Russian Journal of Mathematical Physics, Vol. 11, No. 4, 2004, pp. 456–473.

To Leonid Volevich on the occasion of his 70th birthday

Boundary Problems for Higher Order Hyperbolic Differential Equations in Bounded Domains

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Received October 17, 2004; revised November 3, 2004

Abstract. Arbitrary third-order strictly hyperbolic partial differential equations with constant coefficients in the space $\mathbb{R}^2$ are studied. To any such equation we canonically assign several domains $D \subset \mathbb{R}^2$ and additional boundary operators B such that the (quasi) boundary problem

$$Pu = f \quad \text{in} \quad D, \qquad Bu = g \quad \text{on} \quad \mathcal{M},$$

turns out to be well-posed in the scale of the spaces C^k. The set $\mathcal{M}$ of dimension $\dim \mathcal{M} = 1$ is either the boundary ∂D or some curve on D containing a part of ∂D. The specifics of these problems are that (α) only one boundary condition determines a well-posed problem for a third-order differential operator; (β) a boundary problem for a broad class of *hyperbolic* differential equations is well posed, although the boundary condition is given on the *entire* boundary ∂D. It is also of interest that the problems in question turn out to be *equivalent* to some functional equations which was never investigated earlier.

1. INTRODUCTION

Very little is known yet concerning boundary problems for general partial differential equations. Although the local solvability theory of partial differential equations is advanced enough at present, the questions of the existence of well-posed boundary problems for sufficiently broad classes of differential operators has attracted much less attention. This is especially true of partial differential equations in bounded domains where (with the exception of elliptic equations) a list of well-studied boundary problems seems to be sporadic (the Cauchy and the Goursat problems in characteristic cones, the mixed problem for hyperbolic operators, the latter problem for parabolic operators, the Tricomi problem, and some other tasks). It is clear that at present the general problem of assigning to each differential operator P a domain $D \subset \mathbb{R}^n$ and an additional operator B such that the problem

$$Pu = f \quad \text{in} \quad D, \qquad Bu = g \quad \text{on} \quad \mathcal{M} \subset \overline{D}, \tag{0}$$

is well posed appears to be a utopian dream. However, advances in the study of such problems in the framework of some broad classes of operators does not look hopeless. The present paper relates to this very circle of problems.

In the space $\mathbb{R}^2$, we consider an arbitrary strictly hyperbolic differential operator of third order. To this operator we canonically assign domains D_1 and D_2 of two different types and relate these domains to the corresponding (quasi)boundary operators B_1 and B_2, respectively. As a main result of the paper, we prove that (under some topological restrictions on the domains D_j and the curves $\mathcal{M}_j$) the corresponding problem (0) is uniquely solvable in the scale of spaces C^k. It is remarkable that one of these problems breaks two folklore taboos: *one* boundary condition on the *entire boundary* ∂D for a *hyperbolic* equation of *third order* defines a well-posed boundary problem.

We also note that, instead of integral and pseudodifferential equations which are traditional in the theory of boundary problems, the main technical role in the study of the problems in question is played by *functional equations* (which never arose before in the long-developed theory of these equations). The same functional equations also appear (unexpectedly!) when studying some geometric problems recalling problems in integral geometry and by no means related to partial differential equations (see[P3]).

For the sake of brevity, we restrict ourselves in this paper to homogeneous differential operators with constant coefficients. Nonhomogeneous operators, as well as operators with variable coefficients (not of the most general type), are considered in a paper in preparation.

2. MAIN NOTATION, DEFINITIONS, AND THE FORMULATION OF RESULTS

In this paper, we consider linear homogeneous differential operators $P(\partial)$ with real constant coefficients in the space $\mathbb{R}^2$ of the variables (x, y). Here $\partial = (\partial_x, \partial_y) = (\partial/\partial x, \partial/\partial y)$ is the operator gradient in $C^1(\mathbb{R}^2)$ and $\xi = (\xi_1, \xi_2)$ is the symbol of this operator. Thus, the general form of any mth order operator $P(\partial)$ is

$$P(\partial) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha, \quad \alpha = (\alpha_1, \alpha_2),$$

and its symbol is the polynomial

$$P(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha.$$

To formulate the problems, we deal with, in invariant form we shall now remind the classical definition of strictly hyperbolic differential operators (of order m in the space $\mathbb{R}^n$) and then give an equivalent description of these operators in the space $\mathbb{R}^2$ by using the geometric language (see Proposition 1). This makes it possible to canonically assign, to any *third-order* operator $P(\partial)$ in $\mathbb{R}^2$ of this kind, some bounded domains D and to formulate new boundary problems for $P(\partial)$ in D. The solution of these problems is given in Sections 3 and 4.

We remind that an mth order homogeneous operator $P(\partial)$ in $\mathbb{R}^n$ is said to be *strictly hyperbolic with respect to a vector* $\overline{N}$ if

a) $\sum_{j=1}^{n} \left((\partial/\partial\xi_j)P(\xi)\right)^2 \neq 0 \quad \text{for } \xi \neq 0, \qquad P(\overline{N}) \neq 0,$ and

b) the polynomial

$$\tau \to P_m(\xi + \tau\overline{N})$$

has m real distinct roots in τ for an arbitrary real vector ξ not proportional to $\overline{N}$ (see [H]).

In what follows, it will be convenient to accept the following definition.

Definition. An operator $P(\partial)$ of the above form is said to be *strictly hyperbolic in* $\mathbb{R}^n$ if it is strictly hyperbolic with respect to some vector $\overline{N}$.

It turns out that all strictly hyperbolic differential operators of order m with constant coefficients in the space $\mathbb{R}^2$ admit a very transparent geometric description.

Proposition 1. *Let* $\bar{l}_1, \dots, \bar{l}_m$ *be arbitrary mutually transversal vector fields on* $\mathbb{R}^2$. *Then the* m*th order differential operator of the form*

$$P(\partial) = (\partial/\partial\bar{l}_1) \cdots (\partial/\partial\bar{l}_m) \tag{1}$$

is strictly hyperbolic. The converse is also true: *any* m*th order strictly hyperbolic operator* P *can be represented in the form* (1) *for some mutually transversal vector fields* $\bar{l}_1, \dots, \bar{l}_m$. *These vector fields are determined uniquely up to scalar factors and indexing.*

Proof. To prove the direct assertion, we must show that the polynomial

$$P_m(\xi) = (\bar{l}_1 \cdot \xi)(\bar{l}_2 \cdot \xi) \cdots (\bar{l}_m \cdot \xi), \quad \xi \in \mathbb{R}^2,$$

satisfies the above conditions a) and b) for some vector $\overline{N}$.

To prove the first inequality in a), assume that it fails and that

$$\operatorname{grad} P_m(\xi) = 0 \tag{A}$$

for some nonzero vector $\xi \in \mathbb{R}^2$. It is clear that

$$\operatorname{grad} P_m(\xi) = \bar{l}_1(\bar{l}_2 \cdot \xi) \cdots (\bar{l}_m \cdot \xi) + \cdots + \bar{l}_m(\bar{l}_1 \cdot \xi) \cdots (\bar{l}_{m-1} \cdot \xi). \tag{B}$$

Multiplying this relation by the vector ξ, we see, by virtue of (A), that

$$(\bar{l}_1 \cdot \xi) \cdots (\bar{l}_m \cdot \xi) = 0.$$

Only one of the factors $\bar{l}_j \cdot \xi$ is equal to zero because, since the vectors $\{\bar{l}_k\}_1^m$ are mutually transversal, the relations $\bar{l}_j \cdot \xi = \bar{l}_k \cdot \xi = 0$, $k \neq j$, are impossible for $\xi \neq 0$. Let $\bar{l}_1 \cdot \xi = 0$. It follows from (A) and (B) that

$$\bar{l}_1(\bar{l}_2 \cdot \xi) \cdots (\bar{l}_m \cdot \xi) = 0,$$

and consequently one has $\bar{l}_j \cdot \xi = 0$ for some $j > 1$. As was explained above, this means that $\xi = 0$, which contradicts the assumption $\xi \neq 0$.

Further, to find a desired vector $\overline{N}$, we first assume that one of the vectors $\bar{l}_1, \dots, \bar{l}_m$, say, $\bar{l}_1$, is not orthogonal to any $\bar{l}_j$, $1 \leqslant j \leqslant m$. If we take $\overline{N} = \bar{l}_1$, then the second relation in a) is trivial. Concerning property b), we must show that all roots in τ of the polynomial

$$P_m(\xi + \tau \bar{l}_1) = \prod_{j=1}^{m} \bar{l}_j \cdot (\xi + \tau \bar{l}_1)$$

are real and distinct if $\bar{l}_1 \cdot \xi = 0$. All these roots are

$$\tau_j = -\bar{l}_j \cdot \xi / \bar{l}_j \cdot \bar{l}_1, \quad j = 1, \dots, n.$$

If two of them, say, τ_j and τ_k, coincide, then the vector ξ in question is orthogonal to both vectors

$$\bar{l}_1 \quad \text{and} \quad \bar{l} = \bar{l}_j / \bar{l}_j \cdot \bar{l}_1 - \bar{l}_k / \bar{l}_k \cdot \bar{l}_1.$$

This means that these vectors are proportional, which is impossible because

$$\bar{l}_1 \cdot \bar{l}_1 > 0 \quad \text{and} \quad \bar{l}_1 \cdot \bar{l} = 0.$$

It remains to find a needed vector $\overline{N}$ provided that, for any vector $\bar{l}_j$, there is a (unique) orthogonal vector $\bar{l}_{k_j}$. This can happen only if the number m is even. Let

$$\bar{l}_1 \cdot \bar{l}_2 = 0, \ \dots, \ \bar{l}_{m-1} \cdot \bar{l}_m = 0.$$

Introduce the vectors

$$\overline{N}_k = \bar{l}_1 + \varepsilon_k \bar{l}_2, \quad k = 3, \dots, m.$$

It is clear that $\overline{N}_k \cdot \bar{l}_k = 0$ only if

$$\varepsilon_k = -\bar{l}_1 \cdot \bar{l}_k / \bar{l}_2 \cdot \bar{l}_k, \qquad k = 3, \dots, m.$$

Thus, the vector $\overline{N} = \bar{l}_1 + \varepsilon \bar{l}_2$, where ε differs from each ε_k, is not orthogonal to any vector in the list $\bar{l}_1, \dots, \bar{l}_m$. Consequently,

$$P_m(\overline{N}) \neq 0.$$

To complete the proof of the direct assertion of our proposition, it remains to show that property b) is satisfied for the roots in τ of the polynomial $P_m(\xi+\tau\overline{N})$. However, this can be proved by literally repeating the corresponding part of the proof in the case of $\overline{N}=\bar{l}_1$.

Let us now turn to the proof of converse assertion in the proposition. Let $P(\partial)$ be a homogeneous operator which is strictly hyperbolic with respect to some vector $\overline{N}$ and let $\overline{M}\neq 0$ be a vector orthogonal to $\overline{N}$. Denote by ω the matrix whose columns are $\overline{M}$ and $\overline{N}$. Then the relation

$$P(\xi)=P(\lambda_1\overline{M}+\lambda_2\overline{N})$$

is valid for an arbitrary nonzero vector $\xi\in\mathbb{R}^2$ with

$$(\lambda_1,\lambda_2)=\omega^{-1}\xi. \tag{2}$$

By definition, there are distinct real numbers $\alpha_1,\alpha_2,\dots,\alpha_m$ such that

$$P(\xi)=\lambda_1^m P\big(\overline{M}+(\lambda_2/\lambda_1)\overline{N}\big)=\lambda_1^m A\prod_{j=1}^{m}(\lambda_2/\lambda_1-\alpha_j)=A\prod_{j=1}^{m}(\lambda_2-\alpha_j\lambda_1).$$

Since $\omega^{-1}\,\overline{N}=(0,1)$, it follows from (2) that the constant A here is equal to $P(\overline{N})$, and hence

$$P(\xi)=P(\overline{N})\prod_{j=1}^{m}\overline{L}_j\cdot\lambda,$$

where $\overline{L}_j=(-\alpha_j,1)$. Thus, the desired representation

$$P(\partial)=P(\overline{N})\prod_{j=1}^{m}\partial/\partial\bar{l}_j$$

is proved with

$$\bar{l}_j=(\omega^t)^{-1}\overline{L}_j,\qquad j=1,2,\dots,m.$$

To prove that this representation is unique (in the above sense), we take two representations

$$P(\xi)=\prod_{j=1}^{m}\bar{l}_j\cdot\xi\quad\text{and}\quad P(\xi)=\prod_{j=1}^{m}\overline{m}_j\cdot\xi, \tag{3}$$

where the vectors $\{\bar{l}_j\}_{j=1}^m$, as well as the vectors $\{\overline{m}_j\}_{j=1}^m$, are mutually transversal. Assume that some vector $\bar{l}_k$ is not proportional to any vector $\overline{m}_j$. Then every vector $\bar{l}$ orthogonal to $\bar{l}_k$ is not orthogonal to any $\overline{m}_j$. Therefore, substituting $\bar{l}$ for ξ into both representations of $P(\xi)$ in (3) yields the contradictory relations $P(\bar{l})=0$ and $P(\bar{l})\neq 0$. This completes the proof of Proposition 1.

Let $P(\partial)$ be an arbitrary third-order strictly hyperbolic differential operator in the space $\mathbb{R}^2$. By Proposition 1, this operator admits a unique representation of the form

$$P=(\partial/\partial\bar{l}_1)(\partial/\partial\bar{l}_2)(\partial/\partial\bar{l}_3)$$

with mutually transversal vectors $\bar{l}_j$, $j=1,2,3$. Take an arbitrary point O and consider six semitrajectories of the vector fields $\bar{l}_1$, $\bar{l}_2$, and $\bar{l}_3$ beginning at O. All these rays are nothing but characteristics of the operator P. Take an arbitrary triple of neighboring rays l_1, l_2, and l_3 lying on trajectories of the vector fields $\bar{l}_1$, $\bar{l}_2$, and $\bar{l}_3$, respectively. Let A_1, A_2, and A_3 be arbitrary points on these rays, respectively. We can assume without loss of generality that the ray l_2 lies between l_1

and l_3 (i.e., $\bar{l}_2 = \lambda_1\bar{l}_1 + \lambda_3\bar{l}_3$ with $\lambda_1\lambda_3 > 0$). Then we determine the domain D_1 as a characteristic parallelogram $OA_1O'A_2$. This means that the sides A_1O' and A_2O' are trajectories of the vector fields $\bar{l}_2$ and $\bar{l}_1$, respectively.

In the same situation, we determine the domain D_2 as the curvilinear triangle A_1OA_3 whose third side A_1A_3 is an arbitrary nonsingular C^2-curve Γ (see Fig. 1).

Now everything is ready to formulate the boundary problems we associate with an arbitrary strictly hyperbolic differential operator $P(\partial)$.

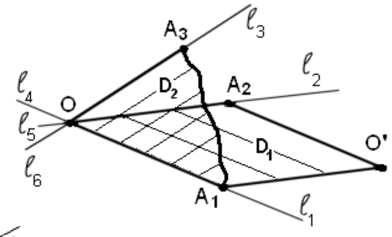

Fig. 1.

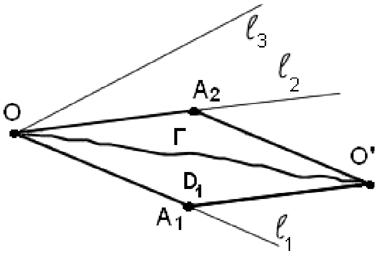

Fig. 2.

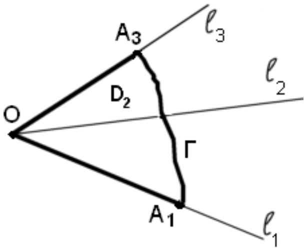

Fig. 3.

First partly characteristic boundary problem. Let D_1 be an arbitrary domain associated with $P(\partial)$. Let $\Gamma = OO'$ be a curvilinear diagonal of D_1, which is assumed to be a nonsingular C^2-curve satisfying the following geometric conditions (see Fig. 2):

(i) the curve Γ is $\bar{l}_1$- and $\bar{l}_2$-convex;[1]

(ii) the curve Γ is transversal to the vector fields $\bar{l}_1$ and $\bar{l}_2$ at the point O;

(iii) the curve Γ is transversal to the vector field $\bar{l}_3$ at any point, and it has no common points with the open intervals OA_1 and OA_2.

Conditions (i)–(iii) are not independent. It can readily be seen that (iii) is a consequence of (i) and (ii). The problem in question looks as follows.

For arbitrarily given functions f in $\overline{D}_1$ and g on $\mathcal{M} = OA_1 \cup OA_2 \cup \Gamma$, find a function u in $\overline{D}_1$ such that

$$\begin{array}{rcl} P(\partial)u & = & f \quad \text{in } D_1, \\ u & = & g \quad \text{on } \mathcal{M}. \end{array} \tag{4}$$

Second partly characteristic boundary problem. This is the problem we associate with domains D_2 (see Fig. 3). Any such domain D_2 is assumed to satisfy the following topological conditions:

(j) the domain D_2 is $\bar{l}_1$- and $\bar{l}_2$-convex;

(jj) the projections

$$\pi_1 : \overline{D}_2 \mapsto l_1 \quad \text{along the vector field } \bar{l}_3,$$

$$\pi_2 : \overline{D}_2 \mapsto \Gamma \quad \text{along the vector field } \bar{l}_2,$$

$$\pi_3 : \overline{D}_2 \mapsto l_3 \quad \text{along the vector field } \bar{l}_1$$

satisfy the condition

$$\pi_1 D_2 = OA_1 \quad \text{and} \quad \pi_3 D_2 = OA_3;$$

(jjj) the curve Γ is transversal to the vector field $\bar{l}_3$.

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[1] For a given smooth vector field $\bar{l}$ in $\mathbb{R}^n$, a set $\omega \subset \mathbb{R}^n$ is said to be $\bar{l}$-*convex* if, for arbitrary points p, q in ω lying on a trajectory of $\bar{l}$, the trajectory (p, q) belongs to ω.

The corresponding problem is formulated in the following way.

Given arbitrarily functions f *in* $\overline{D}_2$ *and* g *on* $\partial D_2 = OA_1 \cup OA_3 \cup \Gamma$, *find a function* u *in* $\overline{D}_2$ *such that*

$$\begin{aligned} P(\partial)u = f \quad &\text{in } \ D_2, \\ u = g \quad &\text{on } \ \partial D_2. \end{aligned} \tag{5}$$

We note that specific trait of the first (quasi) boundary problem is that the second condition includes some *a priori* information about values of an unknown function in the domain D_1. The specifics of the second problem are of a quite different type. This is a typical boundary problem, and it assigns only *one* boundary condition to a *third* order differential operator. On the other hand, this condition is given on the *entire* boundary, although the operator P is *not elliptic*.

Remark. Note that the geometric language used above to describe the operators $P(\partial)$ that we deal with makes it possible to formulate similar problems for strictly hyperbolic differential operators with variable coefficients. The corresponding results (also related to nonhomogeneous operators $P(x, \partial)$) will be published later; the paper is in preparation.

Proceeding with the formulation of the result related to problem (4), we first establish some compatibility conditions which must hold for the function g. Denote by g_1, g_2, and g_3 the restrictions of the function g to the curves OA_1, OA_2, and Γ, respectively. If

$$\{x = x_j(t),\, y = y_j(t);\ 0 \leqslant t \leqslant 1\}, \quad j = 1, 2, 3,$$

are some smooth parametric representations of these curves and $O = (x_j(0), y_j(0))$ for any j, then

$$u\big(x_j(t), y_j(t)\big) = g_j(t), \qquad 0 \leqslant t \leqslant 1,$$

for $j = 1, 2, 3$. Denote by $\overline{\tau}_\Gamma$ the unit tangent vector of Γ at the point O. By the transversality of the vectors $\overline{l}_1$ and $\overline{l}_2$, there are positive constants λ_1 and λ_2 such that

$$\overline{\tau}_\Gamma = \lambda_1 \overline{l}_1 + \lambda_2 \overline{l}_2.$$

This immediately leads to the first compatibility condition

$$g_3'(O) = \lambda_1 g_1'(O) + \lambda_2 g_2'(O). \tag{6}$$

The other condition

$$g_1(O) = g_2(O) = g_3(O) = u(O) \tag{7}$$

is a consequence of the coincidence of the curves OA_1, OA_2, and Γ at the point O.

Theorem 2. *Let* $P(\partial)$ *be an arbitrary homogeneous third-order strictly hyperbolic differential operator with constant coefficients in* $\mathbb{R}^2$. *Let* $\overline{l}_1, \overline{l}_2$, *and* $\overline{l}_3$ *be vector fields appearing in representation* (1), *and let* $\overline{l}_2 = \mu_1 \overline{l}_1 + \mu_3 \overline{l}_3$ *with* $\mu_1, \mu_3 > 0$. *Then, for arbitrary functions* $f \in C(\overline{D}_1)$ *and* $g \in C^2(\mathcal{M})$ *satisfying conditions* (6) *and* (7), *there is a unique solution* $u \in C^2(\overline{D}_1)$ *of problem* (4), *and the inverse operator* $(f, g) \mapsto u$ *is bounded. If* $(f, g) \in C^k(\overline{D}_1) \times C^{2+k}(\mathcal{M})$, *then* $u \in C^{2+k}(\overline{D}_1)$, $k = 1, 2, \ldots$

The relation $g \in C^l(\mathcal{M})$ means that the three functions g_1, g_2, and g_3 are l times continuously differentiable on their domains.

To formulate the result related to problem (5), we need some new concepts and notations.

We introduce two mappings on Γ (i.e., mappings from Γ into itself),

$$\delta_1 = \pi_2 \circ \pi_1, \quad \delta_2 = \pi_2 \circ \pi_3,$$

and consider the noncommutative semigroup Φ_δ of mappings on Γ which is generated by δ_1 and δ_2. The elements of Φ_δ are the mappings in Γ of the form

$$\delta_J = \delta_{j_n} \circ \cdots \circ \delta_{j_1},$$

where $J = (j_1, \dots, j_n)$ is an arbitrary multi-index with all j_ks taking the values 1 and 2 ($\circ$ stands for the composition of mappings).

An *orbit in* Γ generated by Φ_δ is defined as a sequence of points $q_1, \dots, q_n, \dots$ of Γ such that

$$q_{k+1} = \delta_{j_k}(q_k), \quad k = 1, 2, \dots \tag{8}$$

Denote by $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ the set of characteristic points in Γ with respect to the differential operator P. In other words,

$$q \in \mathcal{T}_j \quad \text{if} \quad T_q(\Gamma) \ni \bar{l}_j(q), \quad j = 1, 2,$$

where $T_q(\Gamma)$ is the tangent space of Γ at the point q.

An orbit $(q_1, q_2, \dots, q_n, \dots)$ is said to be $\mathcal{T}$*-proper* if in (8) we have

$$\delta_{j_k} = \delta_1 \quad \text{for} \quad q_k \in \mathcal{T}_1 \qquad \text{and} \qquad \delta_{j_k} = 2 \quad \text{for} \quad q_k \in \mathcal{T}_2.$$

Thus, when moving in D_2 along a $\mathcal{T}$-proper orbit, we leave each point q_k in a direction transversal to Γ.

Finally, an orbit $(q_1, \dots, q_n)$ is said to be *periodic* if $q_1 = q_n$.

Definition. We define the set $\mathfrak{N}_\delta^{\mathcal{T}}$ as the collection of all $\mathcal{T}$-proper periodic orbits lying in the characteristic set $\mathcal{T}$.

Example. If the curve Γ does not contain characteristic points with respect to the operator $P(\partial)$, then $\mathfrak{N}_\delta^{\mathcal{T}} = \varnothing$.

Before formulating the result related to problem (5), note that some compatibility conditions are necessary for a solution of this problem to exist, namely,

$$g_1(O) = g_2(O), \quad g_1(A_1) = g_3(A_1), \quad g_2(A_2) = g_3(A_2), \tag{9}$$

where g_1, g_2, and g_3 are the restrictions of a given function g to the sides OA_1, OA_3, and Γ, respectively.

Theorem 3. *Assume that the vector fields $\bar{l}_1$, $\bar{l}_2$, and $\bar{l}_3$ are constant and at least one of the characteristic sets $\mathcal{T}_1$ and $\mathcal{T}_2$ is finite. Then problem* (5) *has a unique solution $u \in C^{2+k}(\overline{D}_2)$ for arbitrary functions $f \in C^k(\overline{D}_2)$ and $g \in C^{2+k}(\partial D_2)$ satisfying condition* (9) *if and only if $\mathfrak{N}_\delta^{\mathcal{T}} = \varnothing$.*

The proofs of these theorems are given in the next two sections. Theorem 2 is the main new result of this paper, whereas Theorem 3 (in much more general form) is already known (see [P1]). Nevertheless, for the reader's convenience, the latter result is included in the present paper. This enables the reader to compare, in a self-contained form, not only the statements of problems in question but also the rather different methods of their solution in the two cases. The full proof of Theorem 3 is too long; it uses results related to a new class of noncommutative dynamical systems with two generators introduced recently in [P2]. However, in the simplest situation we deal with, the proof given in Section 4 becomes considerably shorter, although the specific features of the proof are preserved.

3. PROOF OF THEOREM 2

It is clear that there is a linear change of variables on $\mathbb{R}^2$ which reduces problem (4) to the problem

$$\begin{aligned} (\partial_x - \partial_y)\partial_x \partial_y u &= f \quad \text{in } D_1, \\ u &= g \quad \text{on } \mathcal{M}. \end{aligned} \tag{10}$$

(For convenience, we preserve the previous notation for the variables, the domain, and the functions.) Here

$$D_1 = \{(x, y) |\ 0 \leqslant x \leqslant X,\ 0 \leqslant y \leqslant Y\},$$

and the set $\mathcal{M}$ consists of the three parts,

$$\mathcal{M}_1 = \{(x,y)|\ 0 \leqslant x \leqslant X,\ y = 0\}, \quad \mathcal{M}_2 = \{(x,y)|\ x = 0,\ 0 \leqslant y \leqslant Y\},$$

and

$$\Gamma = \{(x,y)|\ x = \delta_1(t),\ y = \delta_2(t),\ 0 \leqslant t \leqslant 1\}.$$

The relations $x = \delta_1(t),\ y = \delta_2(t)$ determine a parametric representation of the curve Γ, and the functions δ_1 and δ_2 satisfy the following conditions dictated by geometric properties of Γ:

$$\begin{aligned} &\text{(k)} && \delta_1(0) = \delta_2(0) = 0; \quad \big(\delta_1(1), \delta_2(1)\big) = (X, Y); \\ &\text{(kk)} && \delta_1(t)\delta_2(t) > 0, \quad 0 < t \leqslant 1; \qquad \delta_1'(0)\delta_2'(0) > 0; \\ &\text{(kkk)} && \delta_1'(t) + \delta_2'(t) > 0 \quad \text{and} \quad \delta_1'(t)\delta_2'(t) \geqslant 0. \end{aligned}$$

To prove this fact, we must take into account the fact that the properties (i) and (ii) of Γ postulated above remain valid under linear nonsingular transformations of $\mathbb{R}^2$. Relations (k) are obvious. Relation (kk) follows from properties (i) and (ii) of Γ and, finally, relations (kkk) are nothing but property (i) of Γ. Indeed, since we have now $\overline{l}_1 = (1,0)$, $\overline{l}_2 = (0,1)$, and $\overline{l}_3 = (1,-1)$, the first inequality in (kkk) is equivalent to the transversality of an arbitrary tangent vector $\overline{\tau}(t) = \big(\delta_1'(t), \delta_2'(t)\big)$ of Γ to the vector $\overline{l}_3$, whereas the other inequality follows from (i). The first relation in (kkk) enables us to assume (without loss of generality) that the functions $\delta_1(t)$ and $\delta_2(t)$ satisfy the condition

$$\text{(k)}' \qquad \delta_1(t) + \delta_2(t) = t.$$

Let

$$g = G_1(x) \quad \text{on} \quad \mathcal{M}_1, \quad g = G_2(y) \quad \text{on} \quad \mathcal{M}_3, \quad g = G_3(x,y) \quad \text{on} \quad \Gamma.$$

Introduce the function $\varphi(t) = G_3\big(\delta_1(t), \delta_2(t)\big),\ 0 \leqslant t \leqslant 1$.

By (7), the first necessary compatibility condition related to the functions G_1, G_2, and φ is

$$G_1(0) = G_2(0) = \varphi(0). \tag{11}$$

To formulate condition (6) corresponding to the operator $(\partial_x - \partial_y)\partial_x\partial_y$, note that we now have $\overline{l}_1 = (1,0)$, $\overline{l}_2 = (0,1)$, and $\overline{\tau}_\Gamma = \big(\delta_1'(0), \delta_2'(0)\big)$. This means that the coefficients λ_1 and λ_2 in (6) are $\delta_1'(0)$ and $\delta_2'(0)$, respectively. Hence, the second compatibility condition becomes

$$\varphi'(0) = \delta_1'(0)G_1'(0) + \delta_2'(0)G_2'(0). \tag{12}$$

As the first step in the proof of Theorem 2, *we shall prove this theorem for* $f = 0$. Introduce the function

$$u(x,y) = \int_0^x \Big(\int_0^y F(s+t)dt\Big)ds + G_1(x) + G_2(y) - G_1(0)$$

in D_1, where F is an arbitrary continuous function on the closed interval $I_r = (0, X+Y)$. By (11), the function $u(x,y)$ satisfies the boundary conditions $u = g$ on $\mathcal{M}_1$ and $\mathcal{M}_2$. On the other hand, this function solves the homogeneous differential equation in (10) (if the operator $\partial_x - \partial_y$ is treated as a vector field $(1,-1) \cdot (\partial_x, \partial_y)$).

To solve problem (10) with $f = 0$, it remains to choose a continuous function F such that the function $u(x,y)$ in question satisfies the condition

$$u(x,y) = G_3(x,y) \quad \text{on} \quad \Gamma.$$

Thus, we arrive at the integral equation

$$\int_0^{\delta_1(t)} \Big(\int_0^{\delta_2(t)} F(x+y)dy\Big)dx = H(t), \quad 0 \leqslant t \leqslant 1, \tag{13}$$

for an unknown function $F \in C(I_r)$. Here

$$H(t) = -G_1\big(\delta_1(t)\big) - G_2\big(\delta_2(t)\big) + G_3\big(\delta_1(t), \delta_2(t)\big) + G_1(0).$$

It is of importance that the function $H(t)$ satisfies the conditions

$$H(0) = 0 \quad \text{and} \quad H'(0) = 0. \tag{14}$$

This follows immediately from the compatibility conditions (11) and (12).

The following lemma is the crucial point of the proof of Theorem 2.

Lemma 4. *For an arbitrary function $H(t) \in C^2(I)$ with $I = \{t \mid 0 \leqslant t \leqslant 1\}$ satisfying condition* (14), *there is a unique solution $F(z) \in C(I_r)$ of equation* (13).

Remark. Introduce the linear operator

$$\mathcal{B} : F(z) \to \int_0^{\delta_1(t)} \Big(\int_0^{\delta_2(t)} F(x+y)dy \Big) dx.$$

It can readily be seen that the relation $\mathcal{B}F \in C^2(I)$ holds for all functions $F \in C(I_r)$. Indeed,

$$\begin{aligned}(\mathcal{B}F)'(t) &= \delta_1'(t) \int_0^{\delta_2(t)} F(\delta_1(t)+y)dy + \delta_2'(t) \int_0^{\delta_1(t)} F(\delta_2(t)+x)dx \\ &= \delta_1'(t) \int_{\delta_1(t)}^{(\delta_1+\delta_2)(t)} F(z)dz + \delta_2'(t) \int_{\delta_2(t)}^{(\delta_1+\delta_2)(t)} F(z)dz.\end{aligned}$$

In this form, the existence of the function $(\mathcal{B}F)''(t)$ is clear. Thus, the lemma asserts that the integral operator $\mathcal{B}$ determines an isomorphism between the space $C(I_r)$ and the subspace of functions in $C^2(I)$ satisfying the boundary conditions (14).

Proof. We first note that, due to (k), for an arbitrary continuous function F, the function $(\mathcal{B}F)(t)$ satisfies the boundary condition (14). This makes it possible to conclude that, when differentiating relation (13) twice and using relation (k)$'$, we arrive (after some identical transformations) at an *equivalent* equation of the form

$$F(t) - LF(t) + KF(t) = H''(t). \tag{15}$$

Here L and K are linear operators on the space $C(I)$, namely,

$$L\colon F(t) \mapsto a_1(t)F\big(\delta_1(t)\big) + a_2(t)F\big(\delta_2(t)\big)$$

and

$$K\colon F(t) \mapsto \delta_1''(t)\int_{\delta_1(t)}^t F(y)dy + \delta_2''(t)\int_{\delta_2(t)}^t F(y)dy,$$

where

$$a_1(t) = \big(\delta_1'(t)\big)^2 \quad \text{and} \quad a_2(t) = \big(\delta_2'(t)\big)^2.$$

Let us show that the index of the operator $E - L + K$, where E is the identical mapping, is equal to 0. To begin with, note that

$$\delta_1'(t) + \delta_2'(t) = 1, \quad t \in I,$$

by (k)$'$. Therefore, if $\delta_1'(t)\delta_2'(t) \neq 0$ at any point $t \in I$, then the inequality

$$a_1(t) + a_2(t) < 1, \quad t \in I, \tag{16}$$

holds by (kkk), and the norm $\|L\|$ of the operator L is less than 1. Thus, the operator $E - L$ is invertible. On the other hand, the operator K is compact by the Arzela theorem. Combining these observations with the well known Riesz–Schauder theorem, we see that $\operatorname{ind}(E - L + K) = 0$. By equivalency, it follows that

$$\operatorname{ind} \mathcal{B} = 0. \tag{17}$$

Consider now the case in which $(\delta_1'\delta_2')(t) = 0$ for some points t. Let us show that, although $\|L\| = 1$ in this case, *relation* (17) *remains valid.* To this end, let us prove that *the norm $\|L^m\|$ of the operator L^m is less than* 1 for some positive integer m. As is well known, this leads to the invertibility of the operator $E - L$ (see, e.g., [P3]), and, as above, we arrive at relation (17). To find the needed number m, note that, first of all, for an arbitrary positive integer N, the function $L^N F$ can be represented in the form

$$(L^N F)(t) = \sum_{j_1,\dots,j_N=1}^{2} a_{j_1}(t) a_{j_2}\big(\delta_{j_1}(t)\big) \cdots a_{j_N}\big(\delta_{j_{N-1}} \circ \cdots \circ \delta_{j_1}(t)\big) F\big(\delta_J(t)\big),$$

where

$$\delta_J(t) = \delta_{j_N} \circ \cdots \circ \delta_{j_1}(t)$$

for an arbitrary multi-index $J = (j_1, \dots, j_N)$ (see [P1]). It follows that, for an arbitrary function $F \in C(I_r)$ with $\|F\| = 1$, the inequality

$$|(L^N F)(t)| \leqslant \sum_{j_1,\dots,j_N=1}^{2} a_{j_1}(t) a_{j_2}\big(\delta_{j_1}(t)\big) \cdots a_{j_N}\big(\delta_{j_{N-1}} \circ \cdots \circ \delta_{j_1}(t)\big)$$

holds at each point $t \in I$. Let us prove that, for any fixed $t \in I$, there is a positive integer N and a constant $\gamma = \gamma(N) < 1$ such that the inequality

$$|L^N F(t)| < \gamma \tag{18}$$

is satisfied for the same functions F. If $(\delta_1' \delta_2')(t) \neq 0$, then the assertion is true for $N = 1$, according to (16). If $\delta_j'(t) = 0$ for some j, then we consider the orbits $(t_1, t_2, \dots)$ in I with $t_1 = t$ and $t_{k+1} = \delta_{j_k}(t_k)$ generated by the semigroup Φ_δ (see Section 2) and satisfying the condition $\delta_{j_k}'(t_k) \neq 0$. By conditions (kk) and (k)$'$, each such orbit forms a decreasing sequence of points in I_r convergent to the point O. By the second relation in (kk), the inequality $(\delta_1' \delta_2')(t_N) \neq 0$ holds for some number N. This means that

$$(a_1 + a_2)\big(\delta_{j_{N-1}} \circ \cdots \circ \delta_{j_1}(t)\big) < 1,$$

for some multi-index $J = (j_1, \dots, j_{N-1})$, and, in addition,

$$a_{j_1}(t) a_{j_2}\big(\delta_{j_1}(t)\big) \dots a_{j_{N-1}}\big(\delta_{j_1} \circ \cdots \circ \delta_{j_{N-2}}(t)\big) \neq 0.$$

As was shown in [P3, pp. 51–52], by virtue of the last two relations, inequality (18) follows. Furthermore, by the continuity of all functions in question, inequality (18) is valid at all points in some neighborhood V of the point t under consideration, for the same number N, possibly with a larger constant $\gamma < 1$. The collection of these neighborhoods forms an open covering of the closed set $I \setminus U$, where

$$U = \{t \in I \mid (\delta_1' \delta_2')(t) \neq 0\}.$$

Let $\{U_j\}_{j=1}^k$ be a finite subsystem of these neighborhoods, and let N_j and γ_j be the corresponding constants. By setting $m = \max N_j$ and $\gamma = \max \gamma_j$, we arrive at the desired inequality $\|L^m\| < 1$.

To complete the proof of Lemma 4, it remains to show that the *kernel of the operator $\mathcal{B}$ is empty.* Combining this fact with relation (17) results in the unique solvability of equation (13) for all functions H in question. To prove the desired property of the kernel, let us return to equation (13) and introduce a new unknown function S such that

$$S''(z) = F(z), \quad S(0) = 0, \quad S'(0) = 0, \quad z \in I_r. \tag{19}$$

If we substitute S'' for F in (13), then integration by parts leads to the following functional equation for a new unknown function S:

$$S(t) - S\big(\delta_1(t)\big) - S\big(\delta_2(t)\big) = H(t), \quad t \in I, \tag{20}$$

and this equation is none other than the *Cauchy type functional equation* which was first treated in [P1].

Dividing equation (20) by t, we arrive at the *equivalent* functional equation

$$\Phi(t) - \zeta_1(t)\Phi\big(\delta_1(t)\big) - \zeta_2(t)\Phi\big(\delta_2(t)\big) = H_1(t), \quad t \in I, \tag{21}$$

where

$$\Phi(t) = \begin{cases} S(t)/t, & 0 < t \leqslant 1, \\ 0, & t = 0; \end{cases} \qquad H_1(t) = \begin{cases} H(t)/t, & 0 < t \leqslant 1\,, \\ 0, & t = 0\,, \end{cases}$$

and

$$\zeta_j(t) = \begin{cases} \delta_j(t)/t, & 0 < t \leqslant 1, \quad j = 1, 2\,, \\ \delta_j'(0), & t = 0\,, \end{cases}$$

are continuously differentiable functions due to (14), (19), and (kk). We now claim that *equation* (21) *cannot have more than one continuous solution* $\Phi(t)$ *vanishing at the point* $t = 0$. By equivalency, *this proves the uniqueness of a solution* (*if any*) *of equation* (13), *and hence the relation* $\ker \mathcal{B} = \{0\}$. Indeed, let $\Phi(t)$ be a solution of the homogeneous equation

$$\Phi(t) - \zeta_1(t)\Phi\big(\delta_1(t)\big) - \zeta_2(t)\Phi\big(\delta_2(t)\big) = 0. \tag{22}$$

Write

$$M = \max_{t \in I} \Phi(t),$$

and set

$$t_0 = \min\{\, t \mid \Phi(t) = M \,\}.$$

Assume that $t_0 > 0$. Then $\Phi(t_0) = M$ by the continuity of Φ. Combining this relation with (22) leads to the formulas

$$\Phi\big(\delta_1(t_0)\big) = \Phi\big(\delta_2(t_0)\big) = M,$$

because $\zeta_1(t) + \zeta_2(t) = 1$ for all t, by (k)$'$, and $(\zeta_1\zeta_2)(t) > 0$ for $t > 0$, by (kk). Note that, by (k)$'$ and (kk), the relation

$$0 < \delta_1(t_0) < t_0$$

holds. However, this contradicts to the choice of the point t_0, and hence $\max \Phi = \Phi(0) = 0$. Repeating these arguments literally, we see that $\Phi(0) = \min \Phi$, whence $\Phi(z) \equiv 0$. This completes the proof of the lemma.

Thus, we have proved the first assertion of Theorem 2 in the case of the homogeneous differential equation in (4). To complete the proof of this theorem for $f = 0$, it remains to show that $u \in C^{2+k}(D_1)$ whenever $g \in C^k(\mathcal{M}),\ k = 1, 2, \dots$

It suffices to do this for $k = 1$ and to apply the induction. Let $g \in C^3(\mathcal{M})$, which implies that $G_1(x) \in C^3(\mathcal{M}_1)$, $G_2(y) \in C^3(\mathcal{M}_2)$, and $G_3(x, y) \in C^3(\Gamma)$. As we already know, in this case, a unique solution $u(x, y)$ of problem (4) with $f = 0$ is given by the formula

$$u(x, y) = \int_0^x \left(\int_0^y F(s + t)dt \right) ds + G_1(x) + G_2(y) - G_1(0), \tag{23}$$

where the function $F \in C(I_r)$ solves equation (13) or, equivalently, equation (15). In the case under consideration, the function $H(t)$ belongs to the space $C^3(I)$, and the function F is a continuous solution of the above equations. We must prove that F is a differentiable function. If this is the case, then the existence of continuous derivatives $\partial_x^2 \partial_y u$ and $\partial_x \partial_y^2 u$ becomes clear. By (23), we obtain

$$\partial_x^2 u = \int_0^y F'(x + t)dt + G_1''(x) = \int_x^{x+y} F'(z)dz + G_1''(x),$$

and now the relation $\partial_x^2 u \in C^1(\overline{D}_1)$ becomes evident. To prove the differentiability of the function F, we use relation (15) and write out the function F in the form

$$(E-L)F(t) = \mathcal{H}(t), \tag{24}$$

where $\mathcal{H}(t) = H''(t) - KF(t) \in C^1(I)$. Using the relation $\|L^m\| < 1,\ m \geqslant 1$, which was proved above, we can represent the solution F of problem (24) in the form

$$F = \sum_{k=0}^{\infty} L^k \mathcal{H} = \sum_{p=0}^{\infty} \sum_{q=0}^{m-1} L^{mp+q}\mathcal{H},$$

where the series converges absolutely and uniformly. It remains to show that the differentiated series

$$\sum_{p=0}^{\infty} \biggl(\sum_{q=0}^{m-1} \partial_t L^{mp+q}\mathcal{H} \biggr) \tag{25}$$

converges absolutely and uniformly. As we know,

$$\|L\| \leqslant 1 \quad \text{and} \quad \|L^m\| = \gamma < 1 \quad \text{for some integer} \quad m. \tag{26}$$

Introduce the notation

$$\alpha = \max_{t\in I} \bigl(a_1(t) + a_2(t) \bigr)$$

and

$$\alpha_1 = \max_{t\in I} |a_1'(t)| + \max_{t\in I} |a_2'(t)|.$$

Obviously, $\alpha \leqslant 1$. It follows that

$$\|\partial_t L\mathcal{H}\| \leqslant \|\partial_t \mathcal{H}\| + \alpha_1 \|\mathcal{H}\|, \tag{27}$$

and hence we have

$$\|\partial_t L^q \mathcal{H}\| \leqslant \|\partial_t L^{q-1}\mathcal{H}\| + \alpha_1 \|L^{q-1}\mathcal{H}\| \leqslant \|L^2 \partial_t L^{q-2}\mathcal{H}\| + 2\alpha_1 \|L^{q-2}\mathcal{H}\| \leqslant \cdots \leqslant \|\partial_t \mathcal{H}\| + q\alpha_1 \|\mathcal{H}\|,$$

for an arbitrary positive integer $q < m$. On the other hand, by relations (26) and (27), the inequality

$$\|\partial_t L^m \mathcal{H}\| \leqslant \gamma \|\partial_t \mathcal{H}\| + m\alpha_1 \|\mathcal{H}\|,$$

holds, and hence

$$\|\partial_t L^{mp}\mathcal{H}\| \leqslant \gamma^p \|\partial_t \mathcal{H}\| + c_m p \gamma^{p-1} \|\mathcal{H}\|.$$

Finally,

$$\|\partial_t L^{mp+q}\mathcal{H}\| \leqslant \gamma^p \|\partial_t \mathcal{H}\| + c_m p \gamma^{p-1} \|\mathcal{H}\|,$$

where c_m is a positive constant. It follows that

$$\sum_{q=0}^{m-1} \|\partial_t L^{mp+q}\mathcal{H}\| \leqslant m\gamma^p \|\partial_t \mathcal{H}\| + c_m' p \gamma^{p-1} \|\mathcal{H}\|$$

with some positive constant c_m'. Therefore, the functional series (25) is majorized by a convergent numerical series

$$m\|\partial_t \mathcal{H}\| \sum_{p=0}^{\infty} \gamma^p + c_m' \|\mathcal{H}\| \sum_{p=1}^{\infty} p\gamma^{p-1}$$

and hence the series in question converges absolutely and uniformly.

To complete the proof of Theorem 2, it remains to *find a solution* $u \in C^{2+k}(\overline{D}_1)$, $k = 1, 2, \dots$, *of the differential equation*

$$(\partial_x - \partial_y)\partial_x\partial_y u = f, \quad (x, y) \in D_1, \tag{28}$$

for an arbitrary function $f \in C^k(\overline{D}_1)$. Let us first consider the equation

$$(\partial_x - \partial_y)w(x, y) = f(x, y), \quad (x, y) \in D_1. \tag{29}$$

Introduce the new coordinates (s, t) on $\overline{D}_1$ such that

$$x = (s + t)/2, \qquad y = (t - s)/2.$$

In these new coordinates, equation (29) becomes

$$\partial_s v(s, t) = F(s, t), \qquad (s, t) \in D_1,$$

where $v(s, t) = w\big((s + t)/2, (t - s)/2\big)$ and $F(s, t) = (1/2)f\big((s + t)/2, (t - s)/2\big)$. The curve Γ in the new coordinates admits the parametric representation

$$s = (\delta_1 - \delta_2)(z), \qquad t = (\delta_1 + \delta_2)(z), \qquad 0 \leqslant z \leqslant 1.$$

By property (kkk), the second relation is invertible here, and we arrive at the following explicit form of representing the curve Γ:

$$s = \gamma(t), \quad \text{where} \quad \gamma = (\delta_1 - \delta_2) \circ (\delta_1 + \delta_2)^{-1}.$$

This enables us to write out the function $v(s, t)$ in the form

$$v(s, t) = \int_{\gamma(t)}^{s} F(z, t)dz,$$

and, finally, to represent a solution $w(x, y)$ of equation (29) in the form

$$w(x, y) = \frac{1}{2}\int_{\gamma(x+y)}^{x-y} f\Big(\frac{x + y + z}{2}, \frac{x + y - z}{2}\Big)dz. \tag{30}$$

This form proved to be very convenient when dealing with various problems analogous to those treated in the present paper. Below we use the opportunity to successively use this form.

Further, it is clear that the function

$$u(x, y) = \frac{1}{2}\int_0^x \Big(\int_0^y \Big(\int_{\gamma(s+t)}^{s-t} f\Big(\frac{s + t + z}{2}, \frac{s + t - z}{2}\Big)dz\Big)dt\Big)ds \tag{31}$$

is a particular solution of equation (28) in the domain D_1 for an arbitrary function $f \in C(\overline{D}_1)$. However, the function $w(x, y)$, as a matter of fact, does not satisfy equation (29) in the classical sense if f is only a continuous function because w is not differentiable with respect to x or y. However, if we treat the operator $\partial_x - \partial_y$ as the differentiation along the vector field $\overline{l} = (1, -1)$ (this is exactly what must be done when writing down an operator $P(\partial)$ in question in the form (1)), then the function $u(x, y)$ in (31) solves equation (28).

Denote by Rf the integral on the right-hand side of (31). In problem (4), we introduce a new unknown function

$$U(x, y) = u(x, y) - Rf(x, y).$$

This function must be a solution of the problem

$$\begin{aligned} P(\partial)U &= 0 && \text{in } D_1, \\ U &= g - Rf|_{\mathcal{M}} && \text{on } \mathcal{M}. \end{aligned}$$

As was proved above, this problem is uniquely solvable for arbitrary C^2-functions g and arbitrary continuous functions f *if* $Rf|_{\mathcal{M}}$ *is in* $C^2(M) = C^2(OA_1) \times C^2(OA_2) \times C^2(\Gamma)$. Thus, it remains to show that $Rf \in C^2(\overline{D}_1)$.

The existence of the derivative $\partial_x \partial_y u$ is obvious. To prove the existence of $\partial_x^2 u$, note that

$$\partial_x u(x,y) = \frac{1}{2} \int_0^y \Big(\int_{\gamma(x+t)}^{x-t} f\Big(\frac{x+t+z}{2}, \frac{x+t-z}{2}\Big) dz \Big) dt.$$

To show that this double integral can be differentiated with respect to x, we change the variable in the outer integral by the rule $x + t = p$. This leads to the relation

$$\partial_x u(x,y) = \frac{1}{2} \int_x^{x+y} \Big(\int_{\gamma(p)}^{2x-p} f\Big(\frac{p+z}{2}, \frac{p-z}{2}\Big) dz \Big)\, dp, \tag{32}$$

and, in this form, the possibility to differentiate the right-hand side with respect to x becomes evident. Moreover, it follows from (32) that, if $f \in C^k(\overline{D}_1)$, then the functions

$$\partial_x^{2+k} u,\ \partial_x^{1+k} \partial_y,\ \ldots,\ \partial_x \partial_y^{1+k}$$

are continuous. This proves the last statement of Theorem 2 related to the increasing in smoothness, and simultaneously completes the proof of Theorem 2.

4. PROOF OF THEOREM 3

The full proof of this theorem is too long and is far from being an easy reading. Moreover, it can be found in [P1]. Nevertheless, a particular case of the problem in question is considered here in detail. The main reason for this presentation is to show that, although two boundary problems in question are equivalent to (formally) the same functional equation, the solvability properties of these problems and the methods used in the proof of the corresponding results are completely different.

We will prove this theorem in the simplest situation, namely,

$$\textit{the set } \mathcal{T} \textit{ is empty.} \tag{33}$$

In other words, the curve Γ is not characteristic with respect to the differential operator $P(\partial)$. As in the previous section, the second boundary problem (5) can be reduced to the problem

$$\begin{aligned} (\partial_x + \partial_y)\partial_x \partial_y u &= f && \text{in } D_2, \\ u &= g && \text{on } \partial D_2. \end{aligned} \tag{34}$$

We use the symbols $\mathcal{M}_1$ and $\mathcal{M}_2$ to denote the same geometric objects as in Section 3. Let $g_1(x)$, $g_2(x)$, $g_3(x,y)$ be the restrictions of the function g to $\mathcal{M}_1$, $\mathcal{M}_2$, and Γ, respectively (see Fig. 3). Assume that a pair of C^2-functions $x = \Delta_1(t)$, $y = \Delta_2(t)$, $t \in I$, forms a parametric representation of the curve Γ. Certainly, these two functions can separately be considered as parametric representations of the curves $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. The invariant description of the domain D_2 given in Section 2 leads to the following list of properties of the functions Δ_1 and Δ_2:

(n) $\Delta_1(0) = 0, \quad \Delta_1(1) = X; \qquad \Delta_2(0) = Y, \quad \Delta_2(1) = 0;$

(nn) $\Delta_1'(t) \geqslant 0, \quad \Delta_2'(t) \leqslant 0, \quad t \in I;$

(nnn) $\Delta_1'(t) - \Delta_2'(t) > 0, \qquad t \in I.$

Relation (nn) follows from properties (j) and (jj) of the domain D_2 (see [P3]). Relation (nnn) is an obvious consequence of (jjj). The continuity of the function g leads to the natural compatibility conditions

$$g_1(0) = g_2(0), \quad g_1(X) = g_3(X,0), \quad g_2(Y) = g_3(0,Y). \tag{35}$$

As in Section 3, condition (nnn) makes it possible (by introducing the new variable

$$z = (\Delta_1 - \Delta_2)(t))$$

to restrict all considerations to the pair $\big(\Delta_1(t),\ \Delta_2(t)\big)$ satisfying the additional condition

$$\text{(n)}' \qquad (\Delta_1 - \Delta_2)(t) = t.$$

We preserve the same notation t for the new variable. However, conditions (n)–(nnn) and (n)$'$ should be replaced by new conditions. Moreover, to establish the parallelism between the solvability properties of the boundary problems in question, it is convenient to rename the functions $\Delta_j(t)$ in the following way:

$$\Delta_1(t) = \delta_1(t), \quad -\Delta_2(t) = \delta_2(t), \quad t \in I_r.$$

The new conditions for the new functions take the form

$$\begin{aligned} &(\widetilde{\text{n}}) && \delta_1(-Y) = 0; \quad \delta_1(X) = X; \qquad \delta_2(-Y) = -Y; \quad \delta_2(X) = 0;\\ &(\widetilde{\text{nn}}) && \delta_1'(t) \geqslant 0, \quad \delta_2'(t) \geqslant 0, \quad t \in I_r;\\ &(\widetilde{\text{nnn}}) && \delta_1(t) + \delta_2(t) = t, \qquad\quad t \in I_r. \end{aligned}$$

Note that, in the new notation, condition (33) becomes

$$(\widetilde{\text{n}})' \qquad \delta_1'(t)\delta_2'(t) > 0, \quad t \in I_r,$$

and, by $(\widetilde{\text{nnn}})$, the relation

$$\delta_1'(t) + \delta_2'(t) = 1, \quad t \in I_r, \tag{36}$$

holds.

Following the scheme of the proof of Theorem 2, we first *consider the case in which* $f = 0$ *in* (34). Using the postulated properties of the domain D_2, we can readily see that, for an *arbitrary* function $G \in C(I_r)$, the function

$$u(x,y) = \int_0^x \left(\int_0^y G(s-t)dt\right)ds + g_1(x) + g_2(y) - g_1(0) \tag{37}$$

is a solution of the differential equation in (34) and this solution satisfies the boundary conditions on $\mathcal{M}_1 \cup \mathcal{M}_2$. The necessity to satisfy the boundary condition $u = g_3$ on Γ leads to the integral equation

$$(\mathcal{B}G)(t) := \int_0^{\delta_1(t)} \left(\int_0^{\delta_2(t)} G(x-y)dy\right)dx = H(t), \quad t \in I_r, \tag{38}$$

with respect to the unknown function G (cf. (13)). The right-hand side $H(t)$ in this equation is equal to

$$H(t) = -g_1\big(\delta_1(t)\big) - g_2\big(\delta_2(t)\big) + g_3\big(\delta_1(t), \delta_2(t)\big) + g_1(0),$$

and what is important is that, by (35), the function $H(t)$ satisfies the boundary conditions

$$H(-Y) = H(X) = 0. \tag{39}$$

Note that, for an arbitrary continuous function G, the function $(\mathcal{B}G)(t)$ is twice differentiable and, in view of $(\widetilde{n})$, satisfies the same boundary conditions (39). It follows that, *for any function* $H \in C^2(I_r)$ *vanishing on the boundary* ∂I, *equation* (38) *is equivalent to each of the two equations*

$$(d/dt)\mathcal{B}G = (d/dt)H, \quad t \in I_r, \tag{40}$$

and

$$(d^2/dt^2)\mathcal{B}G = (d^2/dt^2)H, \quad t \in I_r. \tag{41}$$

In the detailed notation, equation (40) becomes

$$\delta_1'(t)\int_0^{\delta_2(t)} G\big(\delta_1(t)-y\big)dy + \delta_2'(t)\int_0^{\delta_1(t)} G\big(x-\delta_2(t)\big)dx = H'(t). \tag{42}$$

Write

$$F(t) = \int_{-Y}^{t} G(s)ds, \quad t \in I_r,$$

and substitute F' for G in (42). Integrating by parts, we reduce the equation thus obtained to the functional equation

$$F(t) - \delta_1'(t)F\big(\delta_1(t)\big) - \delta_2'(t)F\big(\delta_2(t)\big) = H'(t) \tag{43}$$

for the unknown function F satisfying the condition $F(-Y) = 0$.

Let us prove that *this equation cannot have more than one solution.* This fact is equivalent to *the uniqueness of a solution of the integral equation* (38). Assume that a function $F \in C(I_r)$ solves the problem

$$F(t) - \delta_1'(t)F\big(\delta_1(t)\big) - \delta_2'(t)F\big(\delta_2(t)\big) = 0, \quad F(-Y) = 0. \tag{44}$$

Write

$$M = \max_{t\in I_r} F(t),$$

and set

$$F(t_0) = M$$

for some point $t_0 \in I_r$. According to $(\widetilde{n})$ and (36), we see that

$$F\big(\delta_1(t_0)\big) = F\big(\delta_2(t_0)\big) = M.$$

Applying the same argument with respect to the point $t_1 = \delta_1(t_0)$, we arrive at the relation

$$F\big(\delta_1(t_1)\big) = F\big(\delta_1^2(t_0)\big) = M.$$

By repeating this procedure, we obtain a sequence $\{\delta_1^n(t_0)\} = \{(\delta_1 \circ \cdots \circ \delta_1)(t_0)\}$ of points in the interval I_r such that

$$F\big(\delta_1^n(t_0)\big) = M, \quad n = 0, 1, \ldots$$

Since $1 \geqslant \delta_1(t) > t$ by $(\widetilde{n})'$, we see that $\delta_1^n(t_0) \to X$ as $n \to \infty$, and therefore

$$F(X) = M.$$

The spreading of the maximal value of a solution F discussed above remains obviously valid for the minimal value m of the same solution. As a result, we find that $F(X) = m$, and hence $F(t) = \text{const}$. Remembering the boundary condition in (44) yields $F(t) = 0$. Thus, we have proved the uniqueness of a solution of equation (43), and therefore of solutions of problems (38), (40), (41), and (34).

Let us now proceed with the solvability of problem (34) in the case of $f = 0$. To this end, consider equation (41) that, in its expanded form and after some identical transformations, becomes

$$F(t) - LF(t) + KF(t) = \mathcal{H}(t), \quad t \in I_r. \tag{45}$$

Here K and L are linear operators on the space $C(I_r)$, namely,

$$LF(t) = \big(\delta_1'(t)\big)^2 F\big(\delta_1(t)\big) + \big(\delta_2'(t)\big)^2 F\big(\delta_2(t)\big)$$

and

$$KF(t) = \delta_1''(t) \int_0^t F\left(\big(\delta_1(t)\big) - y\right) dy + \delta_2''(t) \int_0^t F\left(x - \delta_2(t)\right) dx,$$

and $\mathcal{H}(t) = H''(t)$. By relations $(\widetilde{n})$ and (36), the norm of the operator L on the space $C(I_r)$ is less than 1. On the other hand, the operator K is a compact operator on the same space, due to the Arzela theorem. This enables us to conclude, using Riesz–Schauder theory, that the index of equation (45) is equal to zero. By equivalency, the same is true for problem (34) with $f = 0$. Together with the uniqueness of solution of the latter problem, this completes the proof of Theorem 3 in the case of the homogeneous differential equation. To pass to the general case, it suffices to produce a solution $u \in C^{2+k}(\overline{D}_2)$ of the differential equation

$$(\partial_x + \partial_y)\partial_x\partial_y u = f$$

for an arbitrary function $f \in C^k(\overline{D}_2)$. This can be done by literally repeating the arguments used when proving the analogous assertion in Theorem 2. The solution $u(x, y)$ thus obtained is of the form

$$u(x, y) = \frac{1}{2}\int_0^x \left(\int_0^y \left(\int_{\zeta(s-t)}^{s+t} f\Big(\frac{s-t+z}{2}, \frac{t-s+z}{2}\Big)dz\right)dt\right)ds$$

with $\zeta = (\delta_1 + \delta_2) \circ (\delta_1 - \delta_2)^{-1}$. As was proved in Section 3, this function belongs to the space $C^2(\overline{D}_2)$, and this is just what we need to complete the proof of Theorem 3 for $k = 0$. The proof for general k is realized by following the scheme used when proving the corresponding assertion of Theorem 2.

5. CONCLUDING REMARKS

An attentive reader has probably paid attention to the fact that both boundary problems turned out to be equivalent to the same integral equation. Moreover, the restrictions to the mappings δ_1 and δ_2 are practically not distinguishable. However, the first of these equations, (13), was solved without any restrictions to the mutual behavior of δ_1 and δ_2, whereas the other equation, (38), relates these mappings by a sufficiently delicate topological condition. The proof of the solvability in the second case is based on results in the theory of dynamical systems with two generators, which is obliged for its origin to the second characteristic boundary problem (see Section 2). The proof of Theorem 3 looks very simple due to the additional (unnecessary) condition (33) which was especially introduced for brevity and for simplicity. Indeed, assume (contrary to (33)) that the sets

$$\mathcal{T} = \{t \in I_r \mid \delta_j'(t) = 0\}, \qquad j = 1, 2,$$

are nonempty. When proving the absence of nontrivial solutions of equation (43), we conclude as a first step that

$$F\big(\delta_1(t_0)\big) = F\big(\delta_2(t_0)\big) = M$$

as $\delta_1'(t_0)\delta_2'(t_0) \neq 0$ and $\delta_1(t_0) + \delta_2(t_0) = 1$. Repeating this trick for the points

$$t_1 = \delta_1(t_0), \quad t_2 = \delta_1(t_1) = (\delta_1 \circ \delta_1)(t_0), \ \ldots,$$

we arrive at the relation $F(X) = M$. However, if $t_0 \in \mathcal{T}_1$, then the relation $F(t_0) = M$ is followed by only one relation

$$F\big(\delta_2(t_0)\big) = M.$$

If the point $t_1 = \delta_2(t_0)$ belongs to the set $\mathcal{T}_2$ by chance, then we can conclude that

$$F\big(\delta_1(t_1)\big) = F\big(\delta_1 \circ \delta_2(t_0)\big) = M,$$

and so on. It becomes clear that the maximal value of a solution F is spread not along the orbit of the monotone function δ_1 (as in the above proof) but along some incomprehensible trajectory. What is to be done? A partial answer is contained in [P2] (see also [P3], where the same question arises in connection with a problem in integral geometry). Which properties of pairs of mappings (δ_1, δ_2) determine diverse solvability properties of the above integral equations or of equivalent functional equations of type (20) (and of various problems in analysis which can be reduced to these equations)? In [P4], we try to answer this question by introducing the concepts of P-configuration (see Fig. 4) and Z-configuration (see Fig. 5) formed by pairs (δ_1, δ_2).

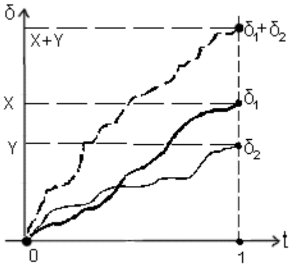

Fig. 4.

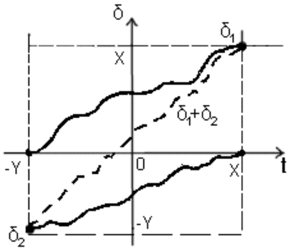

Fig. 5.

(On these figures, the dotted lines represent the graphs of the functions $z = \delta_1(t) + \delta_2(t)$.) These concepts reduce distinctions between diverse pairs in question to distinctions between the sets of fixed points of the mappings δ_1 and δ_2. However, a final solution of the solvability problem for the boundary problems under consideration is far from being complete at present.

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