-y|^{m-\frac{n}{2}}} \log\left(1 + \frac{d(x)}{|x-y|}\right),$$

when $d(x) \le |x-y| \le d(y)$;

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when $|x-y| \le d(x) \le d(y)$;

the remaining cases can be recovered from symmetry.

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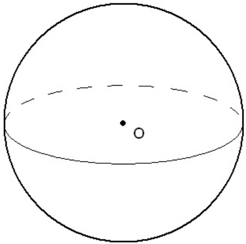
Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 3$, and u be a solution of the Dirichlet problem for the polyharmonic equation with $f \in C_0^\infty(\Omega)$. Then

$$|\nabla^k u| \in L^\infty(\Omega),$$

for $k = m - \frac{n}{2} + \frac{1}{2}$ when n is odd
and $k = m - \frac{n}{2}$ when n is even.

This result is sharp: $\exists\, \Omega \subset \mathbb{R}^n$, $f \in C_0^\infty(\Omega)$ such that $|\nabla^k u| \notin L^\infty(\Omega)$, for $k = m - \frac{n}{2} + \frac{1}{2} + 1$ when n is odd and $k = m - \frac{n}{2} + 1$ when n is even.

Continuity of $\nabla^{m-1} u$



$$\Omega = B(O,1) \setminus \{O\} \subset \mathbb{R}^3$$

Consider

$$u(x) := \eta(x) \sum_{|\alpha|=m-2} c_\alpha D^\alpha |x|^{2m-3}$$

where $\eta \in C_0^\infty(B(O,1/2))$
and $\eta = 1$ in $B(O,1/4)$.

Then u satisfies
$(-\Delta)^m u = f \in C_0^\infty(\Omega)$, $u \in \mathring{W}^{m,2}(\Omega)$.

The assertion $\nabla^{m-1} u \in L^\infty(\Omega)$ is sharp
in the sense that it cannot be replaced by $\nabla^{m-1} u \in C(\overline{\Omega})$.

Question 2: Conditions on Ω ensuring the continuity of $\nabla^{m-1} u$ at a boundary point?

$C^{0,\omega}$ domains

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a $C^{0,\omega}$ domain and $m \geq 2$. If

$$\int_0^1 \frac{t\,dt}{\omega^2(t)} = \infty$$

then every solution to the polyharmonic equation satisfies $\nabla^{m-1} u \in C(\overline{\Omega})$.

Conversely, for every ω such that

$$\int_0^1 \frac{t\,dt}{\omega^2(t)} < \infty$$

there exists a $C^{0,\omega}$ domain and a solution u of the polyharmonic equation such that $\nabla^{m-1} u \notin C(\overline{\Omega})$.

Wiener regularity for $-\Delta$

Let Ω be a bounded domain in $\mathbb{R}^n$,

$$-\Delta u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega), \quad u \in \mathring{W}^{1,2}(\Omega).$$

Definition

We say that a point $O \in \partial\Omega$ is regular if $u(x) \to 0$ as $x \to O$ for any $f \in C_0^\infty(\Omega)$.

Theorem (Wiener, 1924)

The point $O \in \partial\Omega$ is regular if and only if

$$\int_0^1 cap(B_\sigma \setminus \Omega)\sigma^{1-n}\, d\sigma = \infty,$$

where

$$cap(K) = \inf\left\{ \int_{\mathbb{R}^n} |\nabla u|^2\, dx : \ u \in C_0^\infty(\mathbb{R}^n), u > 1 \ \ on \ \ K \right\}$$

is the Wiener (harmonic) capacity of a set K $(n > 2)$.

Regularity with respect to $(-\Delta)^m$

Let Ω be a bounded domain in $\mathbb{R}^n$,

$$(-\Delta)^m u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega), \quad u \in \mathring{W}^{m,2}(\Omega).$$

A point $O \in \partial\Omega$ is regular with respect to $(-\Delta)^m$ if $u(x) \to 0$ as $x \to O$ for any $f \in C_0^\infty(\Omega)$.

Theorem (V.M., 2002)

Let $m = 2$ and $n = 4, 5, 6, 7$,
or $m \geq 3$ and $n = 2m, 2m+1, 2m+2$.
The point $O \in \partial\Omega$ is regular with respect to $(-\Delta)^m$ if and only if

$$\int_0^1 cap(B_\sigma \setminus \Omega)\sigma^{2m-n-1}\, d\sigma = \infty,$$

where

$$cap(K) = \inf\left\{\int_{\mathbb{R}^n} |\nabla^m u|^2\, dx : \ u \in C_0^\infty(\mathbb{R}^n), u > 1 \ \ on \ \ K\right\}, \ \ n > 2m.$$

New polyharmonic capacity

Denote by Y_ℓ^k the spherical harmonics of degree ℓ, $\ell \geq 0$, $|k| \leq \ell$. Let Π denote the space of functions

$$P(x) = \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} b_\ell^k Y_\ell^k(x), \qquad b_\ell^k \in \mathbb{R},$$

equipped with the norm $\|P\|_\Pi = \sqrt{\sum \left(b_\ell^k\right)^2}$.
Then for a compactum $K \subset \mathbb{R}^3 \setminus \{O\}$ and $P \in \Pi$ we define the polyharmonic capacity of K by

$$\mathrm{Cap}_{m,P}(K) = \inf \|\nabla^m u\|^2_{L^2(\mathbb{R}^3)}$$

where the infimum is taken over $u \in \mathring{W}^{m,2}(\mathbb{R}^3 \setminus \{O\})$ such that $u = P$ in a neighborhood of K.

Continuity of $\nabla^{m-1} u$

Let Ω be a bounded domain in $\mathbb{R}^3$, $O \in \partial\Omega$ and

$$(-\Delta)^m u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega), \quad u \in \mathring{W}^{m,2}(\Omega).$$

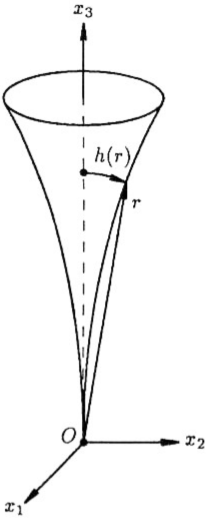
We say that $O \in \partial\Omega$ is $(m-1)$-regular with respect to $(-\Delta)^m$ if $\nabla^{m-1} u(x) \to 0$ as $x \to O$, $x \in \Omega$.

Theorem (S. Mayboroda, V.M.)

- $$\int_0^1 \inf_{P \in \Pi:\, \|P\|=1} \mathrm{Cap}_{m,P}\left(\overline{C_s} \setminus \Omega\right) ds = +\infty \implies (m-1)\text{-regularity}$$
- $$\inf_{P \in \Pi:\, \|P\|=1} \int_0^1 \mathrm{Cap}_{m,P}\left(\overline{C_s} \setminus \Omega\right) ds = +\infty \impliedby (m-1)\text{-regularity}$$

Here $C_s = \{x \in \mathbb{R}^n : s < |x| < 2s\}$.

Exterior of a cusp



$\Omega \cap B(O,1) = \{x \in B(O,1) : \theta > h(r)\}$

h is nondecreasing, $h(2r) \leq ch(r)$

(Ω is exterior of a cusp in a neighborhood of O)

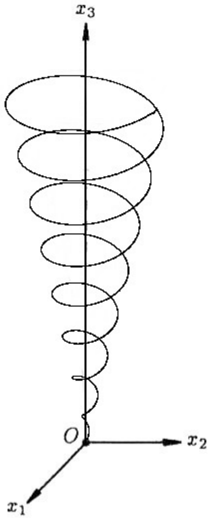
O is $m-1$-regular if and only if

$$\int_0^1 h(s)^2 \frac{ds}{s} = +\infty$$

Compare to Wiener condition for $-\Delta$: the point O is regular if and only if

$$\int_0^1 |\log h(s)|^{-1} \frac{ds}{s} = +\infty$$

Example

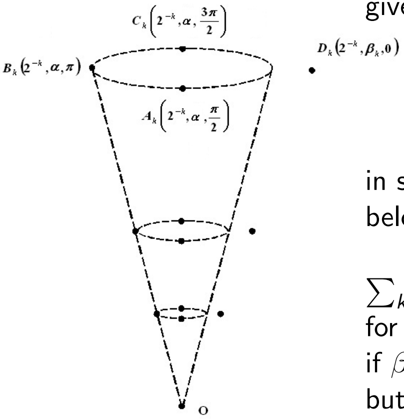


The complement of Ω is a compactum of zero harmonic capacity situated on a circular cone

O is not 1-regular with respect to Δ^2

(e.g. consider the solution $u(x) = |x| \cos\alpha - x_3$ on the cone $\{x : x_3 = |x| \cos\alpha\}$)

Instability of irregularity for Δ^2



The complement of Ω is a compactum given by the points

$$A_k = (2^{-k}, \alpha, \pi/2), \quad B_k = (2^{-k}, \alpha, \pi)$$
$$C_k = (2^{-k}, \alpha, 3\pi/2), \quad D_k = (2^{-k}, \beta_k, 0)$$

in spherical coordinates. They do NOT belong to a common circular cone.

$\sum_k (\beta_k - \alpha)^2 = +\infty \implies O$ is 1-regular,
for example,
if $\beta_k = \beta \neq \alpha \implies O$ is 1-regular
but if $\beta_k = \alpha \implies O$ is not 1-regular.

Therefore, irregularity is not stable under affine transformations

Convex domain

Let Ω be an arbitrary convex domain in $\mathbb{R}^n$, $f \in C_0^\infty(\Omega)$ and $u \in \mathring{W}^{2,2}(\Omega)$ be a solution of the Dirichlet problem for the biharmonic equation. Let $\nabla^2 u$ denote the vector of all second derivatives.

Question 4: Is $|\nabla^2 u|$ bounded in Ω?

- $n = 2$ YES V. Kozlov, V.M., 2004
- $n \geq 3$?

Theorem

Let Ω be a convex domain in $\mathbb{R}^n$ and u be a solution of the Dirichlet problem for the biharmonic equation with $f \in C_0^\infty(\Omega)$. Then

$$|\nabla^2 u| \in L^\infty(\Omega).$$

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Theorem

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$$|\nabla^2 u| \in L^\infty(\Omega).$$

The case of Δ^2 in $\Omega \subset \mathbb{R}^3$: main identity

Let Ω be an arbitrary domain in $\mathbb{R}^3$, $u \in C_0^\infty(\Omega)$ and $v = e^t(u \circ \varkappa^{-1})$. Then for every function g

$$\int_{\mathbb{R}^3} \Delta u(x) \Delta\Big(u(x)|x|^{-1} g(\log|x|^{-1})\Big)\, dx$$

$$= \int_{\mathbb{R}} \int_{S^2} \Big[(\delta_\omega v)^2 g + 2(\partial_t \nabla_\omega v)^2 g + (\partial_t^2 v)^2 g$$

$$-(\nabla_\omega v)^2 \Big(\partial_t^2 g + \partial_t g + 2g\Big) - (\partial_t v)^2 \Big(2\partial_t^2 g + 3\partial_t g - g\Big)$$

$$+\frac{1}{2} v^2 \Big(\partial_t^4 g + 2\partial_t^3 g - \partial_t^2 g - 2\partial_t g\Big)\Big]\, d\omega dt.$$

The case of Δ^2 in $\Omega \subset \mathbb{R}^3$: the choice of weight

Lemma

A bounded solution of the equation

$$\frac{d^4 g}{dt^4} + 2\frac{d^3 g}{dt^3} - \frac{d^2 g}{dt^2} - 2\frac{dg}{dt} = \delta(t),$$

subject to the restriction $g(t) \to 0$ *as* $t \to +\infty$, *is the function*

$$g(t) = -\frac{1}{6}\begin{cases} e^t - 3, & t < 0, \\ e^{-2t} - 3\,e^{-t}, & t > 0. \end{cases}$$

Lemma

For Ω, u, v, g *as above and* $\xi \in \Omega$, $\tau = \log|\xi|^{-1}$

$$\frac{1}{2}\int_{S^{n-1}} v^2(\tau,\omega)\,d\omega \le \int_{\mathbb{R}^n} \Delta u(x)\Delta\Big(u(x)|x|^{-1}g(\log(|\xi|/|x|))\Big)\,dx,$$

The case of Δ^2 in $\Omega \subset \mathbb{R}^3$: local estimates

Theorem

Let Ω be an arbitrary bounded domain in $\mathbb{R}^3$, $O \in \partial\Omega$, $R \in (0, \frac{1}{4}\operatorname{diam}\Omega)$. Let

$$\Delta^2 u = f \text{ in } \Omega, \quad f \in C_0^\infty(\Omega \setminus B_{4R}), \quad u \in \mathring{W}_2^2(\Omega).$$

Then

$$|\nabla u(x)| \leq \frac{C}{R} \left(\fint_{(B_R \setminus B_{R/4}) \cap \Omega} |u(y)|^2 \, dy \right)^{1/2} \quad \text{for every} \quad x \in B_{R/8}.$$

Let Ω be an arbitrary bounded domain in $\mathbb{R}^n$, $O \in \mathbb{R}^n \setminus \Omega$ and $u \in C^2(\bar{\Omega})$, $\quad u\Big|_{\partial\Omega} = 0, \quad \nabla u\Big|_{\partial\Omega} = 0, \quad v = e^{2t}(u \circ \varkappa^{-1}),$ where $\mathbb{R}^n \ni x \stackrel{\varkappa}{\longrightarrow} (t, \omega) \in \mathbb{R} \times S^{n-1}$.

Then for every function g

$$\int_\Omega \Delta u(x) \Delta \left(\frac{u(x) g(\log |x|^{-1})}{|x|^n} \right) dx$$

$$= \int_{\varkappa(\Omega)} \Big[(\delta_\omega v)^2 g + 2(\partial_t \nabla_\omega v)^2 g + (\partial_t^2 v)^2 g$$

$$-(\nabla_\omega v)^2 \left(\partial_t^2 g + n\,\partial_t g + 2n\,g \right) - (\partial_t v)^2 \left(2\partial_t^2 g + 3n\,\partial_t g + \left(n^2 + 2n - 4\right) g \right)$$

$$+ \frac{1}{2} v^2 \left(\partial_t^4 g + 2n\,\partial_t^3 g + \left(n^2 + 2n - 4\right) \partial_t^2 g + 2n(n-2)\,\partial_t g \right) \Big] \, d\omega dt.$$

Convex domains: additional estimates

For every $v \in \mathring{W}_2^2(S_+^{n-1})$,

$$\frac{1}{2n}\int_{S_+^{n-1}} |\delta_\omega v|^2\, d\omega \geq \int_{S_+^{n-1}} |\nabla_\omega v|^2\, d\omega \geq (n-1)\int_{S_+^{n-1}} |v|^2\, d\omega.$$

Therefore, for any convex domain Ω and any function $g \geq 0$ with $\partial_t^2 g + n\partial_t g \leq 0$

$$\int_\Omega \Delta u(x)\Delta\left(\frac{u(x)g(\log|x|^{-1})}{|x|^n}\right)dx$$

$$\geq \int_{\varkappa(\Omega)}\Big[-(\partial_t v)^2\Big(2\partial_t^2 g + 3n\,\partial_t g + \left(n^2-2\right) g\Big)$$

$$+\frac{1}{2}\,v^2\Big(\partial_t^4 g + 2n\,\partial_t^3 g + \left(n^2-2\right)\partial_t^2 g - 2n\,\partial_t g\Big)\Big]\,d\omega dt.$$

Convex domains: choice of g

A bounded solution of the equation

$$\frac{d^4 g}{dt^4} + 2n\frac{d^3 g}{dt^3} + \left(n^2 - 2\right)\frac{d^2 g}{dt^2} - 2n\frac{dg}{dt} = \delta(t)$$

subject to $g(t) \to 0$ as $t \to +\infty$, is

$$g(t) = -\frac{1}{2n\sqrt{n^2+8}}\begin{cases} n\, e^{-1/2(n-\sqrt{n^2+8})t} - \sqrt{n^2+8}, & t < 0, \\ n\, e^{-1/2(n+\sqrt{n^2+8})t} - \sqrt{n^2+8}\, e^{-nt}, & t > 0. \end{cases}$$

Then for g as above and for every $\xi \in \Omega$, $\tau = e^{-|\xi|}$

$$\int_\Omega \Delta u(x) \Delta \left(\frac{u(x) g(\log(|\xi|/|x|))}{|x|^n}\right) dx \geq \frac{1}{2}\int_{\varkappa(\Omega)\cap S^{n-1}} v^2(\tau,\omega)\, d\omega$$

$$+ \int_{\varkappa(\Omega)} \Big(-2\partial_t^2 g(t-\tau) - 3n\partial_t g(t-\tau) - \left(n^2-2\right) g(t-\tau)\Big)(\partial_t v)^2\, d\omega dt$$

Convex domains: Identity 2

Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $O \in \mathbb{R}^n \setminus \Omega$, and $u \in C^4(\bar{\Omega})$, $\quad u\Big|_{\partial\Omega} = 0, \quad \nabla u\Big|_{\partial\Omega} = 0, \quad v = e^{2t}(u \circ \varkappa^{-1})$.
Then

$$2\int_\Omega \Delta u(x)\Delta\left(\frac{u(x)g(\log|x|^{-1})}{|x|^n}\right)dx - \int_\Omega \Delta^2 u(x)\left(\frac{\big(x\cdot\nabla u(x)\big)g(\log|x|^{-1})}{|x|^n}\right)dx$$

$$= \int_{\varkappa(\Omega)}\Big(-\tfrac{1}{2}(\delta_\omega v)^2\partial_t g + (\partial_t\nabla_\omega v)^2\left(\partial_t g + 2ng\right)$$

$$+ (\partial_t^2 v)^2\left(\tfrac{3}{2}\,\partial_t g + 2ng\right) + n(\nabla_\omega v)^2\partial_t g$$

$$+ (\partial_t v)^2\left(-\tfrac{1}{2}\partial_t^3 g - n\partial_t^2 g - \tfrac{1}{2}(n^2+2n-4)\partial_t g - 2n(n-2)g\right)\Big)\,d\omega dt$$

$$- \frac{1}{2}\int_{\varkappa(\partial\Omega)}\left((\delta_\omega v)^2 + 2(\partial_t\nabla_\omega v)^2 + (\partial_t^2 v)^2\right)\,g\,\cos(\nu,t)\,d\sigma_{\omega,t},$$

Convex domains: global estimates

Suppose Ω is a smooth convex domain in $\mathbb{R}^n$, $O \in \partial\Omega$,
$u \in C^4(\bar{\Omega})$, $\quad u\Big|_{\partial\Omega} = 0, \quad \nabla u\Big|_{\partial\Omega} = 0, \quad v = e^{2t}(u \circ \varkappa^{-1})$.
Let $\xi \in \Omega$, $\tau = e^{-|\xi|}$. Combining Identities 1 and 2 implies

$$\frac{1}{2}\int_{\varkappa(\Omega)\cap S^{n-1}} v^2(\tau,\omega)\, d\omega$$

$$\leq (n+1)\int_\Omega \Delta^2 u(x) \left(\frac{u(x)g(\log|\xi|/|x|)}{|x|^n}\right) dx$$

$$-\frac{n}{2}\int_\Omega \Delta^2 u(x) \left(\frac{(x\cdot\nabla u(x))g(\log|\xi|/|x|)}{|x|^n}\right) dx$$

Convex domains: local estimates

Let Ω be a convex domain in $\mathbb{R}^n$, $O \in \partial\Omega$, and fix some $R \in (0, \mathrm{diam}\,(\Omega)/10)$. Suppose u is a solution of the Dirichlet problem $(*)$ with $f \in C_0^\infty(\Omega \setminus B_{10R})$. Then

$$|\nabla^2 u(x)| \leq \frac{C}{R^2} \left(\fint_{(B_{5R} \setminus B_R) \cap \Omega} |u(y)|^2 \, dy \right)^{1/2} \quad \text{for every } x \in B_{R/5} \cap \Omega,$$

In particular,

$$|\nabla^2 u| \in L^\infty(\Omega).$$

Weighted Positivity and Boundary Point Regularity

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Weighted Inequalities

Weighted integral inequalities of the type

$$\int_\Omega Lu \cdot \Psi u \, dx \geq 0, \tag{1}$$

where Ω is a domain in $\mathbb{R}^n$ and L is an elliptic operator, are important in the qualitative theory of elliptic boundary value problems.

These inequalities have been established for certain scalar operators.

Does a similar property hold for systems?

The Elliptic Operator

Let L be a general second-order elliptic operator

$$L_i(x, D_x)u := \sum_{j=1}^{N} \sum_{\alpha,\beta=1}^{n} -A_{ij}^{\alpha\beta}(x) \frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta}$$
$$=: -A_{ij}^{\alpha\beta}(x) D_{\alpha\beta} u_j, \qquad (i = 1, 2, \ldots, N).$$

L is said to be positive with weight $\Psi(x)$ if

$$\int_\Omega Lu \cdot \Psi u \, dx \geq 0,$$

for all real valued, smooth vector functions $u \in C_0^\infty(\Omega)$.

A Necessary Condition

Theorem 1 (Luo and M. 2007)

Assume $\Psi \in C^\infty(\mathbb{R}^n \setminus \{0\})$ *and is homogeneous of order* $2-n$*, i.e.* $\Psi(x) = |x|^{2-n}\Psi(x/|x|)$*. If* L *is positive with weight* Ψ*, then* $L^T(0, D_x)\Psi = \delta M$ *where* δ *is the Dirac delta function,* $L^T(0, D_x)$ *is the formal adjoint of* $L(0, D_x)$*, and* $M \in \mathbb{R}^{N\times N}$ *is a symmetric, semi-positive definite matrix. Furthermore,*

$$\sum_{i,\alpha,\beta} A_{ip}^{\alpha\beta}(r\omega)\xi_\alpha\xi_\beta\Psi_{ip}(\omega) \geq 0, \qquad \forall\xi \in \mathbb{R}^n, \qquad (p = 1, 2, \ldots, N)$$

for all $r > 0,\ \omega \in S^{n-1}$ *such that* $x = r\omega \in \Omega$*.*

Let L be an elliptic operator and Ψ be a weight satisfying the conditions in Theorem 1.

Is L indeed positive with weight Ψ?

This question has been partially answered for the 3D Lamé system

$$Lu = -\Delta u - \alpha \operatorname{grad} \operatorname{div} u,$$

where the weight is the fundamental matrix

$$\Phi_{ij}(x) = \frac{\alpha+2}{8\pi(\alpha+1)}\, r^{-1}\left(\delta_{ij} + \frac{\alpha}{\alpha+2}\,\omega_i\omega_j\right) \qquad (i,j = 1,2,3).$$

Weighted Positivity of the Lamé System

Theorem 2 (Luo and M. 2010)

The 3D Lamé system is positive with weight Φ *when* $\alpha_- < \alpha < \alpha_+$, *where* $\alpha_- \approx -0.194$ *and* $\alpha_+ \approx 1.524$. *It is not positive with weight* Φ *when* $\alpha < \alpha_-^{(c)} \approx -0.902$ *or* $\alpha > \alpha_+^{(c)} \approx 39.450$.

Regularity of a Boundary Point

As an application of the weighted positivity of the Lamé system, we study the regularity of a boundary point w.r.t. the system.

Definition 3

Let Ω be an open set in $\mathbb{R}^3$ and consider the Dirichlet problem

$$Lu = f, \qquad f \in C_0^\infty(\Omega),\ u \in \mathring{H}^1(\Omega). \tag{2}$$

A point $P \in \partial\Omega$ is regular with respect to L if, for any $f \in C_0^\infty(\Omega)$, the solution of problem (2) satisfies

$$\lim_{\Omega \ni x \to P} u(x) = 0.$$

A Sufficient Condition

Theorem 4 (Luo and M. 2010)

Suppose the 3D Lamé system L is positive with weight Φ. *Then* $O \in \partial\Omega$ *is regular with respect to L if*

$$\int_0^1 \mathrm{cap}(\overline{B}_\rho \setminus \Omega)\rho^{-2}\, d\rho = \infty,$$

where $\mathrm{cap}(\cdot)$ *denotes the classical harmonic capacity and* B_ρ *is the open ball centered at O with radius* ρ.