ON THE OVER DETERMINEDNESS OF SOME FUNCTIONAL EQUATIONS

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Abstract. It is shown that some well known functional equations in \(\mathbb{R}^n, n \geq 2\), turn out to be overdetermined. This means that their solutions are uniquely defined if the corresponding relations are fulfilled not in the whole spaces \(\mathbb{R}^n, n \geq 2\), but only at the points of some smooth submanifolds in \(\mathbb{R}^n\).

1. Introduction. One of traditional problems in the general theory of functional equations is to find a general solution of this equation or that. The typical example of such kind is the following problem going back to Cauchy:

Find all continuous functions \(F\) on \(\mathbb{R}\) satisfying the equation

\[ F(x + y) = F(x) + F(y) \quad \text{for all } (x, y) \in \mathbb{R}^2. \]

It is well known that the only linear functions \(F(t) = F(1)t\) solve this problem. Recently it became clear that the information containing in (1) is redundant. To reconstruct a linear function \(F\) it suffices that the equality (1) is fulfilled not in the whole space \(\mathbb{R}^2\), but only at points \((x, y)\) of some curve \(\Gamma\) in \(\mathbb{R}^2\). Thus, the above Cauchy problem turns out to be overdetermined (and just in this sense the title of the present work should be understood). For the first time some class of such curves \(\Gamma\) in connection with the Cauchy equation (1) was described in [Z]. Another class of curves \(\Gamma\) having the same property with respect to equation (1) was discovered by the author in solving some boundary problems for hyperbolic differential equations and some geometric problems analogous to the main problem of Integral geometry (see [P3],[P4]).

In the present work we establish that such overdeterminedness is characteristic for a class of well known functional equations. In addition to equation (1), another three functional equations will be considered. The general solution for each of them is well known (if the corresponding relation is fulfilled at all points of \(\mathbb{R}^2\) or \(\mathbb{R}^3\)). It will be shown that we obtain the same solutions if the above-mentioned relations are fulfilled at points of some curve in \(\mathbb{R}^2\) or some surface in \(\mathbb{R}^3\) only. I conclude this short introduction by the question: are some more (known) functional equations over \(\mathbb{R}^n, n \geq 2\), overdetermined?

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2. The Cauchy functional equation. In the course of this work we deal with a local form of some well-known equations. In connection with the Cauchy equation this means that the original problem is formulated in the following way. Denote by $I$ and $I^\circ$ the sets

$$I = \{t \in \mathbb{R} \mid -1 \leq t \leq 1\}, \quad I^\circ = \{t \in \mathbb{R} \mid -1 < t \leq 1\},$$

and let $K$ be the square

$$K = \{(x,y) \in \mathbb{R}^2 \mid |x \pm y| \leq 1\}.$$ 

It is required to find a function $F \in C(I)$ such that

$$F(x+y) = F(x) + F(y) \quad \text{for all } (x,y) \in K. \quad (2)$$

For completeness we give a short solution of this problem. It follows from (2) that

$$F(0) = 0, \quad F(-z) = -F(z) \quad \text{for all } z \in \mathbb{R},$$

and the relation

$$F\left(\sum_{j=1}^{n} z_j\right) = \sum_{j=1}^{n} F(z_j) \quad (3)$$

holds for every natural $n$ and for all real $z_1, \ldots, z_n$. Setting $z_j = z/n, \quad j = 1, \ldots, n$, in (3) results in the equality

$$F(z/n) = F(z)/n, \quad z \in \mathbb{R}. \quad (4)$$

On the other hand, substituting $z_1 = \ldots = z_m = 1$ in (3) with $n = m$, we find that $F(m) = mF(1)$. Combining this relation and relation (4) with $z = m, \quad |m| \leq n$, results in

$$F(m/n) = (m/n)F(1) \quad \text{for all integers } m, n, |m| \leq n. $$

By continuity, the function $F(z) = zF(1), \quad z \in I$, is a general solution of the problem (2).

Turning to the subject of this work we first define some classes of curves $\Gamma$ we deal with.

Definition We say that real-valued functions $\beta_1, \ldots, \beta_n$ on $I$ form a $Z$-configuration if all $\beta_j$ do not decrease, vanish at the point $t = -1$, and the function $\beta = \sum_{j=1}^{n} \beta_j$ increases on $I$.

We say that maps $\beta_1$ and $\beta_2$ of $I$ into itself form a $P$-configuration if they satisfy the following hypotheses:

(i) both functions do not decrease;

(ii) $\beta_j(t) \neq t, j = 1, 2,$ at points $t \in I$;

(iii) the ranges $\mathcal{R}(\beta_1)$ and $\mathcal{R}(\beta_2)$ have only one common point $t = 0$.

The latter means that

$$\beta_1(-1) = \beta_2(1) = 0, \quad \beta_1(1) = 1, \quad \beta_2(-1) = -1. \quad (5)$$

The Figures 1 and 2 represent typical examples of $Z$- and $P$-configurations, respectively. Dotted lines in both figures are the graphs of functions $z = \beta_1(t) + \beta_2(t)$.

In the $(x_1, x_2)$-plane consider the curves

$$\Gamma = \{(x_1, x_2) \mid x_1 = \beta_1(t), \quad x_2 = \beta_2(t); \quad t \in I\},$$

where the pairs $\beta_1$ and $\beta_2$ form the $Z$- and $P$-configurations, respectively. The graphs of these curves are represented on Figures 1’ and 2’ respectively.
Let $\Gamma = \{(z_1, \ldots, z_n) \mid z_j = \beta_j(t); \ t \in I\}$ be a curve in $\mathbb{R}^n$, and the functions $\beta_1, \ldots, \beta_n$ form the $Z$-configuration. Our first result relates to the functional equation

$$F(\beta(t)) - \sum_{j=1}^{n} F(\beta_j(t)) = 0, \ t \in I. \quad (6)$$

where $F$ is an unknown real-valued function in $D$ and $D = \{y \mid 0 \leq y \leq \beta(1)\}$. This is none other than an equivalent form (3) of the Cauchy equation (2) with $z = (z_1, \ldots, z_n)$ being points of the curve $\Gamma$. Each linear function $F$ satisfies equation (6). We will show that under some conditions there are no other solutions of this equation (cf. [Z]).

Denote by $C_{[m]}(D)$, $m = 1, 2, \ldots$, a subspace in $C(D)$ consisting of functions $F$ such that $F(y)/y m \in C(D)$. All the elements of $C_{[m]}(D)$ are $m$ times differentiable at the point $y = 0$ and $F^{(m)}(0) = \lim_{y \to 0} F(y)/y m$

**Theorem 1.** If all the functions $\beta_1, \ldots, \beta_n$ are continuous on $I$ and $\sum_{t \neq k} (\beta_t \beta_k)(t) \neq 0$ on $I$ then any solution $F(y) \in C_{[1]}(D)$ of equation (6) is a linear function i.e. $F(y) = \gamma y, \ \gamma \in \mathbb{R}$.

**Proof.** Introduce a function $G(y)$ on $D$ which equals $F(y)/y$ at points $y \neq 0$ and equals $F'(0)$ at the point $y = 0$. By hypotheses, this function is continuous on $D$
and satisfies the equation
\[ G(y) - \rho_1(y)G(\delta_1(y)) - \ldots - \rho_n(y)G(\delta_n(y)) = 0, \quad y \in D \setminus \{0\}, \tag{7} \]
where
\[ \delta_j(y) = \beta_j(\beta^{-1}(y)) \quad \text{and} \quad \rho_j(y) = \delta_j(y)/y, \quad j = 1, \ldots, n. \]
Note that all the functions \( \delta_j \) do not decrease and for \( y \neq 0 \)
\[ \sum_{j=1}^{n} \rho_j(y) = 1. \tag{8} \]

To prove the theorem it suffices to show that \( G(y) \equiv \text{const} \).

Let
\[ M = \max_{y \in D} G(y) \quad \text{and} \quad \mathcal{M} = \{ y \in D \mid G(y) = M \} \]
and prove that \( 0 \in \mathcal{M} \). To this end consider the point \( q = \min\{ y \mid y \in \mathcal{M} \} \). It is clear that \( q \in \mathcal{M} \). If \( q \neq 0 \) we substitute \( q \) for \( y \) in (7). In view of (8) and by the hypotheses, \( G(\delta_k(q)) = M \) and \( \delta_k(q) < q \) for some index \( k \). However, this contradicts to the definition of the point \( q \). Applying the same arguments to the minimal value of the solution \( G \) of equation (7) results in the equality \( \min G = G(0) \). Thus, \( \min G = \max G \), and hence \( G \equiv \text{const} \). This completes the proof of Theorem 1.

**Remark** If \( \beta_l(t) \neq 0 \) at some point \( t \) only for a single index \( l \), then \( \beta_k(t) = 0 \) for all indices \( k \neq l \) and for all points \( t \) smaller than some \( T \). But then equation (6) on the interval \( (0, T) \) is none other than the identity \( F(t) = F(t) \), and hence solutions \( F \) of equation (6) may be chosen arbitrarily.

We now pass to the *Cauchy type functional equation*
\[ F(\beta_1(t) + \beta_2(t)) - F(\beta_1(t)) - F(\beta_2(t)) = h, \quad t \in I, \tag{9} \]
in which the functions \( \beta_1 \) and \( \beta_2 \) *form the \( \mathcal{P} \)-configuration*. We assume in addition that these functions are differentiable and satisfy the conditions
\[ \beta'_1(t) + \beta'_2(t) > 0 \quad \text{on \( I \)} \quad \text{and} \quad \beta'_1(-1)\beta'_2(1) \neq 0. \]

Note that equation (9) under these conditions appears in applications to the above-mentioned boundary problems for hyperbolic differential equations and to the integral equations connected with some geometric problems. This equation was introduced for the first time in the author’s paper [P2], where the results related to the homogeneous equation (9) were formulated. It’s clear that the map \( \beta = \beta_1 + \beta_2 \) is a diffeomorphism in \( I \) preserving the boundary \( \partial I \). Consequently, the maps
\[ \delta_1 = \beta_1 \circ \beta^{-1} \quad \text{and} \quad \delta_2 = \beta_2 \circ \beta^{-1} \]
of \( I \) into itself also form the \( \mathcal{P} \)-configuration, and in addition
\[ \delta'_1(t) + \delta'_2(t) = 1 \quad \text{on \( I \)}. \tag{10} \]

Introduce the *critical sets*
\[ \mathcal{T}_1 = \{ t \in I \mid \delta'_2(t) = 0 \}, \quad \mathcal{T}_2 = \{ t \in I \mid \delta'_1(t) = 0 \} \quad \text{and} \quad \mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2. \]
Let \( \Phi_A \) be a noncommutative semigroup generated by the maps \( \delta_1, \delta_2 \), whose elements are all possible maps of \( I \) into itself of the form
\[ \delta_J = \delta_{j_n} \circ \ldots \circ \delta_{j_1}, \quad \text{where} \quad J = (j_1, \ldots, j_n) \quad \text{and all} \quad j_k \quad \text{equal to} \quad 1 \quad \text{or} \quad 2. \]
This semigroup generates naturally a noncommutative dynamical system of a new type, and we will agree about some special terminology related to $\Phi_N$.

1) An ordered set $O = (t_1, \ldots, t_n)$, $n \geq 1$, of points in $I$ is said to be an orbit (of the point $t_1$) if

$$t_{k+1} = \delta_{jk}(t_k), \quad k = 1, 2, \ldots, n,$$

where all $\delta_{jk}$ are equal to either $\delta_1$ or $\delta_2$.

2) An orbit $O$ is called $T$-proper if in (11)

$$\delta_{jk} = \begin{cases} 
\delta_1, \text{ whenever } t_k \in T_1, \\
\delta_2, \text{ whenever } t_k \in T_2, \\
\delta_1 \text{ or } \delta_2 \text{ independently of } t_k, \text{ if } t_k \in I \setminus T.
\end{cases}$$

3) If all the points of an orbit $O$ lie in $T$, then $O$ is called critical orbit.

4) If $t_1 = t_{n+1}$ in $O$, then $O$ is called periodic orbit.

The set of all $T$-proper critical periodic orbits we denote by $\Omega_T$.

Now everything is ready to formulate the main result related to the solvability of the homogeneous Cauchy type functional equation

$$F(\beta_1(t) + \beta_2(t)) - F(\beta_1(t)) - F(\beta_2(t)) = 0.$$  (12)

**Theorem 2.** Let $T_j'$ be the set of limit points of the critical sets $T_j$, $j = 1, 2$. Assume that if both sets $T_1$ and $T_2$ are infinite, then

$$\min\{t \mid t \in T_1'\} > \max\{t \mid t \in T_2'\}.$$

If $\Omega_T = \emptyset$, then all $C^1$-solutions of equation (12) are linear functions $F(z) = \lambda z$.

**Proof:** The change of variable $\beta(t) \mapsto t$ reduces equation (12) to the form

$$F(t) - F(\beta_1(t)) - F(\beta_2(t)) = 0, \quad t \in I.$$  (13)

It is clear that if $F(t)$ solves this equation, then the derivative $G(t) = F'(t)$ satisfies the equation

$$G(t) - \delta_1(t) G(\beta_1(t)) - \delta_2(t) G(\beta_2(t)) = 0, \quad t \in I.$$  (14)

Denote by $M$ the maximum value of $G(t)$ and let $M = \{t \mid G(t) = M\}$. Take an arbitrary point $t_1 \in M$. By virtue of (10), it can be verified easily that

- if $t_1 \in T_1$, then $t_2 = \delta_1(t_1) \in M$;
- if $t_1 \in T_2$, then $t_2 = \delta_2(t_1) \in M$;
- if $t_1 \in I \setminus T$, then both points $t_2 = \delta_1(t_1), t_2 = \delta_2(t_1)$ lie in $M$.

By definition this means that any $T$-proper orbit $O = (t_1, t_2)$ lies in $M$ if $t_1$ does. Continuing to argue in the same way we conclude that if some point $t_1$ lies in $M$, then all $T$-proper orbits $O = (t_1, t_2, \ldots)$ beginning at $t_1$ are situated in $M$. (In other words, the maximal value of any solution $G$ extends along $T$-proper orbits). According to Theorem 1 in [P5], under the hypotheses of Theorem 2, at least one of such orbits converges to the boundary $\partial I$ of the interval $I$. This means that either $G(-1) = M$ or $G(1) = M$ (note that the set $M$ is closed, and the function $G$ is continuous). Arguing in just the same way with respect to the minimal value $m$ of the solution $G$ we conclude that either $G(-1) = m$ or $G(1) = m$. Substituting consecutively $t = -1$ and $t = 1$ in (14) and using relation (10) we obtain $G(-1) = G(1) = G(0)$. Thus, $M = m$ and, hence, $G(t) \equiv const$, whence $F(t) = M$. This completes the proof of Theorem 2.
To illustrate this result let us return to problem (2). The curve \( \Gamma = \{(x, y) \mid x = (t - 1)/2, y = (t + 1)/2; |t| \leq 1\} \), is a side of the square \( K \). The functions \( x(t) = (t - 1)/2 \) and \( y(t) = (t + 1)/2, |t| \leq 1 \), satisfy obviously all the conditions of Theorem 2. Consequently, by this theorem, any solution of the Cauchy type functional equation
\[
F(t) - F((t - 1)/2) - F((t + 1)/2) = 0, \quad t \in I,
\]
is a linear function. Hence, to reconstruct a \( C^1 \) - function \( F \) on \( I \) it is sufficient that the Cauchy relation (2) is fulfilled not everywhere in the square \( K \), but only at the points of one of its sides.

3. The Jensen functional equation. In this section we deal with the following equation for an unknown continuous function \( F \) on \( I \):
\[
F(ax + by) = aF(x) + bF(y), \quad (x, y) \in K, \quad (15)
\]
where \( a \) and \( b \) are positive numbers with
\[
a + b = 1.
\]
As it is well known, any solution of this equation is a function \( F(t) = \lambda t + \mu \) with \( \lambda, \mu \in \mathbb{R} \). Indeed, substituting consecutively \( y = 0 \) and \( x = 0 \) in (15) we find that
\[
F(ax) = aF(x) + bF(0), \quad F(by) = bF(y) + aF(0).
\]
It follows that
\[
F(ax) + F(by) = aF(x) + bF(y) + F(0) = F(ax + by) + F(0).
\]
The change of variables \( x \to x/a, \ y \to y/b \) leads to the equality
\[
F(x + y) - F(x) - F(y) + F(0) = 0.
\]
Setting \( G(t) = F(t) - F(0) \) we arrive at the equation
\[
G(x + y) - G(x) - G(y) = 0,
\]
whence \( G(t) = \lambda t \) and \( F(t) = \lambda t + \mu \).

We are now going to show that analogously to the Cauchy equation (1), the problem (15) is overdetermined. To this end consider the Jensen type functional equation
\[
F(a\beta_1(t) + b\beta_2(t)) = aF(\beta_1(t)) - bF(\beta_2(t)) = 0, \quad t \in I, \quad (16)
\]
where \( C^1 \) - functions \( \beta_1(t), \beta_2(t) \) form a \( P \) - configuration and satisfy the conditions
\[
a\beta_1'(t) + b\beta_2'(t) > 0 \quad \text{on} \ I \quad \text{and} \quad \beta_1'(1)\beta_2'(-1) \neq 0.
\]
Let \( \beta = a\beta_1 + b\beta_2 \) and \( \delta_1 = \beta_1 \circ \beta^{-1}, \delta_2 = \beta_2 \circ \beta^{-1} \). The functions \( \delta_1(t) \) and \( \delta_2(t) \) form a \( P \) - configuration. As in the previous section, we introduce a semigroup \( \Phi_\delta \) of maps in \( I \), the critical sets \( T_1, T_2 \) and \( T \), periodic, \( T \) - proper and critical orbits in \( I \), and finally, we define the set \( \Omega^I_\delta \). The main result of this Section is as follows.

Theorem 3. If the hypotheses in Theorem 2 are fulfilled, then there is no \( C^1 \) - solution of equation (16) distinct from \( F(t) = \lambda t + \mu, \lambda, \mu \in \mathbb{R} \).
Adding all these relations together and using (19) we arrive at the relation
\[ F(t) - aF(\delta_1(t)) - bF(\delta_2(t)) = 0, \quad t \in I. \]
Hence the derivative \( G(t) = F'(t) \) satisfies the equation
\[ G(t) - a\delta'_1(t)G(\delta_1(t)) - b\delta'_2(t)G(\delta_2(t)) = 0, \quad t \in I, \]
where
\[ a\delta'_1(t) + b\delta'_2(t) = 1 \text{ on } I. \]
The latter relation analogous to (10) makes it possible to repeat word for word the proof of the previous theorem and to conclude that all solutions of equation (17) are constants. This proves Theorem 3.

**Remark** As above, it can be verified easily that the result of Theorem 3 remains valid in a wider class of continuous functions \( F \) differentiated at the point \( t = -1 \), if the functions \( \delta_1, \delta_2 \) are continuous and form a \( Z \)-configuration.

4. The quadratic functional equation. The functional equation considered in this section is
\[ F(x + y) + F(x - y) = 2F(x) + 2F(y), \quad (x, y) \in K. \] (18)
It is well known that all continuous solutions \( F \) of this equation on the interval \( I \) are of the form \( F(t) = \lambda t^2, \lambda \in \mathbb{R} \). To prove this we note that in view of (18)
\[ F(0) = 0, \quad F(x) = F(-x) \text{ and } F(2x) = 4F(x), \quad x \in I. \] (19)
Substituting \( y = kx \) in (18), \( k = 1, 2, \ldots, n \), we obtain
\[ F((k + 1)x) - 2F(kx) + F((k - 1)x) - 2F(x) = 0. \]
Adding all these relations together and using (19) we arrive at the relation
\[ F((n + 1)x) - F(nx) = (2n + 1)F(x). \]
One more summation over \( n = 1, 2, \ldots, m \) leads to the relation
\[ F((m + 1)x) = (1 + \ldots + (2m + 1))F(x) = (m + 1)^2F(x). \]
It follows that
\[ F(1) = F(n \cdot \frac{1}{n}) = n^2F(1). \]
Combining the two last equalities results in the relation
\[ F(m/n) = (m/n)^2F(1) \text{ for all } 0 \leq m < n, \]
and, by continuity, we obtain \( F(x) = \lambda x^2, \lambda \in \mathbb{R} \).

The following result shows that the functional equation (18) is also overdetermined.

**Theorem 4.** Let \( F(t) \) be a function from the space \( C_{[2]}(I) \) (i.e. \( F(t)/t^2 \in C(I) \)). If \( F \) satisfies equation (18) at points of the curve \( y = x/2 \) only, then \( F(t) = \lambda t^2 \).

**Proof:** If \( (x, y) \in K \) and \( y = x/2 \), then \(|x| \leq 2/3\), and \( F \) satisfies the equation
\[ F(3x/2) - 2F(x) - F(x/2) = 0, \quad |x| \leq 2/3. \]
The change of variable \( x = 2t/3 \) reduces this relation to the form
\[ F(t) - 2F(2t/3) - F(t/3) = 0, \quad t \in I. \]
According to assumptions, the function $G(t)$ which is equal to $F(t)/t^2$ if $t \neq 0$ and to $F''(0)$, otherwise, is continuous on $I$. It is clear that $G$ satisfies the equation

$$G(t) - (8/9)G(2t/3) - (1/9)G(t/3) = 0, \quad t \in I.$$  

The latter is none other than equation (7) with $\rho_1 = 8/9$, $\rho_2 = 1/9$, $\delta_1 = 2t/3$ and $\delta_2 = t/3$. Arguing as in the concluding part of the proof of Theorem 1 we obtain $\min G = \max G = G(0)$, and hence $F(t) = \lambda t^2$.

5. The generalized Cauchy equation. When solving some integral equations arising in connection with a geometric problem in $\mathbb{R}^n$, $n \geq 2$, (see [P2]) they proved to be reduced (in an equivalent manner) to the following functional equation in $\mathbb{R}^n$:

$$(B_nF)(x_1, \ldots, x_n) := F\left(\sum_{j=1}^n x_j\right) - \sum_{k=1}^n F\left(\sum_{j \neq k} x_j\right) + \ldots + (-1)^n F(0) = 0, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$  

(20)

It turned out that the same equation (and even much more general one) was studied in connection with quite different problems independently and by distinct methods in the papers of S. Mazur and W. Orlicz (Studia Math. 5(1934)) and J. Mozer and M. Struwe (Bol.Soc.Brazil Mat., 1992). It is not difficult to check that all continuous solutions of these equations are polynomials $F(t) = \sum_{k=0}^n a_k t^k$.

The main result of this Section claims that equation (20) is also overdetermined: if a sufficiently smooth function $F$ satisfies equation (20) at points of some smooth surface of codimension one in $\mathbb{R}^n$, then $F$ is a polynomial of degree $n$.

We draw the reader’s attention to the fact that the conditions of Theorem 5 below include some a priori smoothness of $F$ depending on $n$, and the author has no idea whether such a restriction is necessary.

First of all, we will demonstrate (in the case $n = 3$, for the sake of brevity) one of the ways to solve equation (20). Assuming $G(t) = F(t) - F(0)$ we reduce equation (20) to the form

$$G(x_1 + x_2 + x_3) - G(x_1 + x_2) - G(x_1 + x_3) - G(x_2 + x_3) + G(x_1) + G(x_2) + G(x_3) = 0.$$  

Substituting $x_1 = -x_2$ leads to the relation

$$G(x_1 + x_3) + G(x_3 - x_1) = 2G(x_3) + G(x_1) + G(-x_1), \quad (x_1, x_3) \in \mathbb{R}^2.$$  

(21)

Denote by $G^e$ and $G^o$ the even and the odd components of $G$, respectively. By virtue of (21), it follows that

$$G^e(x_1 + x_3) + G^o(x_3 - x_1) = 2G^e(x_3) + 2G^e(x_1)$$

and

$$G^o(x_1 + x_3) + G^o(x_3 - x_1) = 2G^o(x_3).$$

The first of these equations is the quadratic functional equation (18), and hence $G^e(t) = \lambda t^2$. The second one coincides with a Jensen functional equation (15) where $a = b = 1/2$. Consequently, $G^e(t) = \mu t$, and hence

$$F(t) = \lambda t^2 + \mu t + F(0).$$

Introduce some new notation

$$x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n), \quad \partial_j = \partial/\partial x_j$$

and

$$\tau_j \varphi(x) = \varphi(x) |_{x_j = 0}, \quad j = 1, \ldots, n.$$
Let $z = z(x')$ be an arbitrary function continuously differentiable in the domain $D' = \{ x' \mid 0 \leq \sum_{i=1}^{n-1} x_j \leq 1 \}$. Assume that $z(0) = 1$ and $\partial_1 z(x_1, 0, \ldots, 0) \leq 0, \quad 0 \leq x_1 \leq 1$.

**Theorem 5.** If a function $F$ on $I$ is $(n - 1)$ times differentiable and satisfies the equation

$$(B_n F)(x', -z(x')) = 0, \quad x' \in D',$$

then $F(t) = \sum_{k=1}^{n} a_k t^k$ with $a_k$ being constants.

**Proof:** It can be verified directly that

$$\tau_{n-1} \partial_{n-1}(B_n F)(x', -z(x')) = B_{n-1} F'(x''', -z(x''', 0))$$

with $x''' = (x_1, \ldots, x_{n-2})$. This results in

$$\left( \prod_{j=2}^{n-1} \tau_j \partial_j \right) (B_n F)(x', -z(x')) = B_2 F^{(n-2)}(x_1, -z(x_1, 0, \ldots, 0)).$$

It follows that if $F$ satisfies equation (22), then

$$F^{(n-2)}(x_1 - \delta(x_1)) - F^{(n-2)}(x_1) - F^{(n-2)}(-\delta(x_1)) = 0, \quad 0 \leq x_1 \leq 1,$$

where $\delta(x_1) = z(x_1, 0, \ldots, 0)$. Applying Theorem 2 (all the hypotheses of this Theorem are obviously fulfilled now) completes the proof of the theorem.

**REFERENCES**


[Z] M.Zdun, On the Uniqueness of Solutions of the Functional Equation $\varphi(x + f(x)) = \varphi(x) + \varphi(f(x))$, Aequationes Math. (1972), 8, 229 - 232.

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