Quasistatic evolution for Cam-Clay plasticity

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Cam-Clay plasticity

Cam-Clay plasticity is a phenomenological model used to describe the inelastic behaviour of fine grained soils. It exhibits both hardening and softening, depending on the loading conditions.

The reference configuration is a bounded smooth open set $\Omega \subset \mathbf{R}^n$. The variables and constraints of the model are:

- displacement: $u\colon [0,+\infty)\times\Omega \to \mathbf{R}^n$,
- strain: $Eu = \frac{1}{2}(\nabla u + \nabla u^T)\colon [0,+\infty)\times\Omega \to \mathbf{M}^{n\times n}_{sym}$,
- additive decomposition: $Eu = e + p$,
- elastic strain: $e\colon [0,+\infty)\times\Omega \to \mathbf{M}^{n\times n}_{sym}$,
- plastic strain: $p\colon [0,+\infty)\times\Omega \to \mathbf{M}^{n\times n}_{sym}$,
- stress: $\sigma = 2\mu\, e + \lambda\, \mathrm{tr}(e) I$ (μ,λ Lamé constants),
- stress constraint: $\sigma(t,x) \in K(\zeta(t,x))$ for every (t,x),
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The sets $\partial K(\zeta)$ play the role of the yield surfaces. In the typical applications they are homothetic ellipsoids in $\mathbf{M}^{n\times n}_{sym}$ passing through the origin. We assume only $K(\zeta) = \zeta K(1)$ for every $\zeta \geq 1$ (homothety condition), where $K(1)$ is a compact convex subset of $\mathbf{M}^{n\times n}_{sym}$ with $0 \in \operatorname{int} K(1)$ ($0 \in K(1)$ and $K(1)$ with nonempty interior is enough, with a slightly different proof). This implies

$$1 \leq \zeta_1 \leq \zeta_2 \quad \Longrightarrow \quad K(\zeta_1) \subset K(\zeta_2) .$$

Therefore, if $\dot{\zeta}(t,x) > 0$, the set $K(\zeta(t,x))$ expands leading to a hardening response. On the contrary, if $\dot{\zeta}(t,x) < 0$, the set $K(\zeta(t,x))$ shrinks leading to a softening response.

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- equilibrium condition
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Usually the evolution law for the internal variable z is

$$\dot{z}(t,x) = \operatorname{tr}\sigma(t,x)\operatorname{tr}\dot{p}(t,x)\,,$$

and $\operatorname{tr}\sigma \leq 0$ for every $\sigma \in K(\zeta)$ (compressive condition of soil mechanics), so that the hardening or softening behaviour is determined only by the sign of $\operatorname{tr}\dot{p}(t,x)$.

We consider here the nonlocal variant

$$\dot{z}(t,x) = \varrho \star [(\varrho \star \operatorname{tr}\sigma(t,\cdot))\operatorname{tr}\dot{p}(t,\cdot)](x)\,,$$

where $\star$ denotes the convolution with respect to the space variable x, and ϱ is a smooth convolution kernel.

The convolution ensures that a very weak convergence of σ and $\dot{p}$ implies strong convergence of the corresponding z. It gives a nonlocal character to the evolution: the size of the yield surface at a point x is affected by $\operatorname{tr}\sigma$ and $\operatorname{tr}\dot{p}$ in a small neighborhood of x, which is not physically implausible.

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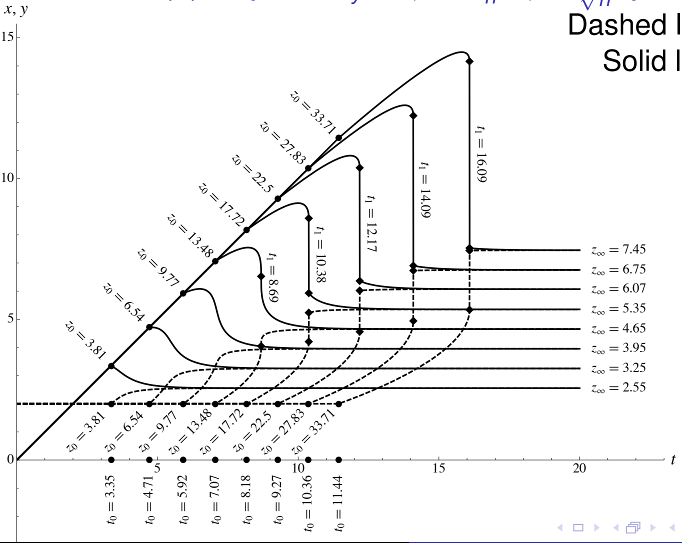
$Ew(t,x) = -\frac{1}{n}I + t\frac{1}{\sqrt{n}}e_D$, $\sigma(t) = e(t) = -x(t)\frac{1}{n}I + y(t)\frac{1}{\sqrt{n}}e_D$,

$\operatorname{tr} e_D = 0$, $K(\zeta) = \{\sigma \in \mathbf{M}^{n\times n}_{sym} : |\sigma + \frac{1}{n}\zeta I| \leq \frac{1}{\sqrt{n}}\zeta\}$.

Dashed lines: $x(t)$.

Solid lines: $y(t)$.



We introduce a notion of generalized solution, based on a viscoplastic approximation of Perzyna-type. Given a viscosity parameter $\varepsilon > 0$, the equations are:

- kinematic admissibility
  $Eu_\varepsilon(t,x) = e_\varepsilon(t,x) + p_\varepsilon(t,x)$ on Ω, $u_\varepsilon(t,x) = w(t,x)$ on $\partial\Omega$,
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If we fix ζ_ε in a suitable function space, the existence of u_ε, e_ε, p_ε, and σ_ε satisfying the first four equations and a prescribed initial condition can be obtained by adapting a result by Suquet.
To fulfill also the evolution law for the internal variable we use the Schauder fixed point theorem.
To obtain an estimate of the viscoplastic approximation we introduce the following functions:

Stored elastic energy: $\mathcal{Q}(e(t)) := \frac{1}{2}\int_\Omega \sigma(t,x) : e(t,x)\,dx$.

Support function: $H(\xi,\zeta) := \sup\{\sigma : \xi \mid \sigma \in K(\zeta)\}$.

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Since $H(\cdot,\zeta)$ is positively homogeneous of degee one, the last definition can be extended to the case where $\dot p(t)$ is a measure on $\overline{\Omega}$, provided $\zeta(t)$ is a continuous function.

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Stored elastic energy: $\mathcal{Q}(e(t)) := \frac{1}{2}\int_\Omega \sigma(t,x) : e(t,x)\,dx$.

Support function: $H(\xi,\zeta) := \sup\{\sigma : \xi \mid \sigma \in K(\zeta)\}$.

Dissipation rate: $\mathcal{H}(\dot p(t),\zeta(t)) := \int_\Omega H(\dot p(t,x),\zeta(t,x))\,dx$.

Since $H(\cdot,\zeta)$ is positively homogeneous of degee one, the last definition can be extended to the case where $\dot p(t)$ is a measure on $\overline{\Omega}$, provided $\zeta(t)$ is a continuous function.

If the kinematic admissiblity, the constitutive equations, and the equilibrium conditions are satisfied, then the regularized flow rule is equivalent to

- modified stress constraint: for a.e. $t > 0$ and a.e. $x \in \Omega$

$$\sigma_\varepsilon(t,x) - \varepsilon \dot{p}_\varepsilon(t,x) \in K(\zeta_\varepsilon(t,x)),$$

- energy-dissipation balance: for every $T > 0$

$$\mathcal{Q}(e_\varepsilon(T)) + \int_0^T \mathcal{H}(\dot{p}_\varepsilon(t), \zeta_\varepsilon(t))dt +$$

$$+ \int_0^T \|\dot{p}_\varepsilon(t)\|_2 \, d_2(\sigma_\varepsilon(t), K(\zeta_\varepsilon(t)))\, dt =$$

$$= Q(e_\varepsilon(0)) + \int_0^T \langle \sigma_\varepsilon(t), E\dot{w}(t)\rangle \, dt,$$

where d_2 denotes the distance in L^2.

We assume that $t \mapsto w(t)$ belongs to $H^1((0,T); H^1(\Omega; \mathbf{R}^n))$, so that all terms are well defined.

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The energy-dissipation balance allows to prove that for every $T > 0$ there exists a constant C_T such that

$$\|\sigma_\varepsilon(t)\|_2 \leq C_T \quad \text{and} \quad \int_0^T \|\dot{p}_\varepsilon(t)\|_1 dt \leq C_T$$

for every $\varepsilon > 0$ and every $t \in [0, T]$. We will rescale the solution using the function

$$s_\varepsilon^\circ(t) := \int_0^t (\|\dot{p}_\varepsilon(\tau)\|_1 + \|E\dot{w}(\tau)\|_2 + 1)\, d\tau\,,$$

and its inverse $t_\varepsilon^\circ := (s_\varepsilon^\circ)^{-1}$.
Define the rescaled functions by $w_\varepsilon^\circ(s) := w(t_\varepsilon^\circ(s))$, $u_\varepsilon^\circ(s) := u_\varepsilon(t_\varepsilon^\circ(s))$, $e_\varepsilon^\circ(s) := e_\varepsilon(t_\varepsilon^\circ(s))$, $\sigma_\varepsilon^\circ(s) := \sigma_\varepsilon(t_\varepsilon^\circ(s))$, $p_\varepsilon^\circ(s) := p_\varepsilon(t_\varepsilon^\circ(s))$, $z_\varepsilon^\circ(s) := z_\varepsilon(t_\varepsilon^\circ(s))$, $\zeta_\varepsilon^\circ(s) := \zeta_\varepsilon(t_\varepsilon^\circ(s))$.

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- kinematic admissibility:
  $Eu^\circ_\varepsilon(s) = e^\circ_\varepsilon(s) + p^\circ_\varepsilon(s)$ on Ω, $u^\circ_\varepsilon(s) = w^\circ_\varepsilon(s)$ on $\partial\Omega$,
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- equilibrium condition:
  $\mathrm{div}\,\sigma^\circ_\varepsilon(s) = 0$,
- energy-dissipation balance: for every $S > 0$

$$\mathcal{Q}(e^\circ_\varepsilon(S)) + \int_0^S \mathcal{H}(\dot p^\circ_\varepsilon(s), \zeta^\circ_\varepsilon(s))ds + $$
$$+ \int_0^S \|\dot p^\circ_\varepsilon(s)\|_2\, d_2(\sigma^\circ_\varepsilon(s), K(\zeta^\circ_\varepsilon(s)))\,ds =$$
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- evolution law for the internal variable:
  $\dot z^\circ_\varepsilon(s) = \varrho \star [(\varrho \star \mathrm{tr}\,\sigma^\circ_\varepsilon(s))\,\mathrm{tr}\,\dot p^\circ_\varepsilon(s)]$.

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As $s^\circ_\varepsilon(t_2) - s^\circ_\varepsilon(t_1) \geq t_2 - t_1$ for every $t_1 < t_2$, we have $0 < t^\circ_\varepsilon(s_2) - t^\circ_\varepsilon(s_1) \leq s_2 - s_1$ for every $s_1 < s_2$. By the Arzelà-Ascoli Theorem we may assume that t°_ε converges uniformly on compact sets to a nondecreasing Lipschitz function $t^\circ\colon [0,+\infty) \to [0,+\infty)$. Define $w^\circ(s) := w(t^\circ(s))$. From the definitions of s°_ε and t°_ε we obtain easily that $\|p^\circ_\varepsilon(s_2) - p^\circ_\varepsilon(s_1)\|_1 \leq |s_2 - s_1|$.

By the Arzelà-Ascoli Theorem we may assume that

$$p^\circ_\varepsilon(s) \rightharpoonup p^\circ(s) \quad \text{weakly}^* \text{ in } M_b(\overline{\Omega}; \mathbf{M}^{n\times n}_{sym}) \text{ for every } s,$$

and $\|p^\circ(s_2) - p^\circ(s_1)\|_1 \leq |s_2 - s_1|$. As $\|e^\circ_\varepsilon(s)\|_2$ is bounded,

$$e^\circ_\varepsilon(s) \rightharpoonup e^\circ(s) \quad \text{weakly in } L^2(\Omega; \mathbf{M}^{n\times n}_{sym}) \text{ for a subsequence.}$$

As $p^\circ(s)$ determines uniquely $e^\circ(s)$, the whole sequence converges and $s \mapsto e^\circ(s)$ is weakly continuous. It follows that

$$u^\circ_\varepsilon(s) \rightharpoonup u^\circ(s) \quad \text{weakly}^* \text{ in } BD(\Omega) \text{ for every } s.$$

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The limit functions satisfy the following equations:

- weak kinematic admissibility (easy):
  $Eu^\circ(s) = e^\circ(s) + p^\circ(s) \;\; \Omega, \;\; p^\circ(s) = (w^\circ(s) - u^\circ(s)) \odot \nu \;\; \partial\Omega,$
- constitutive equations (easy):
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- energy-dissipation inequality (difficult): for every $S > 0$

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where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\boldsymbol{\Omega}; \mathbf{M}^{n\times n}_{sym})$. The idea of the proof could be explained easily if $s \mapsto e^\circ(s)$ were absolutely continuous with values in $L^2(\boldsymbol{\Omega}; \mathbf{M}^{n\times n}_{sym})$ and $u^\circ(s) = w^\circ(s)$ on $\partial\boldsymbol{\Omega}$: it is enough to prove the inequality in differential form

$$\langle \boldsymbol{\sigma}^\circ(s), E\dot{w}^\circ(s)\rangle \leq \langle \boldsymbol{\sigma}^\circ(s), \dot{e}^\circ(s)\rangle + \mathcal{H}(\dot{p}^\circ(s), \zeta^\circ(s)) +$$
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$$\langle \boldsymbol{\sigma}^\circ(s), E\dot{w}^\circ(s)\rangle \leq \langle \boldsymbol{\sigma}^\circ(s), \dot{e}^\circ(s)\rangle + \mathcal{H}(\dot{p}^\circ(s), \zeta^\circ(s)) +$$
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We have to prove that

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The limits $u^\circ(s)$, $e^\circ(s)$, $p^\circ(s)$, $z^\circ(s)$, $\sigma^\circ(s)$, $\zeta^\circ(s)$, and $t^\circ(s)$ satisfy the following conditions, where $w^\circ(s) := w(t^\circ(s))$:

- weak kinematic admissibility:
  $Eu^\circ(s) = e^\circ(s) + p^\circ(s) \;\; \Omega, \;\; p^\circ(s) = (w^\circ(s) - u^\circ(s)) \odot \nu \;\; \partial\Omega,$
- constitutive equations:
  $\sigma^\circ(s) = 2\mu e^\circ(s) + \lambda \mathrm{tr}(e^\circ(s)) I, \qquad \zeta^\circ(s) = V(z^\circ(s)),$
- equilibrium condition: $\mathrm{div}\, \sigma^\circ(s) = 0$,
- energy-dissipation balance: for every $S > 0$

$$\mathcal{Q}(e^\circ(S)) + \int_0^S \mathcal{H}(\dot p^\circ(s), \zeta^\circ(s)) ds +$$
$$+ \int_0^S \|\dot p(s)\|_2 \, d_2(\sigma^\circ(s), K(\zeta^\circ(s))) ds =$$
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- evolution law for the internal variable:
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Moreover they satisfy the following conditions:

- partial stress constraint $\sigma^\circ(s,x) \in K(\zeta^\circ(s,x))$ for a.e. $x \in \Omega$, unless t° is constant near s.
- extended flow rule: $\dot{p}^\circ(s) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ for a.e. s where the stress constraint is not satisfied, and

$$\dot{p}^\circ(s,x) \in N^{ext}_{K(\zeta^\circ(s,x))}(\sigma^\circ(s,x)),$$

where $N^{ext}_{K(\zeta)}(\sigma) := \{\lambda(\sigma - \pi_{K(\zeta)}(\sigma)) : \lambda \geq 0\}$ for $\sigma \notin K(\zeta)$; this is the natural extension of the flow rule to these intervals.

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Using the energy-dissipation balance we can prove that for every $S > 0$ there exists $L_S < +\infty$ such that

$$\|\boldsymbol{\sigma}^\circ(s_2) - \boldsymbol{\sigma}^\circ(s_1)\|_2 \leq L_S(s_2 - s_1)$$

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In the rescaled variable s we observe two different regimes.

- Slow dynamics: in $B := \{s : \sigma^\circ(s) \in K(\zeta^\circ(s)) \text{ a.e. in } \Omega\}$, which corresponds to a continuous evolution in the original variable $t = t^\circ(s)$, the evolution law for p° and z° is

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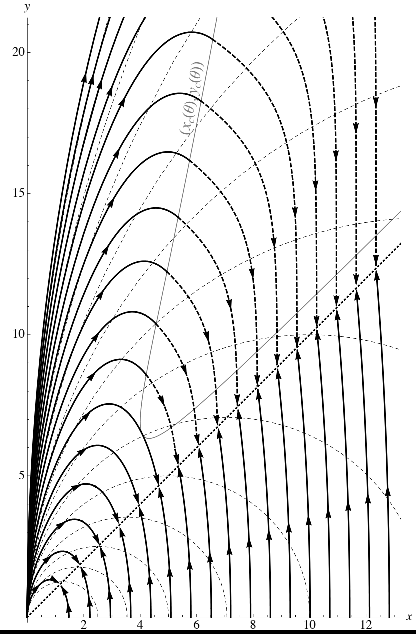

$$Ew(t,x) = -\frac{1}{n}I + t\frac{1}{\sqrt{n}}e_D$$
$$e^\circ(s) = -x(s)\frac{1}{n}I + y(s)\frac{1}{\sqrt{n}}e_D$$
$$\sigma^\circ(s) = e^\circ(s)\,,\ \mathrm{tr}\,e_D = 0$$
$$K(\zeta) = \{|\sigma + \tfrac{1}{n}\zeta I| \le \tfrac{1}{\sqrt{n}}\zeta\}$$

Trajectories $(x(s), y(s))$

Solid lines: slow dynamics

Dashed lines: fast dynamics

Dotted line: equilibrium points