Finite Extinction Time For Non-Negative Solutions Of Some Parabolic Equations

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Basic Problem

$$\begin{cases} \dfrac{du}{dt} + Au \ni 0, \\ u(0) = u_0. \end{cases}$$

Question : is there a finite time T such that $u(T) = 0$?

$\Omega \subset \mathbb{R}^N$ regular bounded domain,

$$\begin{cases} u_t + Lu + a(x)\,|u|^{q-1}u = 0 \text{ in } \Omega \times (0,+\infty), \\ u(0,x) = u_0(x) \text{ on } \Omega, \\ a \geq 0 \text{ a.e., } \; a \in L^\infty(\Omega), \; 0 < q < 1, \end{cases}$$

for Dirichlet or Neumann boundary condition,

$$L = (-1)^m \Delta(\Delta(...)), \; m \text{ times.}$$

Problem : to find sharp conditions on potential $a(x)$, which guarantee the existence of finite T such that arbitrary solution satisfies $u(T,x) \equiv 0$.

Examples (1)

$\Omega \subset \mathbb{R}^N$ regular bounded domain,

$$\begin{cases} u_t + Lu + a(x)\,|u|^{q-1}u = 0 \text{ in } \Omega \times (0,+\infty), \\ u(0,x) = u_0(x) \text{ on } \Omega, \\ a \geq 0 \text{ a.e., } \ a \in L^\infty(\Omega), \ 0 < q < 1, \end{cases}$$

for Dirichlet or Neumann boundary condition.

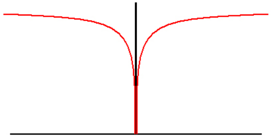
- If $a(x) \geq \gamma > 0$ on Ω then any solution vanishes in a finite time.
- If, by the contrary, $a(x) = 0$ on an open domain $\omega \subset \Omega$, then there exist solutions which never vanish on whole Ω.

Examples (2)

$\Omega \subset \mathbb{R}^N$ regular bounded domain,

$$\begin{cases} u_t + Lu + a(x)\,|u|^{q-1}u = 0 \text{ in } \Omega \times (0,+\infty), \\ u(0,x) = u_0(x) \text{ on } \Omega, \\ a \geq 0 \text{ a.e.},\ a \in L^\infty(\Omega),\ 0 < q < 1, \end{cases}$$

for Dirichlet or Neumann boundary condition.



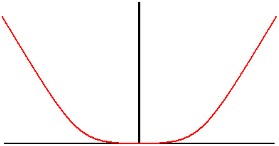
Vertical tangente : $a(x) \approx \text{const.} > 0 \quad \Longrightarrow \quad T < +\infty.$

Examples (3)

$\Omega \subset \mathbb{R}^N$ regular bounded domain,

$$\begin{cases} u_t + Lu + a(x)\,|u|^{q-1}u = 0 \text{ in } \Omega \times (0,+\infty), \\ u(0,x) = u_0(x) \text{ on } \Omega, \\ a \geq 0 \text{ a.e.},\ a \in L^\infty(\Omega),\ 0 < q < 1, \end{cases}$$

for Dirichlet or Neumann boundary condition.



Very flat : $a(x) \approx 0 \quad \Longrightarrow \quad T = +\infty.$

Examples (4)

$\Omega \subset \mathbb{R}^N$ regular bounded domain,

$$\begin{cases} u_t + Lu + a(x)\,|u|^{q-1}u = 0 \text{ in } \Omega \times (0,+\infty), \\ u(0,x) = u_0(x) \text{ on } \Omega, \\ a \geq 0 \text{ a.e.},\ a \in L^\infty(\Omega),\ 0 < q < 1, \end{cases}$$

for Dirichlet or Neumann boundary condition.

$$a(x) = |x| \quad \Longrightarrow \quad T = ?$$

- $a(x) = \text{const.}$: many articles and many authors.
- $a(x) \neq \text{const.}$: results in the framework of semi-classical methods.

KV-method

Equation

- V.A. Kondratiev, L. Véron (1997).
- Y. Belaud, B. Helffer, L. Véron (2001).

$$\begin{cases} u_t - \Delta u + a(x)\,|u|^{q-1}u = 0 \text{ in } \Omega \times (0,+\infty), \\ u(0,x) = u_0(x) \text{ on } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \\ a \geq 0 \text{ a.e.},\ a \in L^\infty(\Omega),\ 0 < q < 1. \end{cases}$$

Theorem

If $L = -\Delta$ *and* $\ln 1/a \in L^p(\Omega)$ *for* $p > N/2$ *then all solutions vanish in a finite time.*

Corollary

If $a(x) = \exp\left(-\dfrac{1}{|x|^\alpha}\right),\ 0 < \alpha < 2 \implies T < +\infty.$

Theorem

- Y. Belaud, A. E. Shishkov (2007).

$$\begin{cases} u_t - \Delta u + a(x)\,|u|^{q-1}u = 0 \text{ in } \Omega \times (0,+\infty), \\ u(0,x) = u_0(x) \text{ on } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \\ a \geq 0 \text{ a.e.},\ a \in L^\infty(\Omega),\ 0 < q < 1. \end{cases}$$

Theorem

If $L = -\Delta$ *and* $a(x) = \exp\left(-\dfrac{\omega(|x|)}{|x|^2}\right)$ *with a nonnegative function* $\omega(s)$ $(\omega(0) = 0)$ *which satisfies the following Dini-like condition* $\displaystyle\int_0^c \frac{\omega(s)}{s}ds < \infty$ *then* $T < +\infty$*.*

- First proof : integral method.

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- Second proof : KV method.

$$\mu_1(h) = \inf \left\{ \int_\Omega |\nabla v|^2 + \frac{a(x)}{h^2}|v|^2 \, dx, v \in W^{1,2}(\Omega), ||v||_{L^2(\Omega)} = 1 \right\}.$$

Regularizing effects L^2, L^∞.

Proposition

Let $(\alpha_n) \downarrow 0$*. If*

$$\sum_n \frac{1}{\mu_1\left(\alpha_n^{\frac{1-q}{2}}\right)} \left(\ln\left(\mu_1\left(\alpha_n^{\frac{1-q}{2}}\right)\right) + \ln\left(\frac{\alpha_n}{\alpha_{n+1}}\right) + 1 \right) < +\infty,$$

then $T < +\infty$*.*

Lieb-Thirring estimate for $\mu_1(h)$ when $h \to 0$.

Sketch Of The Proof

$$\begin{cases} u_t - \Delta u + a(x)\,|u|^{q-1}u = 0 \text{ in } \Omega \times (0,+\infty), \\ u(0,x) = u_0(x) \text{ on } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \\ a \geq 0 \text{ a.e., } \; a \in L^\infty(\Omega), \; 0 < q < 1. \end{cases}$$

$$\mu_1(h) = \inf\left\{ \int_\Omega |\nabla v|^2 + \frac{a(x)}{h^2}|v|^2\,dx, v \in W^{1,2}(\Omega), ||v||_{L^2(\Omega)} = 1 \right\}.$$

- Advantage : use of well-know estimate for $\mu_1(h)$.

- Advantage : use of well-know estimate for $\mu_1(h)$.

- Drawback : regularizing effects from L^2 to L^∞ for u.

Extension to high order operators is almost impossible.

BS-method

(Article submitted)

$$\begin{cases} u_t + Lu + a(x)\,|u|^{q-1}u = 0 \text{ in } \Omega \times (0,+\infty), \\ u(0,x) = u_0(x) \text{ on } \Omega, \\ a \geq 0 \text{ a.e.},\ a \in L^\infty(\Omega),\ 0 < q < 1, \end{cases}$$

for Dirichlet boundary condition, L linear and

$$\int_\Omega Lu.u \geq C_1\,||D_x^m u||^2_{L^2(\Omega)} - C_2\,||u||^2_{L^2(\Omega)},\ \forall u \in W_0^{m,2}(\Omega).$$

Main Result (1)

Theorem

For $N \neq 2m$,

$$\int_0^1 meas\{x \in \Omega : a(x) \leq t\}^{\min\left(\frac{2m}{N},1\right)} \frac{dt}{t} < +\infty \implies T < +\infty.$$

Theorem

For $N = 2m$,

$$\int_0^1 \frac{meas\{x \in \Omega : a(x) \leq t\}}{-\ln meas\{x \in \Omega : a(x) \leq t\}} \frac{dt}{t} < +\infty \implies T < +\infty.$$

Main Result (2)

$$\int_0^1 \text{meas}\{x \in \Omega : a(x) \leq t\}^{\min\left(\frac{2m}{N},1\right)} \frac{dt}{t} < +\infty,$$

for $a(x) = \text{const.} > 0$.

$$\int_0^1 \text{meas}\{x \in \Omega : a(x) \leq t\}^{\min\left(\frac{2m}{N},1\right)} \frac{dt}{t} = +\infty,$$

for $a(x) = 0$.

Main Result (3)

$$a(x) = \exp\left(-\frac{\omega(|x|)}{|x|^{\min(2m,N)}}\right),\ 0 \le \omega(s) \le \omega_0,\ \omega(s) \uparrow .$$

Corollary

For $N \neq 2m$,

$$\int_0^\delta \frac{\omega(t)}{t}\,dt < +\infty \implies T < +\infty.$$

Corollary

For $N = 2m$,

$$\int_0^\delta \frac{\omega(t)}{t\,(-\ln t)}\,dt < +\infty \implies T < +\infty.$$

Definition

$$\lambda_1(h) = \inf \left\{ \int_\Omega Lv.v + a(x)|v|^{1+q}\, dx, v \in W_0^{m,2}(\Omega), ||v||^2_{L^2(\Omega)} = h \right\}.$$

- First eigenvalue of a nonlinear high order operator.
- No estimate for $\lambda_1(h)$ when $h \to 0$.
- No information about a minimizer.

$$\lambda_1(h) = \inf\left\{ \int_\Omega Lv.v + a(x)|v|^{1+q}\, dx, v \in W_0^{m,2}(\Omega), ||v||^2_{L^2(\Omega)} = h \right\}.$$

Proposition (Key-Stone)

If

$$\int_0^1 \frac{1}{\lambda_1(h)} dh < +\infty,$$

then all the solutions vanish in a finite time and in this case,

$$T \leq \frac{1}{2} \int_0^{||u_0||^2_{L^2(\Omega)}} \frac{1}{\lambda_1(h)} dh.$$

Proof Of The Key-Stone Proposition

$$u_t + Lu + a(x)|u|^{q-1}u = 0,$$

$$\frac{1}{2}\frac{d}{dt}(||u||^2_{L^2(\Omega)}) + \int_\Omega Lu.u + a(x)|u|^{1+q}\,dx = 0,$$

$$\int_\Omega Lu.u + a(x)|u|^{1+q}\,dx \geq \lambda_1(||u(.,t)||^2_{L^2(\Omega)}),\ \forall t \geq 0,$$

$$\frac{1}{2}\frac{d}{dt}(||u(.,t)||^2_{L^2(\Omega)}) + \lambda_1(||u(.,t)||^2_{L^2(\Omega)}) \leq 0,$$

$$t \leq \frac{1}{2}\int_{||u(t,.)||^2_{L^2(\Omega)}}^{||u_0||^2_{L^2(\Omega)}} \frac{1}{\lambda_1(h)}dh,$$

$$T \leq \frac{1}{2}\int_0^{||u_0||^2_{L^2(\Omega)}} \frac{1}{\lambda_1(h)}dh.$$

Minimizing Sequence

$$\lambda_1(h) = \inf\left\{\int_\Omega Lv.v + a(x)|v|^{1+q}\,dx,\, v \in W_0^{m,2}(\Omega), ||v||^2_{L^2(\Omega)} = h\right\}.$$

$$\int_\Omega Lu.u \geq C_1\,||D_x^m u||^2_{L^2(\Omega)} - C_2\,||u||^2_{L^2(\Omega)},\ \forall u \in W_0^{m,2}(\Omega).$$

Definition

$$\forall h > 0, \exists v_h \in W_0^{m,2}(\Omega),\ ||v_h||^2_{L^2(\Omega)} = h,$$

and

$$\lambda_1(h) \leq \int_\Omega Lv_h.v_h + a(x)|v_h|^{1+q}\,dx \leq 2\lambda_1(h).$$

$N \geq 2m+1$

Gärding's inequality and Sobolev's inequality :

$$C \leq \left(C_2 + \frac{2\lambda_1(h)}{h} \right) \text{meas} \left\{ x \,:\, |v_h|^{1-q} \left(C_2 + \frac{2\lambda_1(h)}{h} \right) \geq a(x) \right\}^{\frac{2m}{N}} .$$

$$(*) \quad C'\, h^{\eta} \geq |v_h|^{1-q} \left(C_2 + \frac{2\lambda_1(h)}{h} \right),$$

for some $C' > 0$ and $\eta > 0$ may be false :

- v_h is not necessary bounded,
- and even if v_h is bounded, $||v_h||^{1-q}_{L^\infty(\Omega)} \left(C_2 + \frac{2\lambda_1(h)}{h} \right) \rightarrow 0$ may be false.

$N \geq 2m+1$

$O \in \Omega$.

Definition

$$S = \left\{ b \in L^{\infty}(\Omega) \mid \int_0^1 \text{meas}\{x \in \Omega : |b(x)| \leq t\}^{\frac{2m}{N}} \frac{dt}{t} < +\infty \right\}.$$

$$\int_0^1 \text{meas}\{x \in \Omega : a(x) \leq t\}^{\frac{2m}{N}} \frac{dt}{t} < +\infty \iff a \in S.$$

Proposition

$N \geq 2m+1$

$O \in \Omega$.

$$\widetilde{a}(x) = a(x) \exp\left(-\frac{1}{|x|^{\alpha}}\right),\ 0 < \alpha \ll 1.$$

Both functions $a(x)$ and $\widetilde{a}(x)$ satisfy

$$\int_0^1 \operatorname{meas}\{x \in \Omega : a(x) \leq t\}^{\frac{2m}{N}} \frac{dt}{t} < +\infty,$$

$$\int_0^1 \operatorname{meas}\{x \in \Omega : \widetilde{a}(x) \leq t\}^{\frac{2m}{N}} \frac{dt}{t} < +\infty.$$

$\widetilde{a}(x)$ is a good representant of $a(x)$ in the set S.

$N \geq 2m+1$

$O \in \Omega$.

$$\widetilde{a}(x) = a(x) \exp\left(-\frac{1}{|x|^{\alpha}}\right), \; 0 < \alpha \ll 1.$$

$$\widetilde{\lambda}_1(h) = \inf\left\{\int_{\Omega} Lv.v + \widetilde{a}(x)|v|^{1+q}\,dx, v \in W_0^{m,2}(\Omega), ||v||^2_{L^2(\Omega)} = h\right\},$$

$$\widetilde{a}(x) \leq a(x),$$

$$\widetilde{\lambda}_1(h) \leq \lambda_1(h).$$

$$\int_0^1 \frac{1}{\lambda_1(h)} dh \leq \int_0^1 \frac{1}{\widetilde{\lambda}_1(h)} dh.$$

$N \geq 2m+1$

$O \in \Omega$.

$$\widetilde{a}(x) = a(x)\exp\left(-\frac{1}{|x|^{\alpha}}\right),\ 0 < \alpha \ll 1.$$

Definition

$$\forall h > 0, \exists \widetilde{v}_h \in W_0^{m,2}(\Omega),\ ||\widetilde{v}_h||^2_{L^2(\Omega)} = h,$$

and

$$\widetilde{\lambda}_1(h) \leq \int_\Omega L\widetilde{v}_h.\widetilde{v}_h + \widetilde{a}(x)|\widetilde{v}_h|^{1+q}\,dx \leq 2\widetilde{\lambda}_1(h).$$

Lemma

$$\widetilde{\lambda}_1(h) \leq C\,h(-\ln h)^{\frac{2m}{\alpha}}$$

$N \geq 2m+1$

$O \in \Omega$.

$$\widetilde{a}(x) = a(x)\exp\left(-\frac{1}{|x|^{\alpha}}\right),\ 0 < \alpha \ll 1.$$

$$C \ \leq \left(C_2 + \frac{2\widetilde{\lambda}_1(h)}{h}\right) \text{meas}\left\{x\ :\ |\widetilde{v}_h|^{1-q}\left(C_2 + \frac{2\widetilde{\lambda}_1(h)}{h}\right) \geq \widetilde{a}(x)\right\}^{\frac{2m}{N}}.$$

$$\forall\varepsilon > 0,\ h = \int_{\Omega} \widetilde{v}_h^2\, dx \geq \varepsilon\, \text{meas}\{x : \widetilde{v}_h^2 \geq \varepsilon\}.$$

$$\varepsilon = h^{\gamma},\ 0 < \gamma < 1.$$

$N \geq 2m+1$

$O \in \Omega$.

$$C' \leq \left(C_2 + \frac{2\widetilde{\lambda}_1(h)}{h}\right) \operatorname{meas}\left\{x \,:\, h^{\frac{\gamma(1-q)}{2}} \left(C_2 + \frac{2\widetilde{\lambda}_1(h)}{h}\right) \geq \widetilde{a}(x)\right\}^{\frac{2m}{N}}.$$

$$C'\, h^{\eta} \geq h^{\frac{\gamma(1-q)}{2}} \left(C_2 + \frac{2\widetilde{\lambda}_1(h)}{h}\right).$$

$$\int_0^1 \frac{1}{\lambda_1(h)} dh \leq \int_0^1 \frac{1}{\widetilde{\lambda}_1(h)} dh < +\infty.$$

$$\Downarrow$$

$$T < +\infty.$$

$u_t - \Delta(u^m) + a(x)\,u^q = 0$ for Neumann boundary condition.

- Y. Belaud (2001).

Theorem

If $\left(\frac{1}{a}\right)^s \in L^1$ *with* $s > \frac{m-1}{1-q}\frac{N}{2}$ *for* $N \geq 2$ *then all solutions vanish in a finite time.*

Proof : adaptation of the KV-method.
Similar result for

$$u_t - \Delta_p u + a(x)\,u^q = 0.$$

What about

$$(u^\theta)_t - \Delta_p u + a(x)\,u^q = 0$$

for $\theta > q$?

Thank you for your attention.

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Table of contents I

Table of contents II

Table of contents III