Homoclinic and Heteroclinic Connections
for a Semilinear Parabolic Equation

Marek Fila (Comenius University)

with Eiji Yanagida (Tohoku University)
Consider the Fujita equation

\[(E) \quad u_t = \Delta u + |u|^{p-1}u \quad \text{in } \mathbb{R}^N\]

where \( p > 1 \).

**Definition**

- **Global solution**: A solution defined for all \( t \in (\tau, +\infty) \).
- **Ancient solution**: A solution defined for all \( t \in (-\infty, \tau) \).
- **Entire solution**: A solution defined for all \( t \in (-\infty, +\infty) \).

**Liouville-type property**: Any entire solution must be trivial.

**Connecting orbit**: An entire solution that converges to steady states as \( t \to \pm \infty \).
The Liouville property and connections between equilibria constitute a very important theme in the theory of dynamical systems.

(i) $\omega$ limit set consists of entire solutions.

(ii) For gradient-like systems, the attractor consists of equilibria and connecting orbits.

For semilinear parabolic equations in one space dimension, the connection problem has been studied extensively since the beginning of 1980’s.

... Brunovský, Fiedler, Rocha, Fusco, Henry, Smoller, ...
In this talk, we study the existence of connecting orbits for \((E)\). For certain ranges of the exponent \(p\), we prove the existence of

**Heteroclinic orbit** :

Connecting orbit between two different steady states.

**Homoclinic orbit** :

Connecting orbit from the trivial steady state to itself.
Known facts about the Liouville-type property

- $1 < p < p_B := \begin{cases} \frac{N(N + 2)}{(N - 1)^2} & \text{for } N > 1 \\ \infty & \text{for } N = 1 \end{cases}$

There is no positive entire solution.

... Bidaut-Véron (1998), Quittner and Souplet (2007)
1 < p < p_S = \begin{cases} 
\frac{N + 2}{N - 2} & \text{for } N > 2 \\
\infty & \text{for } N \leq 2 
\end{cases} 
\quad \text{(Sobolev exponent)}

There is no positive regular steady state.

... Gidas and Spruck (1983)

There is no positive radial entire solution.

... Poláčik and Quittner (2006), Poláčik, Quittner and Souplet (2007)

There is no radial entire solution with finite number of zeros.

... Bartsch, Poláčik and Quittner (2010)
There exists a one-parameter family of positive radial steady states.

\[ \varphi = \varphi_\alpha(r), \quad r = |x|, \quad \alpha > 0, \] where \( \varphi_\alpha(r) \) satisfies

\[
\begin{cases}
(\varphi_\alpha)_{rr} + \frac{N - 1}{r}(\varphi_\alpha)_r + (\varphi_\alpha)^p = 0, \\
\varphi_\alpha(0) = \alpha, \quad (\varphi_\alpha)_r(0) = 0.
\end{cases}
\]

For each \( \alpha > 0 \), the solution \( \varphi_\alpha \) is strictly decreasing in \( |x| \) and satisfies \( \varphi_\alpha(|x|) \to 0 \) as \( |x| \to \infty \).
Profile of a steady state
\[ p_S \leq p < p_{JL} = \begin{cases} \frac{(N - 2)^2 - 4N + 8\sqrt{N - 1}}{(N - 2)(N - 10)} & \text{for } N > 10 \\ \infty & \text{for } N \leq 10 \end{cases} \]

(Joseph-Lundgren exponent)

Every positive radial steady state is unstable. No results on the Liouville-type property.

\[ p_{JL} \leq p < \infty \]

Every positive radial steady state is stable in some weighted space.

• $p_{JL} < p < \infty$

If $u$ is an ancient solution such that

$$0 < u(x, t) < \varphi_\infty(|x|) \quad \text{for all } x \in \mathbb{R}^N,$$

and satisfies some more conditions, then $u$ must be a steady state.

... Poláčik and Yanagida (2005)

Here $\varphi_\infty$ is a singular steady state given by

$$\varphi_\infty(r) := L r^{-2/(p-1)} , \quad r = |x|,$$

where $L = L(p, N)$ is a certain positive constant. The singular steady state is defined for $p > \frac{N}{N - 2}$. 
$\phi_\infty(r) = L r^{-2/(p-1)}$

Singular steady state
Summary of known facts about connecting orbits:

- $1 < p < p_B$ ... There is no positive connecting orbit.
- $1 < p < p_S$ ... There is no radial connecting orbit.

Our new results:

$\begin{align*}
\text{\textbullet} & \quad p_S < p < p_L := \\
& \quad \begin{cases}
\frac{N - 4}{N - 10} & \text{for } N > 10 \\
\infty & \text{for } N \leq 10
\end{cases} \\
\text{(Lepin exponent)}
\end{align*}$

There is a positive radially symmetric homoclinic orbit.

- $p_S \leq p < p_{JL}$

There is a positive radially symmetric heteroclinic orbit connecting $\varphi_\alpha$ to 0.
Theorem 1  (Existence of a homoclinic orbit)

Let $p_S < p < p_L$. Then there exists a (positive and radially symmetric) entire solution of (E) with the following properties:

(i) There exists a positive constant $C_0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} = C_0 (-t)^{-1/(p-1)} + h.o.t.$$ as $t \to -\infty$.

(ii) The solution tends to 0 with the order

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \sim t^{-1/(p-1)}$$ as $t \to +\infty$.

Remark. As far as we know, this is the first example of a homoclinic orbit for a scalar parabolic equation.
$\phi_\infty(r)$
\[ \phi_\infty(r) \]
\phi_\infty(r)
$\phi_\infty(r)$
Outline of the proof of Theorem 1

Step 1: Consider the existence of a positive \textit{backward self-similar solution}.

Step 2: Show the \textit{instability} of the backward self-similar solution.

Step 3: Show the existence of an \textit{ancient solution} that converges to the backward self-similar solution from below as $t \to -\infty$.

Step 4: Show that the ancient solution \textit{exists globally} in time and converges to a forward self-similar solution.
**Backward self-similar solution**

For a solution $u$ of (E) defined for $t \in (-\infty, 0)$, we set

$$w(y, s) = (-t)^{1/(p-1)}u(x, t), \quad y = (-t)^{-1/2}x, \quad s = - \log(-t).$$

Then we obtain the following equation for $w$:

$$(W) \quad w_s = \Delta w - \frac{1}{2}y \cdot \nabla w - \frac{1}{p-1}w + |w|^{p-1}w, \quad y \in \mathbb{R}^N, \ s \in \mathbb{R}.$$  

Any radially symmetric steady state $w = b(r)$, $r = |y|$, must satisfy

$$(B) \quad \begin{cases} 
    b_{rr} + \frac{N-1}{r}b_r - \frac{r}{2}b_r - \frac{1}{p-1}b + |b|^{p-1}b = 0, \\
    b(0) = \beta, \quad b_r(0) = 0,
\end{cases}$$

for some $\beta > 0$. If $b(r) > 0$ for all $r > 0$, then

$$u = B(x, t) := (-t)^{-1/(p-1)}b((-t)^{-1/2}|x|), \quad x \in \mathbb{R}^N, \ t < 0,$$

is called a (positive radial) **backward self-similar solution** of (E).
Lemma 1. Let $p_S < p < p_L$. Then there exists $\beta_\star > 0$ such that the solution $b(r)$ of (B) has the following properties:

(i) If $\beta = \beta_\star$, then $b(r)$ is positive for all $r > 0$, monotone decreasing in $r > 0$, and intersects exactly twice with $\varphi_\infty$.

(ii) If $\beta < \beta_\star$, then $b(r)$ vanishes at some finite $r$.

Proof. This fact was proved by Lepin (1988,1990), Galaktionov and Vázquez (1997). Recently, Mizoguchi (2009) showed that for $p > p_L$, such a solution does not exist. \qed
The function $b(r)$. 

$\phi_\infty(r)$
Lemma 2. Let $p_S < p < p_L$. Then the solution of (B) with $b(0) = \beta_*$ is exponentially unstable as a solution of (W).

Proof. By using the intersection with the singular solution, it can be shown that the eigenvalue problem

\[
\begin{cases}
\mu W = W_{rr} + \frac{N - 1}{r} W_r - \frac{r}{2} W_r - \frac{1}{p - 1} W + pb^{p-1} W, & r > 0, \\
W(0) = 1, \quad W_r(0) = 0.
\end{cases}
\]

has a positive eigenvalue $\mu > 0$ with a positive eigenfunction. \qed
Lemma 3. Let $p_s < p < p_L$, and let $b$ be the solution of (B) with $b(0) = \beta_*$. Then there exists an ancient solution of (W) such that $w$ is monotone decreasing in $s$ and $\|w(\cdot, s) - b(|\cdot|)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $s \to -\infty$.

Proof. In the supercritical case, we can not apply standard methods of the theory of dimensional dynamical systems. To show the existence of an ancient solution of (W) that converges to $b$ exponentially as $s \to -\infty$, we modify the method of Fukao, Morita and Ninomiya (2004) based on a comparison technique.

The ancient solution $w$ of (W) corresponds to an ancient solution of (E) that converges to the trivial solution as $t \to -\infty$. \qed
Lemma 4. The ancient solution given as above converges to the trivial solution with the order $t^{-1/(p-1)}$.

Proof. Applying a result of F, King, Winkler and Yanagida (2008) concerning the asymptotic behavior of solutions, we can show that the solution approaches a forward self-similar solution of (E):

$$u = F_\gamma(x, t) := t^{-1/(p-1)} f_\gamma(t^{-1/2}|x|), \quad x \in \mathbb{R}^N, \ t > 0,$$

where $f = f_\gamma(r)$, $r = |y|$, be a solution of

$$\begin{cases}
  f_{rr} + \frac{N - 1}{r} f_r + \frac{r}{2} f_r + \frac{m}{2} f + |f|^{p-1} f = 0, & r > 0, \\
  f(0) = \gamma, \quad f_r(0) = 0,
\end{cases}$$

with some $\gamma > 0$. \hfill \Box
If $u$ is a homoclinic orbit then

$$u^\lambda(x, t) := \lambda^{2/(p-1)}u(\lambda x, \lambda^2 t),$$

also is a homoclinic orbit for any $\lambda > 0$.

**Theorem 2 (Convergence to a singular homoclinic orbit)**

Let $u$ be the homoclinic orbit constructed as above. Then $u^\lambda(x, t)$ approaches a singular homoclinic orbit as $\lambda \to \infty$:

(i) For any $\tau < 0$, $u^\lambda(x, t)/B(x, t) \to 1$ as $\lambda \to \infty$ uniformly in $(x, t) \in \mathbb{R}^N \times (-\infty, \tau)$.

(ii) For any $\tau > 0$, $u^\lambda(x, t) \to F_\gamma(x, t)$ as $\lambda \to \infty$ uniformly in $(x, t) \in \mathbb{R}^N \times (\tau, \infty)$. 
In other words, as $\lambda \to \infty$, the homoclinic solution $u^\lambda$ approaches a singular homoclinic orbit that consists of a backward self-similar solution and a forward self-similar solution. Such a singular homoclinic orbit was found by Galaktionov and Vázquez (1997), and later studied by Naito (2009).

cf. Singular connections for other equations

$u^\lambda(x,t)$

Singular connection

$\|u\|$
Other results

Theorem 3  (Existence of heteroclinic orbits)

Let $p_S \leq p < p_{\text{JL}}$. For every $\alpha > 0$, there exists a entire solution of (E) with the following properties:

(i) There exist positive constants $C_0$ and $\mu_0 > 0$ such that

$$\|u(\cdot, t) - \varphi_{\alpha}(\cdot)\|_{L^\infty(\mathbb{R}^N)} = C_0 \exp(\mu_0 t) + h.o.t. \text{ as } t \to -\infty.$$

(ii) The solution tends to 0 with the order

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \sim t^{-(N-2)/2} \text{ if } p = p_S,$$

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \sim t^{-1/(p-1)} \text{ if } p_S < p < p_{\text{JL}},$$

as $t \to +\infty$. 
Theorem 4 (Connection from 0 to $\infty$)

Let $p_S \leq p < p_{JL}$. For every $\alpha > 0$, there exists an ancient solution $u$ of (E) with the following properties:

(i) There exist positive constants $C_0$ and $\mu_0 > 0$ such that

$$\|u(\cdot, t) - \varphi_\alpha(|\cdot|)\|_{L^\infty(\mathbb{R}^N)} = C_0 \exp(\mu_0 t) + h.o.t. \text{ as } t \to -\infty.$$  

(ii) The solution blows up at the origin at some $t = T < \infty$ and the blow-up is of Type I: There exists $C > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C(T - t)^{-1/(p-1)} \text{ for all } T - 1 < t < T.$$
Theorem 5  (Nonexistence of ancient solutions)

Let $p_S \leq p < p_{JL}$. Then there is no ancient solution of (E) such that

$$0 < u(x, t) \leq \varphi_\infty(|x|) \quad \text{for all } t < \tau.$$
Summary of the existence of connecting orbits

• $1 < p < p_B = \frac{N(N + 2)}{(N - 1)^2}$ $\implies$ No positive connecting orbit

• $1 < p < p_S = \frac{N + 2}{N - 2}$ $\implies$ No radial connecting orbit

• $p_S \leq p < p_{JL} = \frac{(N - 2)^2 - 4N + 8\sqrt{N - 1}}{(N - 2)(N - 10)}$ $\implies$ $\exists$ Radially symmetric heteroclinic orbit

• $p_S < p < p_L = \frac{N - 4}{N - 10}$ $\implies$ $\exists$ Radially symmetric homoclinic orbit
Open questions

\begin{itemize}
  \item $p_L \leq p < \infty \implies \text{No radially symmetric homoclinic orbits} \, ??$
  \item $p_{JL} \leq p < \infty \implies \text{No radially symmetric heteroclinic orbits} \, ??$
  \item $N = 2$ and $p_B \leq p < \infty \implies \text{No homoclinic orbit} \, ??$
\end{itemize}
Thank you for your attention!