

Dedicated to the memory of Leonid Volevich

Identifying Functions Determined by Linear Functional Operators

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Abstract. The new identifying problem is formulated for general linear functional operators $\mathcal{P}F = \sum c_j F \circ a_j$ which significantly generalizes in particular the well-known Ulam stability problem. The results obtained can be very useful when processing experimental data of any kind as they enable to determine with high precision the structure of a compactly supported Banach-valued function F by using a rather restricted information concerning $\mathcal{P}F$.

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1. INTRODUCTION

We study the general linear functional operator $\mathcal{P}F = \sum c_j F(a_j)$, where a_j and c_j are given functions and F is a compactly supported Banach-valued function of one variable. The best known and well-studied operators of this kind are the Cauchy operator

$$\mathfrak{C}F := F(x + y) - F(x) - F(y), \quad (x, y) \in \mathbb{R}^2,$$

the Jensen operator

$$\mathfrak{J}F := F(\alpha x + \beta y) - \alpha F(x) - \beta F(y), \quad (x, y) \in \mathbb{R}^2,$$

where α and β are positive constants with $\alpha + \beta = 1$,

and the quadratic operator

$$\mathfrak{Q}F := F(x + y) + F(x - y) - 2F(x) - 2F(y), \quad (x, y) \in \mathbb{R}^2.$$

Quite recently (see [3–6]), significantly more general operators \mathcal{P} were considered from different points of view in connection with a set of problems in diverse fields of analysis (including integral geometry, boundary problems for hyperbolic differential equations, and integral equations). Moreover, one of these operators turned out to be closely related to a purely physical problem of linearized flow of a chemically relaxing gas. Therefore, along with traditional problems concerning existence, uniqueness, and qualitative properties of solutions to the equation $\mathcal{P}F = H$, the question of approximate solutions (i.e., solutions of the inequality $|\mathcal{P}F| < \varepsilon$) becomes especially important. In the simplest situation of the Cauchy operator \mathfrak{C} , this problem (in a slightly different setting) was posed for the first time by Ulam [1] and solved by Hyers [2]. Their result can be formulated as follows. For an arbitrary $\varepsilon > 0$ and $F \in C(\mathbb{R})$, the inequality

$$|\mathfrak{C}F(x, y)| < \varepsilon \quad \text{for all } (x, y) \in \mathbb{R}^2 \tag{1}$$

implies the estimate

$$|F - \varphi|_C < \varepsilon \quad \text{for some } \varphi \in \mathcal{K} \tag{2}$$

with \mathcal{K} the one-dimensional subspace in $C(\mathbb{R})$ formed by the linear functions $F(z) = \lambda z$. In other words, any solution F to problem (1) is an *almost element* of the subspace \mathcal{K} .

Recently (see [7]), it was shown that, if I is a compact interval and $C_{\langle \tau \rangle}(I)$ is a subspace of all functions in $C(I)$ having Hölder smoothness at a single chosen point only, then the same result remains valid for the operators \mathcal{P} , even with nonlinear a_j and nonconstant c_j , under the following significantly weaker condition:

$$|\mathcal{P}F|_{\Gamma} |_{\langle \tau \rangle} < c\varepsilon, \tag{3}$$

where Γ is a curve in $I \times I$ and c is a constant (independent of ε and F).

In this paper we solve the analogous problem for a wide class of the operators \mathcal{P} including the above classical operators $\mathfrak{C}, \mathfrak{J}, \mathfrak{Q}$. Namely, with any such operator \mathcal{P} we associate a corresponding finite-dimensional space $\mathcal{K}_{\mathcal{P}}$ and a sufficiently smooth curve $\Gamma_{\mathcal{P}}$ providing the validity of the implication (3) \Rightarrow (2) with $\Gamma_{\mathcal{P}}$ and $\mathcal{K}_{\mathcal{P}}$ instead of Γ and \mathcal{K} , respectively. Such conclusions may be very useful, in particular, when processing the experimental data of any kind, as they reduce the search of a needed function F whose values at some points of a curve Γ are known, to testing the only elements of the subspace $\mathcal{K}_{\mathcal{P}}$.

2. MAIN NOTIONS AND DEFINITIONS

We begin with some standard notation. The kernel and the range of any linear operator L are denoted by $\ker L$ and $\mathcal{R}L$, respectively. We denote by I the interval $\{t \mid -1 \leq t \leq 1\}$. Given a Banach space B with the norm $|\cdot|_B$ we denote by $|F| = \sup_{t \in I} |F(t)|_B$ the norm in the space $C(I, B)$.

Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a domain, and Γ a one-dimensional submanifold in D . For an arbitrary function $w \in C(D)$ we denote by w_{Γ} the restriction of w to Γ . Let $\zeta: I \rightarrow \Gamma$ be a one-to-one C -map (parameterization of Γ). We set $w_{\Gamma}^{\zeta} = w_{\Gamma} \circ \zeta$. Introduce the operator

$$\mathcal{P}_{\Gamma}: F \rightarrow \sum_{j=0}^N c_{j\Gamma}^{\zeta}(z) F \circ a_{j\Gamma}^{\zeta}(z), \quad z \in I,$$

which can be interpreted as the restriction of the operator \mathcal{P} to Γ and plays the important role in the following.

The problem we deal with in this work is formulated in the most general form as follows.

IDENTIFYING PROBLEM. For a given operator \mathcal{P} of the above form, find a finite-dimensional subspace $\mathcal{K} \subset C(I, B)$ and a subset $\Gamma \subset D$, as small as possible, such that a function $F \in C(I, B)$ for which the values of $\mathcal{P}F$ are sufficiently small at the points of Γ is as close as desired to \mathcal{K} .

The solution of any such problem makes it possible to determine the structure of a function F with a high precision using the sufficiently small information about $\mathcal{P}F$. When dealing with different applied problems, such a possibility may be of the fundamental significance.

Definition 1 Given an operator \mathcal{P} , a term $c_k F \circ a_k$ is called *leading term* of \mathcal{P} if the function a_k maps D onto I .

Definition 2 A curve $\Gamma \subset D$ is called \mathcal{P} -*admissible* if for a leading term $c_k F \circ a_k$ the $c_{k\Gamma}$ does not vanish and the $a_{k\Gamma}$ maps Γ one-to-one onto I .

The following function spaces turn out to be very useful when studying the problems in question. Let $\tau > 0$. If $\tau \in \mathbb{N}$, then, by definition,

$$C_{\langle \tau \rangle}(I, B) = \left\{ F \mid F(t) = \sum_{j=0}^{\tau} b_j t^j + t^{\tau} \varphi(t), \quad t \in I \right\}, \quad (4)$$

where $\varphi \in C(I, B)$ and $\varphi(0) = 0$. If $\tau \notin \mathbb{N}$ and $[\tau]$ stands for the integral part of τ , then the elements of $C_{\langle \tau \rangle}(I, B)$ are all functions

$$F(t) = \sum_{j=0}^{[\tau]} b_j t^j + t_{+}^{\tau} \varphi_{+}(t) + t_{-}^{\tau} \varphi_{-}(t), \quad t \in I, \quad (5)$$

where

$$t_+ = \begin{cases} t, & t \geq 0 \\ 0, & t < 0, \end{cases} \quad \text{and} \quad t_- = \begin{cases} 0, & t > 0 \\ -t, & t \leq 0, \end{cases}$$

$\varphi_{\pm}(t)$ stand for continuous functions on I with $\varphi_+ = \text{const}$ for $t \leq 0$ and $\varphi_- = \text{const}$ for $t \geq 0$. These spaces endowed by the norms

$$|F|_{\langle \tau \rangle} = \sum_{j=0}^{\tau} |b_j| + |\varphi|$$

and

$$|F|_{\langle \tau \rangle} = \sum_{j=0}^{[\tau]} |b_j| + |\varphi_+| + |\varphi_-|,$$

respectively, are Banach spaces.

Remark Alternatively, $C_{\langle 1+r \rangle}(I, B)$ is the space of all B -valued continuous functions on I differentiable at the point $t = 0$, whose derivatives satisfy the Hölder condition of order r at $t = 0$.

3. RESULTS: FORMULATION AND PROOFS

The operator we deal with in the present paper is

$$\mathcal{P}F := F(a_1x + a_2y) + F(b_1x + b_2y) - A_1F(x) - A_2F(y), \quad (x, y) \in D,$$

with

$$D = \{(x, y) \mid |a_1x + a_2y| \leq 1, |b_1x + b_2y| \leq 1, |x| \leq 1, |y| \leq 1\}.$$

We associate with \mathcal{P} the matrix

$$\mathcal{M} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

and assume that the numerical parameters of \mathcal{P} satisfy the following conditions:

- (i) $a_j \geq 1, j = 1, 2; \quad b_1 > 0, b_2 < 1;$
- (ii) $A_1 + A_2 > 2, \quad A_1A_2 > 0.$

Let $\lambda = (\lambda_1, \lambda_2)$ be a unique solution of the equation

$$\mathcal{M}\lambda = (1, \lambda_2)$$

and let ρ be a unique z -solution of the equation

$$A_1\lambda_1^z + \tilde{A}_2\lambda_2^z = 1$$

with $\tilde{A}_2 = A_2 - 1$. Introduce the curve

$$\Gamma = \{(x, y) \in D \mid x = \lambda_1t, y = \lambda_2t; \quad t \in I\}$$

and define the seminorm

$$\langle w \rangle_{\Gamma, \tau} = |w_{\Gamma}^{\zeta}|_{\langle \tau \rangle}, \quad \tau > 0,$$

on the space $C_{\langle \tau \rangle}(D, B)$. The following theorem - the main result of the work - completely solves the above identifying problem.

Theorem 1 1° The curve Γ lies in D and is admissible.

2° The subspace $\mathcal{K}_\rho = (\ker \mathcal{P}_\Gamma) \cap C_{\langle \rho \rangle}(I, B)$ consists of the functions $\sigma_1 t_+^\rho + \sigma_2 t_-^\rho$ with σ_1, σ_2 being arbitrary elements of B , if $\rho \notin \mathbb{N}$, and $\mathcal{K}_\rho = \{\sigma t^\rho\}_{\sigma \in B}$, if $\rho \in \mathbb{N}$.

3° For an arbitrary $\tau > \rho$, the à priori estimates

$$\inf_{\sigma \in B} |F - \sigma t^\rho|_{\langle \tau \rangle} \leq c |\mathcal{P}_\Gamma F|_{\langle \tau \rangle}, \quad F \in C_{\langle \tau \rangle}(I, B),$$

if $\rho \in \mathbb{N}$, and

$$|F|_{\langle \tau \rangle} \leq c |\mathcal{P}_\Gamma F|_{\langle \tau \rangle}, \quad F \in C_{\langle \tau \rangle}(I, B),$$

if $\rho \notin \mathbb{N}$, hold with a constant c not depending on F .

In particular, if $a_1 = a_2 = b_1 = 1$, $b_2 = -1$, and $A_1 = A_2 = 2$, the \mathcal{P} is nothing but the well-known quadratic operator. The reader can easily check that in this situation

$$\lambda_1 = \frac{2}{3}, \quad \lambda_2 = \frac{1}{3}, \quad \rho = 2,$$

and therefore Γ is the straight line

$$x = \frac{2}{3}t, \quad y = \frac{1}{3}t, \quad t \in I,$$

situated in D . By the Theorem, if $\langle \mathcal{P}F \rangle_{\Gamma, \tau} < \varepsilon$, for some $\tau > \rho$, then

$$|F(t) - \sigma t^2|_{\langle \tau \rangle} < c\varepsilon$$

for some σ and c . In other words, in any space $C_{\langle \tau \rangle}(I, B)$, $\tau > 2$, the function $\varphi = \sigma t^2$ is a good approximation for the solution of the inequality $\langle \mathcal{P}F \rangle_{\Gamma, \tau} < \varepsilon$. Thus, this solves the identifying problem for the quadratic operator \mathcal{Q} in the framework of the subspaces $C_{\langle \tau \rangle}$ of the space C .

Proof. 1° To begin with we note that the unique solution $\lambda = (\lambda_1, \lambda_2)$ to the equation

$$\mathcal{M}\lambda = (1, \lambda_2) \tag{6}$$

exists by (i) and satisfies the inequalities

$$0 < \lambda_1 < 1, \quad 0 < \lambda_2 < 1. \tag{7}$$

Indeed, it suffices to check that both λ_1 and λ_2 are positive. Then the relations $\lambda_j < 1$, $j = 1, 2$ follow from

$$a_1 \lambda_1 + a_2 \lambda_2 = 1, \tag{8}$$

by (i). According to (6)

$$b_1 \lambda_1 + (b_2 - 1) \lambda_2 = 0,$$

whence

$$\lambda_2 = \frac{b_2 - 1}{a_1 b_2 - a_2 b_1 - a_1}.$$

By (i), we see that $\lambda_2 > 0$, and, therefore, $\lambda_1 > 0$.

We now show that the curve

$$\Gamma = \{(x, y) \mid x = \lambda_1 t, \quad y = \lambda_2 t; \quad t \in I\}$$

is admissible with respect to the coefficient

$$a(x, y) = a_1 x + a_2 y.$$

In other words, the Γ is situated in D and the a maps Γ one-to-one to I . The first assertion follows from the obvious convexity and central symmetry of D as the point $\Lambda = (\lambda_1, \lambda_2)$ in Γ corresponding to $t = 1$ lies in D by (8). The second one is trivial as

$$a : (\lambda_1 t, \lambda_2 t) \rightarrow t,$$

by (8), for all $t \in I$. This proves 1°.

2° First of all, we establish the correctness of the definition of number ρ . By property (ii) of the parameters A_1, A_2 and property (7) of λ_1, λ_2 , the function

$$\Phi(z) := A_1 \lambda_1^z + \tilde{A}_2 \lambda_2^z$$

is monotone decreasing. The existence of the unique ρ with $\Phi(\rho) = 1$ follows from

$$\Phi(0) = A_1 + \tilde{A}_2 > 1 \quad \text{and} \quad \Phi(\infty) = 0.$$

It is verified directly that for the Γ in question

$$(\mathcal{P}_\Gamma F)(t) = F(t) - A_1 F(\lambda_1 t) - \tilde{A}_2 F(\lambda_2 t), \quad t \in I.$$

To describe the space \mathcal{K}_ρ , we consider separately the cases $\rho \in \mathbb{N}$ and $\rho \notin \mathbb{N}$. We represent an arbitrary function F from \mathcal{K}_ρ in the form (4), if $\rho \in \mathbb{N}$, and in the form (5), if $\rho \notin \mathbb{N}$. Let $\rho \notin \mathbb{N}$ and $\mathcal{P}_\Gamma F = 0$ for the function F in (5) with $\tau = \rho$. Let us find the $b_j, \varphi_+(t)$ and $\varphi_-(t)$ ensuring the relation $F \in \ker \mathcal{P}_\Gamma$. To this end we rewrite the relation $\mathcal{P}_\Gamma F = 0$ in a detailed form

$$\begin{aligned} & \sum_{j=0}^{[\rho]} b_j (1 - A_1 \lambda_1^j - \tilde{A}_2 \lambda_2^j) t^j \\ & - t_+^\rho [\varphi_+(t) - A_1 \lambda_1^\rho \varphi_+(\lambda_1 t) - \tilde{A}_2 \lambda_2^\rho \varphi_+(\lambda_2 t)] \\ & - t_-^\rho [\varphi_-(t) - A_1 \lambda_1^\rho \varphi_-(\lambda_1 t) - \tilde{A}_2 \lambda_2^\rho \varphi_-(\lambda_2 t)] = 0, \end{aligned}$$

for all $t \in I$. By monotonicity of the function $\Phi(z)$ and by the choice of the number ρ all values $1 - A_1 \lambda_1^j - \tilde{A}_2 \lambda_2^j, j = 1, \dots, [\rho]$, are nonzero. It is possible only if *all* $b_j = 0$. Furthermore, supports of the function φ_+ and φ_- are mutually disjoint. Therefore,

$$\varphi_\pm(t) - A_1 \lambda_1^\rho \varphi_\pm(\lambda_1 t) - \tilde{A}_2 \lambda_2^\rho \varphi_\pm(\lambda_2 t) = 0, \quad t \geq 0.$$

Let us show that *both functions φ_+ and φ_- are constants*. By definition of φ_\pm , it suffices to show that $\varphi_\pm = \text{const}$ for $t \geq 0$.

Let

$$M := \max_{0 \leq t \leq 1} \varphi_+(t) = \varphi_+(t_0), \quad t_0 \in [0, 1].$$

As $A_1 \lambda_1^\rho + \tilde{A}_2 \lambda_2^\rho = 1$, both numbers $\varphi_+(\lambda_1 t_0)$ and $\varphi_+(\lambda_2 t_0)$ are equal to M and, therefore, $\varphi_+(\lambda_1^n t_0) = M$, for all natural n . It follows that

$$\varphi_+(0) = M.$$

In just the same way we conclude that

$$\varphi_+(0) = \min_{0 \leq t \leq 1} \varphi_+(t),$$

and this proves that $\varphi_\pm = \text{const}$ for $t \geq 0$.

Returning to representation (5) of an arbitrary element $F \in \mathcal{K}_\rho$, $\rho \notin \mathbb{N}$, we arrive at the following result:

$$\mathcal{K}_\rho = \{\sigma_+ t_+^\rho + \sigma_- t_-^\rho\}_{\sigma_\pm \in B, \sigma_\pm \in B}.$$

Let now $\rho \in \mathbb{N}$. Then, in just the same way, using (4) for the representation of an arbitrary element F from \mathcal{K}_ρ , we arrive at the relations $b_j = 0$ only for $j = 0, \dots, \rho - 1$. However, as $A_1 \lambda_1^\rho + \tilde{A}_2 \lambda_2^\rho = 1$, all terms of the first sum in (4) vanish. As above, we find that for φ in (4)

$$t^\rho [\varphi(t) - A_1 \lambda_1^\rho \varphi(\lambda_1 t) - \tilde{A}_2 \lambda_2^\rho \varphi(\lambda_2 t)] = 0, \quad t \in I.$$

Using the above approach with max and min results in the relation

$$\varphi(t) \equiv \text{const in } I.$$

whence $\varphi(t) \equiv 0$, by the postulated property $\varphi(0) = 0$. Therefore, if $\rho \in \mathbb{N}$, then

$$\mathcal{K}_\rho = \{\sigma t^\rho\}_{\sigma \in B}.$$

This completes the proof of the part 2° of the theorem.

3° The proof is completely based on Proposition 1 in [P5]. For the completeness, we remind it. Let $L : B_1 \rightarrow B_2$ be a closed linear operator between Banach spaces and $\mathcal{K} = \ker L$. If the range $\mathcal{R}(L)$ is closed, then there is a positive constant c such that the à priori estimate

$$\inf_{\varphi \in \mathcal{K}} |F - \varphi|_{B_1} < c |LF|_{B_2} \tag{9}$$

holds for all elements $F \notin \mathcal{K}$. In our case the spaces B_1 and B_2 coincide with $C_{\langle \tau \rangle}(I, B)$ and the operator

$$\mathcal{P}_\Gamma : F(t) \rightarrow F(t) - A_1 F(\lambda_1 t) - \tilde{A}_2 F(\lambda_2 t).$$

plays the role of L . As the closeness of the operator \mathcal{P}_Γ follows from its continuity, to obtain estimate (9) it suffices, by the above Proposition, to prove solvability of the equation

$$F(t) - A_1 F(\lambda_1 t) - \tilde{A}_2 F(\lambda_2 t) = H(t) \tag{10}$$

in $C_{\langle \tau \rangle}(I, B)$ for an arbitrary H from some closed subspace in $C_{\langle \tau \rangle}(I, B)$. To this end we compare the asymptotic expansions of the left hand side in (10) with the analogous expansions

$$\sum_{j=0}^{\rho} b_j t^j + t^\tau h(t) \quad \text{and} \quad \sum_{j=0}^{[\rho]} b_j t^j + t_+^\tau h_+(t) + t_-^\tau h_-(t)$$

of the function H corresponding to the cases $\rho \in \mathbb{N}$ and $\rho \notin \mathbb{N}$, respectively. In both situations we obtain immediately that

$$c_j = b_j / (1 - A_1 \lambda_1^j - \tilde{A}_2 \lambda_2^j) \text{ for } j = 0, \dots, [\rho] - 1,$$

and

$$b_{[\rho]} = c_{[\rho]} \left(1 - A_1 \lambda_1^{[\rho]} - \tilde{A}_2 \lambda_2^{[\rho]} \right).$$

It follows that if $\rho \notin \mathbb{N}$, then all coefficients c_j , $0 \leq j \leq [\rho]$, are defined uniquely by the given function H . But if $\rho \in \mathbb{N}$, then c_j are defined by H only for $0 \leq j \leq [\rho] - 1$, whereas $b_\rho = 0$. The latter means that the range of the operator \mathcal{P}_Γ coincides with the closed subspace of functions $H \in C_{\langle \tau \rangle}$ with $H^{(\rho)}(0) = 0$ (what does not obstruct using Proposition 1 from [7]). In both cases the desired functions φ_\pm and φ from representations (4) and (5) are nothing but solutions of the equations

$$\varphi_\pm(t) - A_1 \lambda_1^\tau \varphi_\pm(\lambda_1 t) - \tilde{A}_2 \lambda_2^\tau \varphi_\pm(\lambda_2 t) = h_\pm(t), \quad t \in I_\pm,$$

and

$$\varphi(t) - A_1\lambda_1^\tau\varphi(\lambda_1t) - A_2\lambda_2^\tau\varphi(\lambda_2t) = h(t), \quad t \in I,$$

in which $\varphi, h \in C(I, B)$, $\varphi_\pm, h_\pm \in C(I_\pm, B)$ and $I_+ = \{t \in I, t \geq 0\}, I_- = \{t \in I, t \leq 0\}$. All these equations have the same operator form

$$f - \mathcal{A}f = h,$$

where

$$\mathcal{A}: f(t) \rightarrow A_1\lambda_1^\tau f(\lambda_1t) + \tilde{A}_2\lambda_2^\tau f(\lambda_2t)$$

is the linear operator in $C(I_\pm, B)$ with the norm

$$\|\mathcal{A}\| \leq A_1\lambda_1^\tau + \tilde{A}_2\lambda_2^\tau < 1.$$

The latter inequality follows by the choice of the value ρ and by (7). Applying the classical result in functional analysis (the invertibility of the operator $E - \mathcal{A}$, E the identical operator) results in the unique solvability of equations in question for an arbitrary functions $h_\pm \in C(I_\pm, B)$. Thus, we have proved solvability in the space $C_{\langle\tau\rangle}(I, B)$ of equation (10) for all elements H from a close subspace in $C_{\langle\tau\rangle}(I, B)$ and hence the validity of a priori estimate (9). It remains to establish that the subspace \mathcal{K} in (9) coincides with $\{0\}$, when $\rho \notin \mathbb{N}$, and coincides with \mathcal{K}_ρ , when $\rho \in \mathbb{N}$. Let $\mathcal{K} = \ker \mathcal{P}_\Gamma \cap C_{\langle\tau\rangle}(\mathcal{I}, \mathcal{B})$. If $\rho \notin \mathbb{N}$, then $\tau > \rho$ and hence $\mathcal{K} \subset \mathcal{K}_\rho$. It follows, by 2°, that any function $\psi \in \mathcal{K}$ can be represented in a form

$$\psi = \sigma_1 t_+^\rho + \sigma_2 t_-^\rho$$

with σ_1, σ_2 from B . On the other hand, being an element of \mathcal{K} this function ψ has a form $\psi = t^\tau \zeta(t)$ with $\zeta \in C(I, B)$. However, it is clear that the relation

$$t^\tau \zeta(t) = \sigma_1 t_+^\rho + \sigma_2 t_-^\rho, \quad t \in I,$$

may be true under condition $\tau > \rho$ only if $\sigma_1 = \sigma_2 = 0$. Therefore, $\mathcal{K} = \{0\}$.

Let $\rho \in \mathbb{N}$. It is obvious, that $\mathcal{K} \subset \mathcal{K}_\rho$, as $\tau > \rho$. On the other hand, $\mathcal{K}_\rho \subset \mathcal{K}$, because $\mathcal{K}_\rho = \{\sigma t^\rho\}_{\sigma \in B}$, by 2°, and $t^\rho \in C^\infty$. Therefore, $\mathcal{K} = \mathcal{K}_\rho$ and this completes the proof of the theorem.

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